

Moduli of p -divisible groups

(after Fargues, Fontaine, Scholze, Weinstein)

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September 13, 2019

Lecture I: Moduli of p -divisible groups

1 Background:

- 1 Finite flat group schemes
- 2 p -divisible groups
- 3 Isogenies

2 More background:

- 1 Adic rings
- 2 Formal groups
- 3 Liftings

3 The Lubin Tate tower \mathcal{M}_∞

- 1 The Lubin Tate moduli space
- 2 Drinfeld level structure and the tower
- 3 The case $h = 1$

Motivation

A an abelian variety / k field of $char. \neq \ell$

$$Gal(\bar{k}/k) \curvearrowright T_\ell A = \varprojlim A(\bar{k})[\ell^n] \simeq \mathbb{Z}_\ell^{2g}$$

- Tate/Faltings: $T_\ell A$ determines isogeny class of A when k is a finite/number field.
- Ogg-Néron-Shafarevitch: k a number field, then $T_\ell A$ unramified at $v \nmid \ell \Leftrightarrow A$ has good reduction at v .

In $char. p$: $T_\ell A$ inadequate to study *deformations* or *variation in families* (ℓ -adic and p -adic topologies incompatible). On the other hand $0 \leq rk T_p A \leq g$ (not enough information).

Solution: Consider $A[p^\infty]$ as a p -divisible group!

Theorem (Serre-Tate)

Given A/k , the category of deformations of A is naturally equivalent to the category of deformations of $A[p^\infty]$.

Finite flat group schemes (ffgs)

- G/R finite flat (commutative) group scheme, $G = \text{Spec}(A)$, locally $A \simeq R^n$ as a module, $n = \text{rk}(G)$.
- Group structure \leftrightarrow (cocommutative) Hopf algebra structure on A

$$A \xrightarrow{m^*} A \otimes_R A, \quad A \xrightarrow{i^*} A, \quad A \xrightarrow{e^*} R.$$

Example

- (1) Γ finite abelian group, $A = R^\Gamma = \prod_{\gamma \in \Gamma} R$, constant $\underline{\Gamma} = \text{Spec}(A)$.
- (2) $\mu_n = \text{Spec}(R[X]/(X^n - 1))$, $m^*(X) = X \otimes X$, $i^*(X) = X^{-1}$, $e^*(X) = 1$.
- (3) R : \mathbb{F}_p -algebra, $\alpha_p = \text{Spec}(R[X]/(X^p))$, $m^*(X) = X \otimes 1 + 1 \otimes X$, $i^*(X) = -X$, $e^*(X) = 0$.
- (4) \mathcal{A}/R abelian scheme, $G = \mathcal{A}[m]$, $\text{rk}(G) = m^{2g}$.

- **Functor of points:** $G(-) : \text{Alg}_R \rightarrow \text{Ab}$, $S \mapsto G(S)$, fppf sheaf.
Caution: $G(S)$ need not be finite, but killed by $n = \text{rk}(G)$.

- **Cartier duality:** $A^\vee = \text{Hom}_R(A, R)$ finite flat module, coalgebra (algebra) structure of $A \rightsquigarrow$ algebra (coalgebra) structure on A^\vee by duality.

$$G^\vee = \text{Spec}(A^\vee), \quad G^{\vee\vee} = G.$$

- Represents the functor

$$G^\vee(S) = \text{Hom}_{S\text{gps}}(G_S, \mathbb{G}_{m,S}).$$

E.g. $\mu_n^\vee \simeq \underline{\mathbb{Z}/n\mathbb{Z}}$, $\alpha_p^\vee \simeq \alpha_p$, $\mathcal{A}[m]^\vee \simeq \mathcal{A}^t[m]$ (Weil pairing).

- **Lie algebra:**

$$\text{Lie}(G) = \ker(G(R[\varepsilon]) \rightarrow G(R))$$

an R -module ($[r] : a + b\varepsilon \mapsto a + rb\varepsilon$). Check: $\text{Lie}(G) \simeq$

$$\{\text{derivations of } G/R \text{ centered at } 0\} \simeq \text{Hom}_R(\omega_{G/R}, R)$$

where $\omega_{G/R} = \Omega_{A/R} \otimes_{A, e^*} R$.

- **Étale and connected:** G is *étale* $\Leftrightarrow \omega_{G/R} = 0 \Leftrightarrow \exists R \rightarrow S$ finite étale s.t. G_S is constant. G is *connected* if A has no idempotents other than 0, 1.
- R Henselian local ring (e.g. complete), *lifting idempotents* \rightsquigarrow **connected-étale exact sequence**

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{et} \rightarrow 0.$$

- If R is a *perfect field*, the sequence *splits canonically*:

$$G^{red} \hookrightarrow G, \quad G^{red} \simeq G^{et}.$$

- **The category** Ffgs_R is additive, but in general not abelian (unless R is a field). A sequence

$$0 \rightarrow G' \xrightarrow{\alpha} G \xrightarrow{\beta} G'' \rightarrow 0$$

is a SES if α is a closed immersion, β is faithfully flat and $\alpha = \ker(\beta)$ (equiv. SES as fppf sheaves).

Definition

A p -divisible group of height h over R is a system (G_n, i_n, p_n) where G_n is a ffgs of rank p^{nh} , $i_n : G_n \hookrightarrow G_{n+1}$ is a closed immersion identifying G_n with $G_{n+1}[p^n]$, and $p_n : G_n \twoheadrightarrow G_{n-1}$ is faithfully flat and satisfies $p_{n+1} \circ i_n = i_{n-1} \circ p_n = [p]_{G_n}$.

- $G = \lim_{\rightarrow} G_n$, $G_n = G[p^n]$ as an **fppf sheaf**
- **Examples:** $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$, μ_{p^∞} , $\mathcal{A}[p^\infty]$ hts 1, 1, 2g
- Notion of **SES**
- **Cartier-Serre duality:** $(G_n, i_n, p_n)^\vee = (G_n^\vee, p_{n+1}^\vee, i_{n-1}^\vee)$, e.g.

$$\mu_{p^\infty}^\vee \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}, \quad \mathcal{A}[p^\infty]^\vee \simeq \mathcal{A}^t[p^\infty].$$

- **Lie algebra:** Suppose $p^N = 0$. Since $G_n = G_{n+1}[p^n]$,

$$\text{Lie}(G_n) = \text{Lie}(G_{n+1})[p^n] = \text{Lie}(G_{n+1})$$

if $n \geq N$. Call this common module $\text{Lie}(G)$.

- **Facts:** (1) $\text{Lie}(G)$ is *locally free* of rank $d \leq h = \text{ht}(G)$. Call $d = \dim(G)$ the **dimension**.

$$(2) \text{Lie}(G_S) = \text{Lie}(G)_S, \quad (3) \dim(G) + \dim(G^\vee) = \text{ht}(G).$$

- **Connected-étale exact sequence:** R Henselian local ring \rightsquigarrow

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{\text{et}} \rightarrow 0.$$

Splits if $R =$ perfect field of char. p . Same for $G^\vee \rightsquigarrow$

$$G = G^{\text{mult}} \times G^{\text{biloc}} \times G^{\text{et}}$$

$G^{\text{mult}, \vee}$ étale, $G^{\text{biloc}, \vee}$ connected.

- $\text{Hom}(G, G')$ is a flat \mathbb{Z}_p -module. Let

$$q\text{Hom}(G, G') = \text{Hom}(G, G')[1/p].$$

The category of p -divisible groups up to isogeny has the same objects but Hom is replaced by $q\text{Hom}$. A *quasi-isogeny* is an isomorphism in this category. An *isogeny* is a quasi-isogeny which is a homomorphism.

- If G and G' are isogenous then they have the same height and dimension, and the kernel of any isogeny is a ffgs. Any quasi-isogeny has $ht \in \mathbb{Z}$ and $ht(f' \circ f) = ht(f) + ht(f')$.
- Important isogenies if R is an \mathbb{F}_p -algebra: **Frobenius** and **Verschiebung**

$$F_G : G \rightarrow G^{(p)}, \quad V_G : G^{(p)} \rightarrow G.$$

Here $G^{(p)} = G \times_{R, \phi} R$, $\phi(x) = x^p$, $F_G =$ relative Frobenius morphism, $V_G = (F_{G^\vee})^\vee$.

Definition

An *adic ring* is a complete and separated topological ring R for which \exists ideal I s.t.

$$R \simeq \varprojlim R/I^n$$

(topologically). Any such I is an *ideal of definition*, and R is " I -adic". Adic = category of adic rings and continuous hom's.

- **Examples:** R discrete ($I = 0$); completion of any ring w.r.t. a f.g. ideal; $(\mathbb{Z}_p[[u]], I = (p, u))$; $(\mathbb{Z}_p\langle u \rangle, I = (p))$.
- J is also an ideal of definition if $I^m \subset J$, $J^n \subset I$.
- If R is I -adic, $J \subset I$ and J^n is closed for all n (e.g. R noetherian) then R is complete and separated in the J -adic topology as well.
- If $R \in \text{Adic}$, $\text{Nil}(R) =$ ideal of topologically nilpotent elements.

Definition

Let $R \in \text{Adic}$, $A = R[[X_1, \dots, X_d]]$. A d -dimensional (commutative) formal group law over R is $\Phi(X, Y) \in (A \hat{\otimes}_R A)^d$ such that

- 1 $\Phi(X, 0) = X$
- 2 $\Phi(X, Y) = \Phi(Y, X)$
- 3 $\Phi(\Phi(X, Y), Z) = \Phi(X, \Phi(Y, Z))$.

- $\exists ! \iota(X) \in A^d$ without constant term s.t. $\Phi(X, \iota(X)) = 0$.
- **Examples:** $\hat{\mathbb{G}}_a$: $\Phi(X, Y) = X + Y$; $\hat{\mathbb{G}}_m$:
 $\Phi(X, Y) = X + Y + XY$; \mathcal{A}/R abelian scheme, $A = \hat{\mathcal{O}}_{\mathcal{A}, 0}$.
- $\mathcal{G}_\Phi : \text{Adic}_R \rightarrow \text{Ab}$, $\mathcal{G}_\Phi(S) = (\text{Nil}(S)^d, [+]_\Phi)$. By Yoneda, determines Φ up to isomorphism.
- A **formal group** \mathcal{G} over R is a functor $\text{Adic}_R \rightarrow \text{Ab}$ which, locally on R , is of the form \mathcal{G}_Φ .

p -divisible formal groups

- Let $R \in \text{Adic}_{\mathbb{Z}_p}$. \mathcal{G}_Φ is p -divisible if $[p]^* : A \rightarrow A$ is finite flat. (Examples: $\hat{\mathbb{G}}_m$ because $[p]^*(X) = pX + \dots + X^p$, but not $\hat{\mathbb{G}}_a$ where $[p]^*(X) = pX$).
- *Fact:* \mathcal{G} p -divisible $\Rightarrow \deg[p]^* = p^h$, $h = ht(\mathcal{G})$, otherwise $ht = \infty$.

Theorem (Tate, Messing)

Let \mathcal{G} be a p -divisible formal group. Define

$$G = \mathcal{G}(p) = (\mathcal{G}[p^n], i_n, p_n)$$

as presheaves on Adic_R . Then (i) G is a p -divisible group (ii) The functor $\mathcal{G} \rightarrow \mathcal{G}(p)$ is fully faithful from the category “ p -divisible formal groups” onto a full subcategory “formal p -divisible groups”. (iii) If R is local complete \mathfrak{m}_R -adic then G/R is formal iff it is connected (iv) the functor preserves height and dimension (where we define $\text{Lie}(G) = \lim_{\leftarrow N} \text{Lie}(G_R/p^N R)$).

- Let $R \in \text{Adic}_{\mathbb{Z}_p}$, $S \in \text{Adic}_R$, I_S an ideal of def'n. If G is a p -divisible group over R we **re-define**

$$G(S) = \varprojlim G(S/I_S^n).$$

- Assume R local complete \mathfrak{m}_R -adic. If S is discrete or G is étale, have not changed the def'n. If G is connected, and $G = \mathcal{G}(p)$ then $G(S) = \mathcal{G}(S)$.
- **Example:** $\mu_{p^\infty}(\mathcal{O}_C) = \varprojlim \mu_{p^\infty}(\mathcal{O}_C/p^n) = \varprojlim (1 + \mathfrak{m}_C \text{ mod } p^n) = 1 + \mathfrak{m}_C = \hat{\mathbb{G}}_m(\mathcal{O}_C)$. Here $C = \mathbb{C}_p$.
- **Notation:** G, G'_R . Write $\text{Hom}_S(G, G')$ for $\text{Hom}_{S_{\text{gps}}}(G_S, G'_S)$.

Theorem (Lazard, 1955)

Let $R \in \text{Adic}$, $J \subset R$ a closed ideal. Then any formal group over R/J lifts to a formal group over R .

Reason: there exists a universal d -dimensional formal group and it is defined over a free polynomial ring over \mathbb{Z} .

Theorem (Rigidity of quasi-isogenies)

Let \mathcal{F}, \mathcal{G} be p -divisible formal groups over $R \in \text{Adic}_{\mathbb{Z}_p}$, J a closed topologically nilpotent ideal in R . Then (i)

$$\text{Hom}_R(\mathcal{F}, \mathcal{G}) \hookrightarrow \text{Hom}_{R/J}(\mathcal{F}, \mathcal{G})$$

is injective. (ii) Assume $J^2 = 0$ and $p^N J = 0$. Then for any $\alpha \in \text{Hom}_{R/J}(\mathcal{F}, \mathcal{G})$, $p^N \alpha$ lifts to $\text{Hom}_R(\mathcal{F}, \mathcal{G})$. (iii) If J and p are nilpotent

$$q\text{Hom}_R(\mathcal{F}, \mathcal{G}) = q\text{Hom}_{R/J}(\mathcal{F}, \mathcal{G}).$$

Caution:

- $End_R(\hat{G}_a) \rightarrow End_{R/J}(\hat{G}_a)$ not injective (\hat{G}_a not p -div).
- $End_{\mathcal{O}_C}(\mathcal{G}) \rightarrow End_{\mathcal{O}_C/\mathfrak{m}_C}(\mathcal{G})$ not injective (\mathfrak{m}_C not top. nilp.).
- $qEnd_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow qEnd_{\mathbb{F}_p}(\mathcal{G})$ not surjective ($p\mathbb{Z}_p$ not nilpotent).

Example (One dimensional formal groups)

k algebraically closed field, char. p . For any $h \geq 1 \exists!$ 1-dimensional formal p -divisible group H_0 of height h .

- *Construction:* H/\mathbb{Z}_{p^h} Lubin-Tate formal group law with $[p]_H = pX + X^{p^h} \rightsquigarrow H_0 = H \times_{\mathbb{Z}_{p^h}} k$.
- *Endomorphisms:* Let $\mathcal{O}_D = \mathbb{Z}_{p^h}[\Pi]$, $\Pi^h = p$, $\Pi a = \sigma(a)\Pi$, the maximal order in the division algebra D of invariant $1/h$ with center \mathbb{Q}_p . Then

$$End_k(H_0) = \mathcal{O}_D.$$

- *Exercise:* H_0 defined over \mathbb{F}_p . Find End over \mathbb{F}_p (\mathbb{F}_{p^h}), and which of them lift to endomorphisms of H over \mathbb{Z}_p (\mathbb{Z}_{p^h}).

The Lubin-Tate moduli space

- Let k, H_0 as above, $W = W(k)$. Let $\mathcal{C}_k \subset \text{Adic}_W$ be the full subcategory of local complete noetherian rings with residue field k . Consider the deformation functor $\mathcal{M}^0 : \mathcal{C}_k \rightarrow \text{Sets}$

$$\mathcal{M}^0(R) = \{(G, \iota) \mid G/R \text{ 1 dim } p \text{ div gp, } \iota : G \times_R k \simeq H_0\} / \simeq.$$

- Rigid: $\text{Aut}(G, \iota) = \{1\}$
- $\mathcal{O}_D^\times \curvearrowright \mathcal{M}^0$ via $\delta(G, \iota) = (G, \delta \circ \iota)$
- Variant: ι quasi-isogeny, $\mathcal{M} = \bigsqcup_{ht(\iota)=i} \mathcal{M}^i$, $D^\times \curvearrowright \mathcal{M}$, $\Pi(\mathcal{M}^i) = \mathcal{M}^{i+1}$.

Theorem (Lubin-Tate)

\mathcal{M}^0 is representable by $\text{Spf}(A_0)$ where

$$A_0 = W[[u_1, \dots, u_{h-1}]].$$

Remark

(i) $h = 1$: a unique deformation = $\hat{\mathbb{G}}_m$. (ii) Action of \mathcal{O}_D^\times non-trivial!

Sketch: Let v_1, v_2, \dots be variables. Define $b_i \in p^{-i}\mathbb{Z}_p[v_1, v_2, \dots]$

$$b_0 = 1, \quad pb_i = v_i + b_1 v_{i-1}^p + b_2 v_{i-2}^{p^2} + \dots + b_{i-1} v_1^{p^{i-1}}$$

$$f = \sum_{i=0}^{\infty} b_i X^{p^i}, \quad F(X, Y) = f^{-1}(f(X) + f(Y)).$$

Lemma (Lazard, Hazewinkel)

(i) $F \in \mathbb{Z}_p[\underline{v}][[X, Y]]$ is a universal 1-dimensional formal group law over \mathbb{Z}_p -algebras. (ii)

$$\log_F = f \equiv X + p^{-1}v_h X^{p^h} \pmod{(v_1, \dots, v_{h-1}, X^{p^{h+1}})}$$

$$[p]_F \equiv v_h X^{p^h} \pmod{(p, v_1, \dots, v_{h-1}, X^{p^{h+1}})}.$$

Corollary

If R is an \mathbb{F}_p algebra and G is obtained from F by $v_i \mapsto 0$ ($1 \leq i < h$), $v_h \mapsto 1$ then G has height h .

- Let $H_{/A_0}^{univ}$ be obtained from F by $v_i \mapsto u_i$ ($1 \leq i < h$), $v_h \mapsto 1$, $v_i \mapsto 0$ ($h < i$). Identify $H_k^{univ} = H_0$. **Need to show** that for every $(G, \iota) \in \mathcal{M}^0(R)$, $R \in \mathcal{C}_k$, $\exists! \varphi : A_0 \rightarrow R$ and a unique isomorphism

$$G \simeq H^{univ} \times_{A_0, \varphi} R$$

lifting $\iota : G_k \simeq H_0$.

- Infinitesimal deformations:** Identify $\mathcal{M}^0(k[\varepsilon]) \simeq$

$$\text{Ext}^1(H_0, \underline{\text{Lie}}(H_0)) \simeq \text{Ext}^1(H_0, \hat{\mathbb{G}}_a) \otimes_k \text{Lie}(H_0) \simeq \text{Lie}(H_0^\vee) \otimes_k \text{Lie}(H_0)$$

(Ext group to be discussed later), so of dimension $h-1$ (in general $(h-d)d$). Identify it with the tangent space to $A_0 \otimes_W k$. This essentially shows that \mathcal{M}^0 is representable by a quotient A_0/\mathfrak{a} .

- Deformation problem is unobstructed:** Show $\mathfrak{a} = 0$.

Follows from $\text{Ext}^2(H_0, \hat{\mathbb{G}}_a) = 0$.

- Remark: $\text{Ext}^i(H_0, \hat{\mathbb{G}}_a) \simeq H^{i+1}(H_0, \hat{\mathbb{G}}_a)_s$, certain cohomology groups.

Remarks on Lubin-Tate's paper

- Lubin-Tate paper (1966) is only 10 pages long. Recommended!
- May replace formal groups by “**formal A -modules**” (A a CDVR), p (uniformizer) by π , p (degree) by q etc. See Gross-Hopkins, Drinfeld. Works in the function field case too, theory of Drinfeld modules.
- When all $u_i = 0$ one gets the “**canonical lifting**”. If $R = W$ get the Lubin-Tate formal group of height h over W . More generally, for every $[L : \mathbb{Q}_p] < \infty$ and uniformizer π of L get a unique-up-to-isomorphism “**Lubin-Tate formal group**” over \mathcal{O}_L associated with π . It plays an important role in Class Field Theory. Over \widehat{L}^{nr} the dependence on π disappears.
- Let $(G, \iota) \in \mathcal{M}^0(R)$. Let $End_R(G) = \mathcal{O} \xrightarrow{\iota} \mathcal{O}_D = End_k(H_0)$. The pairs $(G', \iota') \in \mathcal{M}^0(R)$ with $G' \simeq G$ are classified by $\mathcal{O}_D^\times / \mathcal{O}^\times$ under the action of \mathcal{O}_D^\times on $\mathcal{M}^0(R)$. Note $\mathcal{O} \supset \mathbb{Z}_p$. [L-T] $\Rightarrow \exists$ elliptic curves without CM whose p -divisible group has $End \not\supseteq \mathbb{Z}_p$.

Definition

A *Drinfeld level- n structure* on $(G, \iota) \in \mathcal{M}(R)$ is a homomorphism of ffgs/ R

$$\alpha_n : (\mathbb{Z}/p^n\mathbb{Z})^h \rightarrow G[p^n]$$

such that $\sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^h} \alpha_n(x) = G[p^n]$ as Cartier divisors.

- Each $\alpha_n(x) : \text{Spec}(R) \rightarrow G[p^n] \longleftrightarrow$ ideal I_x in the Hopf algebra of $G[p^n]$. The condition is $\prod_{x \in (\mathbb{Z}/p^n\mathbb{Z})^h} I_x = 0$.
- Equivalently, if $\mathcal{G}(p) \simeq G$,

$$[p^n]^*(X) \sim \prod_{x \in (\mathbb{Z}/p^n\mathbb{Z})^h} (X - \alpha_n(x))$$

in $R[[X]]$ (generate the same ideal). Note $\alpha_n(x) \in \mathfrak{m}_R$.

- E.g. $\alpha_n : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mu_{p^n}$ is a Drinfeld structure $\Leftrightarrow \Phi_{p^n}(\alpha_n(1)) = 0$ (the cyclotomic polynomial).

The Lubin-Tate tower

- $\mathcal{M}_n(R) = \{(G, \iota, \alpha_n) \mid (G, \iota) \in \mathcal{M}(R), \alpha_n \text{ level } n \text{ structure}\}$.
- $(g_1, g_2) \in D^\times \times GL_h(\mathbb{Z}/p^n\mathbb{Z}) \curvearrowright \mathcal{M}_n(R)$

$$(g_1, g_2)(G, \iota, \alpha_n) = (G, g_1 \circ \iota, \alpha_n \circ g_2^{-1}).$$

Theorem (Drinfeld)

- (i) $\mathcal{M}_n^0 = \text{Spf}(A_n)$ is representable.
- (ii) A_n is a regular complete local ring, finite flat over A_0 .
- (iii) if $M^0 = \text{Spa}(A_0, A_0)$ and $M_n^0 = \text{Spa}(A_n, A_n)$ (the adic spaces associated to these formal schemes) and

$$M_\eta^0 = M^0 \times_{\text{Spa}(W, W)} \text{Spa}(W[1/p], W), \quad M_{n, \eta}^0 = \dots$$

are their generic fibers, then $M_{n, \eta}^0 \xrightarrow{\pi_\eta} M_\eta^0$ is Galois étale of Galois group $GL_h(\mathbb{Z}/p^n\mathbb{Z})$.

Caution: $M_\eta^0 (M_{n, \eta}^0)$ is not an affinoid, but an “open polydisk”.

- $A_\infty = (\varprojlim A_n)^\wedge$ (I -adic completion, $I = (p, u_1, \dots, u_{h-1})$)
non-noetherian but I f.g. so A_∞ is complete and separated.
Then

$$\mathcal{M}_\infty^0 = \mathrm{Spf}(A_\infty) = \varprojlim \mathcal{M}_n^0.$$

- $M_\infty^0 = \mathrm{Spa}(A_\infty, A_\infty)$ is an adic space. (*A point to check*: its structure presheaf is sheafy, follows from the fact that A_∞ is a perfectoid ring).
- $M_{\infty, \eta}^0$ the generic fiber of M_∞^0 (open set of valuations where $|p| \neq 0$), is **the analytic Lubin-Tate space at the infinite level**.
- In Scholze's terminology

$$M_{\infty, \eta}^0 \sim \varprojlim M_{n, \eta}^0.$$

The Lubin-Tate tower, $h = 1$

- $A_0 = W$, unique deformation is $\hat{\mathbb{G}}_m$.
- α_n level- n structure iff $\Phi_{p^n}(\alpha_n(1)) = 0$ so

$$A_n = W[X]/(\Phi_{p^n}) = W[\zeta_{p^n}], \quad A_\infty = \mathcal{O}_L, \quad L = \widehat{\mathbb{Q}_p^{ab}}.$$

Proposition

L is a perfectoid field, i.e. $\phi : \mathcal{O}_L/p \rightarrow \mathcal{O}_L/p$ is surjective.

Proof. ϕ is surjective on W/p and $\phi(\zeta_{p^{n+1}}) = \zeta_{p^n}$.

- Tilt: $L^\flat = \bar{\mathbb{F}}_p((t^{1/p^\infty}))$
- First instance of: “the Lubin-Tate tower at the infinite level”
 $M_{\infty, \eta}^0$ is a perfectoid space.
- Action of \mathcal{O}_D^\times (resp. $GL_1(\mathbb{Z}_p)$) via the (resp. inverse of) cyclotomic character $\chi_{\text{cyc}} : \text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p^{nr}) \simeq \mathbb{Z}_p^\times$.

- 1 The universal covering
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 - 2 The Tate module
 - 3 Logarithms
 - 4 A simple description of \mathcal{M}_∞
- 2 A tour of crystalline Dieudonné theory
 - 1 The universal vectorial extension
 - 2 The Grothendieck-Messing crystal
 - 3 Dieudonné modules
 - 4 F -Isocrystals

The universal covering of G

- $R \in \text{Adic}_{\mathbb{Z}_p}$, G/R a p -div gp. Recall $G(S) = \lim_{\leftarrow} G(S/I_S^n)$.

Definition

$\tilde{G}(S) = \lim_{\leftarrow \times p} G(S) = \{(x_0, x_1, \dots) \mid x_i \in G(S), [p]_G(x_{i+1}) = x_i\}$.

Presheaf on Adic_R , values in \mathbb{Q}_p -vector spaces.

- Examples: $G = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$, $\tilde{G} = \underline{\mathbb{Q}_p}$; $G = \mu_{p^\infty}$, $\tilde{G}(\mathcal{O}_C) = 1 + \mathfrak{m}_C$.
- If $G \sim G'$ then $\tilde{G} \simeq \tilde{G}'$.

Lemma (Crystalline nature of \tilde{G})

$I \subset S$ closed topologically nilpotent $\Rightarrow \tilde{G}(S) = \tilde{G}(S/I)$.

Proof. Let $y = (y_0, y_1, \dots) \in \tilde{G}(S/I)$, $z_i \in G(S)$ lifting y_i . Then $x_i = \lim_{j \rightarrow \infty} [p^j](z_{i+j})$ exists, is independent of the lifting, and defines the unique $x \mapsto y$.

Corollary. Let $T \twoheadrightarrow S \in \text{Adic}_R$ be a (pro-)nilpotent thickening. For any lift G' of G to T , $\widetilde{G}'(T) = \widetilde{G}(S)$. Write $\widetilde{G}'(T) = \widetilde{G}(T)$.

Proposition

Assume R perfect \mathbb{F}_p -algebra (ϕ bijective), G formal. Then \widetilde{G} is (locally on R) representable by a formal scheme

$$\widetilde{G} = \text{Spf}(R[[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]])$$

Key idea: May replace $\lim_{\leftarrow \times p}$ by $\lim_{\leftarrow \times F}$ and get isomorphic groups. For this need to consider

$$G \xleftarrow{F} G^{(p^{-1})} \xleftarrow{F} G^{(p^{-2})} \xleftarrow{F} \dots$$

so R has to be perfect.

- $R = \mathcal{O}_C$, $k = \mathcal{O}_C/\mathfrak{m}_C = \bar{\mathbb{F}}_p$.

Theorem (Isotriviality)

There exists a quasi-isogeny

$$\rho : G \times_{\mathcal{O}_C} \mathcal{O}_C/p \dashrightarrow G_k \times_k \mathcal{O}_C/p.$$

Much deeper than “rigidity of quasi-isogenies” because $\mathfrak{m}_C/(p)$ not nilpotent - relies on theorems of Fargues and “full-faithfulness” result of Scholze-Weinstein. Crucial ingredient: C is perfectoid.

Corollary

$\tilde{G} \simeq \text{Spf}(\mathcal{O}_C[[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]])$, hence its associated analytic space $(\text{Spa}(\dots, \dots)_\eta)$ is a perfectoid (a “perfectoid open unit polydisk”).

Proof. Apply (i) crystalline nature of \tilde{G} (ii) isotriviality + invariance under isogenies: $\tilde{G}(S) = \tilde{G}(S/p) \simeq \tilde{G}_k(S/p)$, but G_k is already defined over a perfect field.

- **Tate module:** $T_p G = \varprojlim_{\leftarrow \times p} G[\rho^n]$, $V_p G = T_p G[1/p] \hookrightarrow \tilde{G}$.
- Example: When R is a perfect field

$$T_p \mathbb{G}_m = \text{Spf}(R[[X^{1/p^\infty}]]) / (X) = \text{Spec}(R[X^{1/p^\infty}] / (X)).$$

- *Warning:* If G' is a lifting to a nilpotent thickening $T \twoheadrightarrow S$, $\tilde{G}'(T) = \tilde{G}(S)$, but the subspace $V_p G'(T)$ very much depends on the lifting.
- *Goal:* an exact sequence (S flat over \mathbb{Z}_p)

$$\text{(LOG)} \quad 0 \rightarrow V_p G(S) \rightarrow \tilde{G}(S) \xrightarrow{\theta} \text{Lie}(G_S)[1/p].$$

- *Spoiler:* we shall later recover this sequence (when S is a perfectoid) as the **global sections of a “modification of vector bundles”** on the Fargues-Fontaine curve.

- G/R formal p -div $\iff \mathcal{G} = \text{Spf}(A)$, $A = R[[X_1, \dots, X_d]]$.
- $\omega_{G/R} = \oplus R dX_i|_0 \simeq \{\omega \in \Omega_{A/R} \mid m^*(\omega) = \omega \otimes 1 + 1 \otimes \omega\} =$
translation invariant differentials (*all closed*: $d\omega = 0$).
- R flat over \mathbb{Z}_p : Then $\forall \omega \exists! \lambda_\omega \in A[1/p]$ without constant term, $d\lambda_\omega = \omega$, $\lambda_\omega \in \text{Hom}_{R[1/p]}(G, \hat{G}_a)$ (*formal Poincaré lemma*).
- $\text{Hom}(\omega_{G/R}, \text{Hom}(G, \hat{G}_a)) = \text{Hom}(G, \text{Hom}(\omega_{G/R}, \hat{G}_a))$.
- $\underline{\text{Lie}}(G) = \text{Hom}(\omega_{G/R}, \hat{G}_a) \rightsquigarrow \log_G \in \text{Hom}_{R[1/p]}(G, \underline{\text{Lie}}(G))$.
- Let $\theta = \log_G \circ pr_0$ where $pr_0 : \tilde{G} \rightarrow G$ is $x \mapsto x_0$. Then (LOG) is exact.
- If $pr_0 : \tilde{G}(S) \rightarrow G(S)$ is surjective θ is surjective too.
- θ will be related to the “ θ ” of Fontaine’s rings.

A simple description of \mathcal{M}_∞

- $R \in \mathcal{C}_k$, $(G, \iota) \in \mathcal{M}(R)$, $\iota : G \times_R k \dashrightarrow H_0$, $ht(H_0) = h$. Let H/W be the Lubin-Tate group (“canonical lifting”) of H_0 .

Theorem (Hedayatzadeh, 2015)

There exists a canonical alternating multilinear $\lambda_n : G[p^n]^h \rightarrow \mu_{p^n}$ satisfying a universal property. $\rightsquigarrow \lambda_G : \tilde{G}^h \rightarrow \tilde{\mathbb{G}}_m$.

- Via ι : $\tilde{G}(S) = \tilde{G}(S/\mathfrak{m}_S) \stackrel{!}{\simeq} \tilde{H}(S/\mathfrak{m}_S) = \tilde{H}(S)$. Get

$$\exists! \lambda(\iota) \in \mathbb{Q}_p^\times, \text{ord}_p(\lambda(\iota)) = ht(\iota), \lambda_H \circ \iota = \lambda(\iota) \cdot \lambda_G.$$

- $\tilde{H} = \text{Spf}(W[[X^{1/p^\infty}]])$, $\tilde{\mathbb{G}}_m = \text{Spf}(W[[T^{1/p^\infty}]])$, $\lambda_H \rightsquigarrow$

$$T^{1/p^n} \mapsto \text{compatible } \delta^{1/p^n} \in R[[X_1^{1/p^\infty}, \dots, X_h^{1/p^\infty}]]$$

$\delta(\dots, X_i[+]_H X'_i, \dots) = \delta(\dots, X_i \cdots)[+]_{\mathbb{G}_m} \delta(\dots, X'_i, \dots)$ and alternating condition.

A commutative diagram of formal schemes over $\mathrm{Spf}(W)$ ($L = \widehat{\mathbb{Q}_p^{ab}}$)

$$(LT_\infty) \quad \begin{array}{ccc} \mathcal{M}_\infty & \longrightarrow & \bigsqcup_{ht(\iota)=i} \mathrm{Spf}(\mathcal{O}_L) \\ \downarrow & & \downarrow \underline{t} \\ \tilde{H}^h & \xrightarrow{\lambda_H} & \tilde{\mathbb{G}}_{m,W} \end{array}$$

- Right $\underline{t} \iff T^{1/p^n} \mapsto \left(\lim_m (\zeta_{p^m} - 1)^{p^{m-n-i}} \right)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} \mathcal{O}_L$
- Top $(G, \iota, \alpha_\infty) \in \mathcal{M}_\infty^0 \mapsto (\zeta_{p^n} \mapsto \lambda(\iota) \cdot \lambda_{n,G}(\alpha_{n,1}, \dots, \alpha_{n,h}))$.
- Left $(G, \iota, \alpha_\infty) \mapsto (\alpha_{n,1}, \dots, \alpha_{n,h})_{n=1}^\infty \in \tilde{\mathbb{G}}^h \stackrel{\iota}{\simeq} \tilde{H}^h$.

Theorem (Weinstein, 2016)

(i) The diagram is cartesian. (ii) Action of $g \in GL_h(\mathbb{Q}_p)$: on \tilde{H}^h via right action of g^{-1} on row vectors, on $\tilde{\mathbb{G}}_{m,W}$ via $\det g^{-1}$, similarly on $\bigsqcup_{ht(\iota)=i} \mathrm{Spf}(\mathcal{O}_L)$ (p shifts between copies, \mathbb{Z}_p^\times acts like inverse cyclotomic character) (iii) Action of D^\times : via $D = q\mathrm{End}(H_0)$ on $\tilde{H} = \tilde{H}_0$, via Nrd on right column.

Remark. ($i = 0$) $\mathrm{Spf}(\mathcal{O}_L) \simeq T_p \hat{\mathbb{G}}_{m,W}^{\mathrm{prim}} \subset V_p \hat{\mathbb{G}}_{m,W} \hookrightarrow \tilde{\mathbb{G}}_{m,W}$. When $h = 1$ the bottom row is the identity. In general \mathcal{M}_∞^0 is the fiber of λ_H at the \mathcal{O}_L -point \underline{t} of $\tilde{\mathbb{G}}_{m,W}$. Unlike the horizontal maps, the vertical maps *do not make sense at finite levels*.

Corollary

(i) The group $(GL_h(\mathbb{Q}_p) \times D^\times)^{\det=Nrd}$ acts on the cartesian diagram (trivially on $\tilde{\mathbb{G}}_{m,W}$).

(ii) Explicitly, let $t^{1/p^n} = \underline{t}^*(T^{1/p^n})$, $\mathcal{M}_\infty = \mathrm{Spf}(A_\infty)$,

$$A_\infty = \mathcal{O}_L[[X_1^{1/p^\infty}, \dots, X_h^{1/p^\infty}]]/(\delta^{1/p^n} - t^{1/p^n}).$$

(iii) $\phi(x) = x^p$ is surjective on A_∞/p .

(iv) $M_{\infty,\eta}$ is a perfectoid space.

Scholie: The Lubin-Tate tower at the *infinite level* is *infinitely simpler* than at finite levels, and in addition is a perfectoid!

The universal vectorial extension

G/R p -div gp, $p^N = 0$ in R . The sequence of fppf sheaves on Alg_R

$$0 \rightarrow G[p^n] \rightarrow G \xrightarrow{p^n} G \rightarrow 0$$

is exact. Applying $R\text{Hom}(-, \mathbb{G}_a)$ get a SES

$$0 \rightarrow \text{Hom}(G, \mathbb{G}_a)/p^n \rightarrow \text{Hom}(G[p^n], \mathbb{G}_a) \rightarrow \text{Ext}(G, \mathbb{G}_a)[p^n] \rightarrow 0.$$

- $\text{Hom}(G, \mathbb{G}_a) = 0$ since G is p -divisible but $p^N \mathbb{G}_a = 0$.
- $n \geq N \Rightarrow \text{Ext}(G, \mathbb{G}_a) = \text{Hom}(G[p^n], \mathbb{G}_a) = \{a \in A_n \mid m_G^*(a) = a \otimes 1 + 1 \otimes a\} = \text{Lie}(G^\vee[p^n]) = \text{Lie}(G^\vee) = \text{Hom}(\omega_{G^\vee}, R)$.
- Similarly for any R -module $\text{Ext}(G, \underline{M}) \simeq \text{Hom}(\omega_{G^\vee/R}, M)$.
- Taking $M = \omega_{G^\vee/R}$ and the identity \rightsquigarrow “**universal**” extension

$$0 \rightarrow \underline{\omega}_{G^\vee/R} \rightarrow EG \rightarrow G \rightarrow 0$$

from which *any* (fppf sheaf) extension of G by a vector-group \underline{M} is gotten by a unique push-out.

The Grothendieck-Messing crystal MG

Take $\underline{Lie}(-)$. Get a SES of vector groups ($MG = \underline{Lie}(EG)$)

$$0 \rightarrow \omega_{G^\vee/R} \rightarrow MG \rightarrow \underline{Lie}(G) \rightarrow 0.$$

- $\forall S \in \text{Alg}_R$, $MG(S)$ a locally free module, $rk(MG) = ht(G)$.
- **Goal:** enhance MG to a *crystal* of modules on the crystalline site. *Need:* MG classifies *rigidified extensions* of G^\vee by \mathbb{G}_a .
- A **rigidification** of an extension E of G by \mathbb{G}_a is a splitting

$$0 \rightarrow \mathbb{G}_a \rightarrow Lie(E) \xrightarrow{\leftarrow} Lie(G) \rightarrow 0.$$

Any two rigidifications differ by a homomorphism from $Lie(G)$ to \mathbb{G}_a , i.e. by an element of $\omega_{G/R}$. The group of rigidified extensions $Ext^{\natural}(G^\vee, \mathbb{G}_a)$ sits in an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \omega_{G^\vee/R} & \rightarrow & Ext^{\natural}(G^\vee, \mathbb{G}_a) & \rightarrow & Ext(G^\vee, \mathbb{G}_a) \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & \omega_{G^\vee/R} & \rightarrow & MG(R) & \rightarrow & Lie(G) \rightarrow 0. \end{array}$$

- **Big crystalline site** over $R \in \text{Alg}_{\mathbb{Z}_p}$: Objects are diagrams

$$\begin{array}{ccc} T & \xrightarrow{pd} & S \\ & & \uparrow \\ & & R \end{array}$$

$S \in \text{Alg}_R$, T a nilpotent divided powers thickening of S . If S is \mathbb{Z}_p -flat: $x \in I = \ker(T \twoheadrightarrow S) \Rightarrow x^n/n! \in I$, and $\exists N$ s.t. $(x_1^{n_1}/n_1!) \cdots (x_r^{n_r}/n_r!) = 0$ if $x_i \in I$, $\sum n_i \geq N$. Morphisms “preserve the pd structure”.

- Coverings of $T \xrightarrow{pd} S$: $\{(T_i \xrightarrow{pd} S_i) \rightarrow (T \xrightarrow{pd} S)\}$ s.t. $\text{Spec}(T) = \bigcup \text{Spec}(T_i)$ a Zariski cover, $S_i = S \otimes_T T_i$.
- Structure sheaf $\mathcal{O}(T \xrightarrow{pd} S) = \mathcal{O}(T)$.

Theorem (Grothendieck-Messing)

If $(T \xrightarrow{pd} S)$ as above and G'_T is a lifting of G_S to T then MG'_T depends functorially only on G . Denote it by $MG(T \twoheadrightarrow S)$.

- A locally free coherent sheaf, $MG(S) = M(G_S)$ is $MG(S \twoheadrightarrow S)$.

Explanation (Katz): $R \in \text{Adic}_{\mathbb{Z}_p}$, \mathbb{Z}_p -flat. Let \mathcal{F}/R a p -divisible formal group, $\mathcal{F} = \text{Spf}(R[[X_1, \dots, X_d]])$.

- $H_{dR}^1(\mathcal{F}/R) = \{[\eta] \mid \eta \text{ closed, } m_{\mathcal{F}}^*(\eta) - \eta \otimes 1 - 1 \otimes \eta \text{ exact}\}$
translation invariant cohomology classes.
- $\omega_{\mathcal{F}/R} = \{\eta \mid m_{\mathcal{F}}^*(\eta) = \eta \otimes 1 + 1 \otimes \eta\} \hookrightarrow H_{dR}^1(\mathcal{F}/R)$, because R is p -adic, and \mathcal{F} is p -divisible, so η exact and translation invariant $\Rightarrow \eta = 0$. (Logarithms need $R[1/p]$.)

$$\begin{array}{ccccccc}
 0 & \rightarrow & \omega_{\mathcal{F}/R} & \rightarrow & H_{dR}^1(\mathcal{F}/R) & \xrightarrow{\partial} & H^2(\mathcal{F}; \mathbb{G}_a)_s & \rightarrow & 0 \\
 & & \parallel & & \parallel & & \parallel & & \\
 0 & \rightarrow & \omega_{\mathcal{F}/R} & \rightarrow & \text{Ext}^1(\mathcal{F}; \mathbb{G}_a) & \rightarrow & \text{Ext}(\mathcal{F}; \mathbb{G}_a) & \rightarrow & 0
 \end{array}$$

- Bottom row identified, when $G^\vee \xleftarrow{\sim} \mathcal{F}$, with

$$0 \rightarrow \omega_{G^\vee/R} \rightarrow MG(R) \rightarrow \text{Lie}(G) \rightarrow 0.$$

- We explain the map ∂ :

$$H^2(\mathcal{F}; \mathbb{G}_a)_s = \frac{\{\Delta(X, Y) \in R[[X; Y]] \mid \text{symm.}, \delta(\Delta) = 0\}}{\{\delta(f) = f(X[+]Y) - f(X) - f(Y)\}}$$

$$\delta(\Delta) = \Delta(Y, Z) - \Delta(X[+]Y, Z) + \Delta(X, Y[+]Z) - \Delta(X, Y).$$

The map from $H_{dR}^1(\mathcal{F}/R)$ is: find a primitive $f(X) \in R[1/p][[X]]$ for η , let $\Delta = \delta(f)$. $[\eta]$ is translation invariant $\Rightarrow \Delta$ is integral: $\delta(\eta) = d\Delta$. Set $\partial([\eta]) = [\Delta]$.

The identification $H^2(\mathcal{F}; \mathbb{G}_a)_s \simeq \text{Ext}(\mathcal{F}; \mathbb{G}_a)$ is standard, that of $H_{dR}^1(\mathcal{F}/R) \simeq \text{Ext}^{\flat}(\mathcal{F}; \mathbb{G}_a)$ requires only a little more work.

Lemma

Let $\mathcal{F}', \mathcal{F}''$ be liftings of \mathcal{F} to $T \xrightarrow{pd} R$. Let $\varphi: \mathcal{F}' \rightarrow \mathcal{F}''$ be a morphism of pointed Lie varieties reducing to the identity on R . Then (i) $\varphi^*: H_{dR}^1(\mathcal{F}''/T) \simeq H_{dR}^1(\mathcal{F}'/T)$ (preserving the invariance under the group law). (ii) φ^* is independent of φ . (iii) Similarly, if φ reduces to an endomorphism φ_0 of \mathcal{F} , φ^* is a homomorphism that depends only on φ_0 .

Proof.

($d = 1$) Let $\eta = df$, $f \in T[1/p][[X]]$, represent $[\eta] \in H_{dR}^1(\mathcal{F}''/T)$.

Let $I = \ker(T \xrightarrow{pd} R)$, $\varphi_1, \varphi_2 \in T[[Y]]$, $\varphi_i(0) = 0$, $\varphi_1 \equiv \varphi_2 \pmod{I}$.

Then by Taylor

$$\varphi_2^*(\eta) - \varphi_1^*(\eta) = d \left(\sum_{n=1}^{\infty} f^{(n)}(\varphi_1) \cdot \frac{(\varphi_2 - \varphi_1)^n}{n!} \right)$$

and $(\dots) \in T[[Y]]$ since I has divided powers and $f^{(1)}$ is already integral. This shows (ii) $\varphi_2^*([\eta]) = \varphi_1^*([\eta])$. A similar argument proves (i) and (iii). □

- Explains phrase: “ $MG'_T(T)$ depends functorially only on G ”.
- It is *blatantly false* that φ^* maps $\omega_{\mathcal{F}''/T}$ to $\omega_{\mathcal{F}'/T}$.
- The proof highlights the use of “divided powers”.
- Also: relation between crystalline and de Rham cohomology.
- For a proof when R is not \mathbb{Z}_p -flat see Messing's thesis.

Dieudonné modules

k perfect field, $\text{char. } p$, $W = W(k)$, σ the Frobenius automorphism.
 G/k p -div gp. Its Dieudonné module is

$$D(G) = M(G^\vee)(W \twoheadrightarrow k).$$

- *Contravariant*, free W -module $\text{rk } h = ht(G)$.
- $F_G : G \rightarrow G^{(p)}$, dual isogeny $V_{G^\vee} : G^{\vee(p)} \rightarrow G^\vee$. By functoriality of $M(-)$ get $F : D(G)^{(p)} \rightarrow D(G)$, i.e. σ -linear $D(G) \rightarrow D(G)$. Similarly, $V_G : G^{(p)} \rightarrow G \rightsquigarrow \sigma^{-1}$ -linear V .

$$F \circ V = V \circ F = p.$$

- $(D(G), F, V)$ - an F -crystal over k . Form an additive category.

Theorem (Dieudonné-Manin)

$D(-)$ is an anti-equivalence between pdivgp_k and Fcryst_k .

- $M(G^\vee)(k) = D(G)/pD(G) (\simeq H_{dR}^1(\mathcal{A}/k)$ if $G = \mathcal{A}[p^\infty]$).
- $\omega_{G/k} \simeq VD(G)^{(p^{-1})}/pD(G)$.
- Original equivalent def'n: $D(G) = \text{Hom}_k(G, CW)$ where CW is the group of co-Witt vectors $\curvearrowright F, V$.

F -isocrystals (N, F, V) - N a f.dim. $W[1/p]$ -vector space, F, V as above. An equivalence of categories between “ p -div gps up to isogeny” and “ F -isocrystals containing an invariant F -crystal”.

- Standard example: $(r, s) = 1, s > 0, \lambda = r/s$. Let $N_\lambda = \sum_{i=1}^s W[1/p]e_i, Fe_i = e_{i+1} (i < s), Fe_s = p^r e_1$. Call λ the (Frobenius) *slope*.

Theorem

Let k be alg. closed. The category of F -isocrystals over k is semisimple. Its simple objects are the N_λ . An F -isocrystal contains an F -crystal iff all its slopes are contained in $[0, 1]$.

- $End(N_\lambda) = D_{-\lambda}$, division ring over \mathbb{Q}_p with invariant $-\lambda \pmod{1}$. (If $N = D(G)[1/p]$ this means $qEnd(G) \simeq D_\lambda$.)
- Lubin-Tate case $\lambda = 1/h$.
- *Exercise:* If $0 \leq r \leq s$ extend e'_i by $e'_{i+ms} = p^m e'_i$, define the F -crystal

$$M_\lambda = \sum_{i=1}^s We'_i, Fe'_i = e'_{i+r}, Ve'_i = e'_{i+s-r}.$$

Then N_λ has a lattice isomorphic to M_λ (but there are others).

- Let k be perfect. Call N *isoclinic of slope λ* if $N \otimes_k \bar{k} \simeq N_\lambda^n$.

Proposition (Slope decomposition)

Let k be perfect and N an F -isocrystal over k . Then $N = \bigoplus_{\lambda \in \mathbb{Q}} N(\lambda)$ where $N(\lambda)$ is isoclinic of slope λ .

Let $\lambda_1 < \lambda_2 < \dots < \lambda_n$ be the slopes of N . Then the **Newton polygon** $NP(N)$ is convex, starts at $(0,0)$, and has slopes λ_i with horizontal length $rk(N(\lambda_i))$. Break points are in \mathbb{Z}^2 .

- 1 The Grothendieck-Messing (GM) period map
 - 1 The quasi-logarithm and a big diagram
 - 2 Rapoport-Zink deformation spaces
 - 3 The Grothendieck-Messing period map
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- 2 The Hodge-Tate (HT) period map
 - 1 The Hodge-Tate decomposition
 - 2 Hodge-Tate period map
- 3 Example: Drinfeld's p -adic symmetric domain
 - 1 Rapoport-Zink spaces with PEL structure
 - 2 The Drinfeld moduli problem

Quasi logarithms and a big diagram

- $R, S \in \text{Adic}_{\mathbb{Z}_p}$, $\pi : S \xrightarrow{pd} R$, $S \simeq \lim_{\leftarrow} S/(\ker \pi)^n$. Assume S flat over \mathbb{Z}_p , e.g. $\mathcal{O}_C \rightarrow \mathcal{O}_C/p$.
- Let G/S lift G_0/R . Both \tilde{G} and MG have a “crystalline nature”. We relate them. Top maps to bottom via log’s (not shown).

$$\begin{array}{ccccccc}
 & \omega_{G^v/S} & \hookrightarrow & EG_0(S) & \cdots & \rightarrow & G(S) \\
 & \nearrow \alpha_G & & \nearrow s_G & & \nearrow & \downarrow \\
 T_p G(S) & \hookrightarrow & \tilde{G}_0(S) & \xrightarrow{pr_0} & G(S) & & \log_G \\
 \downarrow & & & & & & \downarrow \\
 \vdots & & & & & & \\
 \downarrow & \omega_{G^v/S, \mathbb{Q}} & \hookrightarrow & MG_0(S)_{\mathbb{Q}} & \cdots & \rightarrow & \text{Lie}(G)_{\mathbb{Q}} \\
 & \nearrow & & \nearrow q\log & & \nearrow & \\
 V_p G(S) & \hookrightarrow & \tilde{G}_0(S) & \xrightarrow{\theta} & \text{Lie}(G)_{\mathbb{Q}} & &
 \end{array}$$

- $s_G(x_0, x_1, \dots) = \lim [p^n]_{EG}(\xi_n)$, if $EG(S) \ni \xi_n \mapsto x_n \in G(S)$.
- $\alpha_G = s_G|_{T_p G(S)}$ has the following interpretation:

$$x \in T_p G(S) \rightsquigarrow x_n \in G(S)[p^n] = \text{Hom}_S(G^\vee[p^n], \hat{G}_m)$$

$$\rightsquigarrow \text{Lie}(x) \in \text{Hom}(\text{Lie}(G^\vee), \hat{G}_a) = \omega_{G^\vee/S}.$$

- $\text{qlog}_G = \log_{EG} \circ s_G$. If $G \leftarrow \mathcal{G}$, fix coordinates on $E\mathcal{G}$, let $x = (x_0, x_1, \dots)$ and ξ_n as above, then

$$\text{qlog}_G(x) = \lim_m \lim_n \frac{1}{p^m} [p^{n+m}]_{E\mathcal{G}}(\xi_n).$$

- $\theta = \log_G \circ pr_0 = pr_{\text{Lie}(G)}^{MG} \circ \text{qlog}_G$.
- The maps s_G, qlog_G are morphisms of crystals; θ, α_G depend on G and will be related to the GM / HT period maps.

Rapoport-Zink deformation spaces

Generalize Lubin-Tate space: k alg. closed char. p , $W = W(k)$, H_0/k p -div gp, $ht = h$, $\dim = d$. Set

$$M_0 = MH_0(W \twoheadrightarrow k) = D(H_0^\vee) \simeq W^h.$$

- $S \in \text{Nilp}_W = W$ -algebras on which p is loc. nilp., e.g. \mathcal{O}_C/p^N .

$$\mathcal{D}(S) = \{(G, \iota) \mid G/S \text{ } p \text{ div gp, } \iota : G \times_S S/p \xrightarrow{q.i.} H_0 \times_k S/p\}.$$

- If $S = p$ -adic W -algebra, let $\mathcal{D}(S) = \lim_{\leftarrow} \mathcal{D}(S/p^N)$.
- If S also flat over W , ι induces $MG(S)_{\mathbb{Q}} \simeq M_0 \otimes_W S_{\mathbb{Q}}$.
- Lubin-Tate case: $d = 1$, H_0 formal, $R \in \mathcal{C}_k$ and $S = R/\mathfrak{m}_R^N$, $\mathcal{D}(S) = \mathcal{M}(S)$, since by rigidity of quasi-isogenies

$$q\text{Hom}_{S/p}(G, H_0) \simeq q\text{Hom}_{R/\mathfrak{m}_R}(G, H_0),$$

but note $\mathcal{D}(k[[u]]) \neq \mathcal{M}(k[[u]])$.

- In general, a quasi-isogeny $G_0 \rightarrow H_0$ of height 0 need not be an isomorphism, so can not replace $\xrightarrow{q.i.}$ by \simeq , even on \mathcal{D}^0 .

Example

1) $d = 1, h = 2, H_0 = \underline{\mathbb{Q}_p/\mathbb{Z}_p} \times \mu_{p^\infty}$. Since $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ and μ_{p^∞} do not deform,

$$\mathcal{D}^0(S) = \text{Ext}_S(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}) \simeq \widehat{\mathbb{G}_m}(S)$$

(“Serre-Tate canonical coordinate”). Note $\mathcal{D}^0(k)$ is a point.

2) $d = 2, h = 4, H_0 = H_{1/2} \times H_{1/2}$ where $H_{1/2}$ is the 1-dimensional formal group of height 2 (the formal group of a supersingular elliptic curve). $\mathcal{D}^0(k)$ will be infinite because there are $\mathbb{P}^1(k)$'s of pairwise non-isomorphic G isogenous to H_0 (Moret-Bailly families).

Example

3) Lubin-Tate case: H_0 the unique 1-dimensional ht h formal p -div gp over k . Then $\mathcal{D}^0 = \mathcal{M}^0$ and again $\mathcal{D}^0(k)$ is a point.

Theorem (Drinfeld, Rapoport-Zink)

The functor \mathcal{D} is “representable” by a formal scheme over W whose ideal of definition is locally finitely generated. Every irreducible component of its (reduced) special fiber is proper over k .

- The period map π_{GM} will be a map of *analytic spaces* (over $W[1/p]$) from \mathcal{D}_η^{ad} (the generic fiber of the adic space associated with the formal scheme \mathcal{D}) to $Gr(d, M_0)_\eta^{ad}$. For simplicity we only describe it on (C, \mathcal{O}_C) -points.

The Grothendieck-Messing period map

- Take $S = \mathcal{O}_C$. For $(G, \iota) \in \mathcal{D}(\mathcal{O}_C)$ we have a quotient map

$$M_0 \otimes_W C \xrightarrow{\iota^{-1}} MG(\mathcal{O}_C)_{\mathbb{Q}} \twoheadrightarrow Lie(G_C)$$

from our fixed $M_0 \otimes_W C \simeq C^h$ onto a d -dimensional vector space.

- This defines a “period map” from the moduli space to a Grassmanian

$$\pi_{GM}(G, \iota) \in Gr(d, M_0)(C) \simeq Gr(d, h)(C).$$

- *Fact:* The period map $\pi_{GM} : \mathcal{D}_{\eta}^{ad} \rightarrow Gr(d, M_0)_{\eta}^{ad}$ is an *étale analytic* map.
- *Example (Dwork):* $d = 1, h = 2, H_0 = \underline{\mathbb{Q}_p/\mathbb{Z}_p} \times \mu_{p^\infty}$. Then $\mathcal{D} = \widehat{\mathbb{G}}_m, \mathcal{D}_{\eta}^{ad}$ is the open unit disk Δ around 1, and $\pi_{GM} : \Delta \rightarrow \mathbb{A}^1 \subset \mathbb{P}^1$ is $q \mapsto \log(q)$.

- If $(G, \iota) \in \mathcal{D}(W)$ (*unramified*) the Grothendieck-Messing theorem allows to identify the *deformation* with its *period*:

Theorem (Grothendieck-Messing)

The assignment $G \mapsto \text{Lie}(G)$ is a bijection between the liftings G of H_0 to W (up to strict isomorphism) and the liftings of $MH_0(k) \twoheadrightarrow \text{Lie}(H_0)$ to a free quotient $M_0 \twoheadrightarrow L$ over W .

- As a corollary, in the Lubin-Tate case, the map sending $(G, \iota) \in \mathcal{M}^0(W)$ to $\pi_{GM}(G, \iota) \in \text{Gr}(1, M_0)(W) \simeq \mathbb{P}^{h-1}(W)$ is one-to-one, and its image is the W -points of the residue disk R_x in $\mathbb{P}^{h-1}(W)$ reducing to $x = [MH_0(k) \twoheadrightarrow \text{Lie}(H_0)]$.

However:

- The relation between the Lubin-Tate coordinates $(u_1, \dots, u_{h-1}) \in \mathfrak{m}_W^{h-1}$ and the projective-space coordinates on the residue disk R_x is the period morphism.

- If K is a finite *ramified* extension of $W[1/p]$, $\mathcal{O}_K \rightarrow k$ is no longer a pd thickening, so the theorem does not apply. We still have $\mathcal{D}^0(\mathcal{O}_K) \simeq \mathcal{M}^0(\mathcal{O}_K) \approx \mathfrak{m}_K^{h-1}$, but a quasi-isogeny of height 0 over \mathcal{O}_K/p is *not necessarily an isomorphism*, so $(G, \iota) \in \mathcal{D}^0(\mathcal{O}_K)$ only provides a map

$$M_0 \otimes_W K \simeq MG(\mathcal{O}_K)_{\mathbb{Q}} \rightarrow \text{Lie}(G_K),$$

i.e. a point of $Gr(1, M_0)(K) \simeq \mathbb{P}^{h-1}(K)$. Since it is not defined integrally, we can not talk about its reduction.

- The resulting period map from $\mathcal{D}^0(\mathcal{O}_K)$ to $\mathbb{P}^{h-1}(K)$ is not 1 : 1 in general, and its image is not confined any more to a residue disk. In the Lubin-Tate case (but not in general), when K is replaced by C it is even surjective, and its fibers are infinite.

The period map (Lubin-Tate case)

- $\pi_{GM} : \mathcal{M}(\mathcal{O}_C) \rightarrow \mathbb{P}^{h-1}(C)$ studied by Gross-Hopkins.

Theorem (Gross-Hopkins)

(i) π_{GM} is D^\times -equivariant.

(ii) $\pi_{GM}(G, \iota) = \pi_{GM}(G', \iota') \Leftrightarrow \exists f : G \xrightarrow{q.i.} G', \iota' \circ \bar{f} = \iota.$

(iii) $\pi_{GM}^0 : \mathcal{M}^0(\mathcal{O}_C) \rightarrow \mathbb{P}^{h-1}(C)$ is surjective.

(iv) $M_{\infty, \eta} \rightarrow M_\eta \rightarrow \mathbb{P}_C^{h-1}$ gives $\mathbb{P}_C^{h-1} = M_{\infty, \eta} / GL_h(\mathbb{Q}_p).$

- Part (i) follows from the definitions. Action of D^\times on $\mathbb{P}^{h-1}(C)$ is via the (projective) regular representation. The element Π acts (in appropriate coordinates) like

$$\begin{pmatrix} 0 & & & & p \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \dots & \dots & \\ & & & 1 & 0 \end{pmatrix}.$$

- The “if” part in (ii) follows from the definitions. The “only if” follows from Grothendieck-Messing.
- Part (iii) is proved in [G-H] by a detailed analysis of π_{GM} “in coordinates”.
- Part (iv) follows from (ii) and (iii).

In general, the image of π_{GM} is restricted by the notion of “weak admissibility”. Given an exact sequence

$$0 \rightarrow \text{Fil} \rightarrow M_{0,C} \rightarrow W \rightarrow 0$$

with associated filtration $\text{Fil}^0 = M_{0,C} \supset \text{Fil}^1 = \text{Fil} \supset \text{Fil}^2 = 0$, $\underline{N} = (N, \text{Fil}) = (M_{0,C}, \text{Fil})$ becomes a *filtered F-isocrystal*.

- If N' is a sub- F -isocrystal let $Fil' = Fil \cap (N' \otimes_{W[1/p]} \mathbb{C})$, $\underline{N}' = (N', Fil')$.
- For any filtered F -isocrystal \underline{N} define

$$t_{Newton}(\underline{N}) = v_p(\det(F))$$

(independent of the matrix representing F , since this matrix is unique up to σ -conjugation),

$$t_{Hodge}(\underline{N}) = \sum i \dim gr_{Fil}^i = \dim Fil.$$

- Call $\underline{N} = (N, Fil)$ *weakly admissible* if for any sub F -isocrystal $N' \subset N$

$$t_{Hodge}(\underline{N}') \leq t_{Newton}(\underline{N}')$$

with equality for $\underline{N}' = \underline{N}$.

- Given H_0/k , the *weakly admissible period domain* is an open subspace $\mathfrak{F}^{wa} \subset Gr(d, M_0)_\eta^{ad}$ such that $\mathfrak{F}^{wa}(C)$ consists of all d -dimensional quotients

$$M_{0,C} \rightarrow W$$

for which \underline{N} is weakly admissible.

Theorem

- (i) The image of $\pi_{GM} : \mathcal{D}_\eta^{ad} \rightarrow Gr(d, M_0)_\eta^{ad}$ factors through \mathfrak{F}^{wa} .
- (ii) The image contains all the classical points of \mathfrak{F}^{wa} (points whose residue field is a finite extension of $K_0 = W[1/p]$).

Remarks: (i) is easy. (ii) (Colmez-Fontaine) “weakly admissible filtered isocrystals are admissible”. We shall later relate it to the geometry of the Fargues-Fontaine curve. Hartl describes the non-classical points in $\mathfrak{F}^a = \text{Im}(\pi_{GM})$. In general $\mathfrak{F}^a \neq \mathfrak{F}^{wa}$.

The Hodge-Tate decomposition

Recall the map $\alpha_G : T_p G(R) \rightarrow \omega_{G^v/R}$. Let $R = \mathcal{O}_C$ and let $-(1)$ denote Tate twist. The following theorem was the beginning of p -adic Hodge theory, 50 years ago.

Theorem (Tate)

(i) *There is an exact sequence*

$$0 \rightarrow \mathrm{Lie}(G_C)(1) \xrightarrow{\alpha_{G^v}^v(1)} T_p G(\mathcal{O}_C) \otimes_{\mathbb{Z}_p} C \xrightarrow{\alpha_G} \omega_{G^v/C} \rightarrow 0.$$

(ii) *(Hodge-Tate decomposition) If G is defined over \mathcal{O}_K where $K \subset C$ is a complete discrete valuation field, then the sequence splits canonically (respecting $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$ action)*

$$T_p G(\mathcal{O}_C) \otimes_{\mathbb{Z}_p} C \simeq \omega_{G^v/C} \oplus \mathrm{Lie}(G_C)(1).$$

- The map $\alpha_{G^\vee}^\vee(1)$ sends $Lie(G_C)(1) = Hom(\omega_{G/C}, T_p\mu_{p^\infty} \otimes C)$ to

$$Hom(T_p G^\vee \otimes C, T_p\mu_{p^\infty} \otimes C) \simeq T_p G \otimes C.$$

- To get (ii) from (i) invoke Tate's theorems that $H^0(\Gamma_K, C(i)) = H^1(\Gamma_K, C(i)) = 0$ if $i \neq 0$ and both cohomology groups are 1-dimensional if $i = 0$. In the absence of Galois action, there is no canonical splitting of (i).
- Let $G = \mathcal{A}[p^\infty]$. Dualizing, (i) is equivalent to the existence of a spectral sequence (Faltings: the Hodge-Tate spectral sequence)

$$E_{i,j}^2 = H^i(\mathcal{A}, \Omega_{\mathcal{A}/C}^j)(-j) \Rightarrow H_{et}^{i+j}(\mathcal{A}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C.$$

Compare with the Hodge spectral sequence that starts with $E_{i,j}^1 = H^j(\mathcal{A}, \Omega_{\mathcal{A}/C}^i)$. This applies to any proper smooth variety over C .

The Hodge-Tate period map

- The fact that the Hodge-Tate decomposition is not valid in families, only the HT *filtration*, leads to the HT *period map*, just as over \mathbb{C} the fact that only the Hodge *filtration* varies holomorphically in families lies behind the classical period map to classifying spaces of Hodge structures.

The **Hodge-Tate period map**. Consider (Lubin-Tate case)

$$(G, \iota, \alpha_\infty) \in \mathcal{M}_\infty(\mathcal{O}_C).$$

Use $\alpha_\infty : \mathbb{Z}_p^h \simeq T_p G(\mathcal{O}_C) \rightsquigarrow (\alpha_G \otimes 1) \circ (\alpha_\infty \otimes 1) : C^h \rightarrow \omega_{G^\vee/C}$, whose kernel is a *line*. Mapping $(G, \iota, \alpha_\infty)$ to this line is

$$\pi_{HT} : \mathcal{M}_\infty(\mathcal{O}_C) \rightarrow \mathbb{P}^{h-1}(C).$$

Unlike π_{GM} , π_{HT} is defined only on \mathcal{M}_∞ . It goes *canonically* to $\mathbb{P}^{h-1}(C)$ while π_{GM} landed in $\mathbb{P}(M_0)(C) \simeq \mathbb{P}^{h-1}(C)$.

- *Fact:* π_{HT} comes from an *analytic* map $M_{\infty, \eta} \rightarrow (\mathbb{P}^{h-1})_{\eta}^{ad}$. In our “basic” case (but not always), it is also *étale*.
- For $\delta \in D^{\times}$, $\pi_{HT} \circ \delta = \pi_{HT}$ (obvious).
- π_{HT} intertwines the actions of $GL_h(\mathbb{Q}_p)$ on $M_{\infty, \eta}$ and \mathbb{P}^{h-1} (obvious).

A global detour ($h = 2$): *modular curves at the infinite level.* Let Y_n be the (open) modular curve of level p^n over \mathbb{Q}_p and Y_{∞} the scheme $\lim_{\leftarrow} Y_n$. A point of $Y_{\infty}(C)$ is an elliptic curve E/C equipped with an isomorphism $\alpha_{\infty} : \mathbb{Z}_p^2 \simeq T_p E$. As above, we get $\pi_{HT} : Y_{\infty}(C) \rightarrow \mathbb{P}^1(C)$. Let $\mathfrak{X} = \mathbb{P}^1(C) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ (the Drinfeld p -adic upper half plane).

Theorem

The map $\pi_{HT} : Y_{\infty}(C) \rightarrow \mathbb{P}^1(C)$ is surjective. We have $\pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p)) = Y_{\infty}(C)^{ord}$ (the pairs (E, α_{∞}) where E has bad, or good ordinary reduction) and $\pi_{HT}^{-1}(\mathfrak{X}) = Y_{\infty}(C)^{ss}$ (the pairs where E has good supersingular reduction).

- Note the anomaly: at infinite level the “fat” set $Y_\infty(C)^{ord}$ gets mapped to the “meager” $\mathbb{P}^1(\mathbb{Q}_p)$ and the meager $Y_\infty(C)^{ss}$ fills up its complement \mathfrak{X} .
- If E is ordinary, $G = E[p^\infty] \rightsquigarrow T_p G^0$, the Tate module of the “kernel of reduction”, a line in $T_p G$, spans $\ker(\alpha_G \otimes 1)$. This proves $\pi_{HT}(E, \alpha_\infty) \in \mathbb{P}^1(\mathbb{Q}_p)$. Conversely, if E is defined over a CDVF K and $\pi_{HT}(E, \alpha_\infty) \in \mathbb{P}^1(\mathbb{Q}_p)$ then $\Gamma_K \curvearrowright T_p G$ is potentially reducible, so E is ordinary. This proves the theorem, except for the surjectivity. In general:

Theorem

(i) *The image of π_{HT} is the Drinfeld p -adic symmetric domain*

$$\mathfrak{X}(C) = \mathbb{P}^{h-1}(C) \setminus \bigcup_{a \in (\mathbb{P}^{h-1})^*(\mathbb{Q}_p)} H_a.$$

(ii) *π_{HT} induces $M_{\infty, \eta}/D^\times \simeq \mathfrak{X}(C)$ (on the level of C -points, so far).*

Rapoport-Zink spaces with PEL structure

- Recall k alg. closed field, char. p , $W = W(k)$, $H_{0/k}$ p -div gp, $\dim d$, ht h , $M_0 = MH_0(W \rightarrow k)$ the covariant Dieudonné module.
- Ignore **Level** (L) - treated in [R-Z] by “multi-chains of lattices”.
- **Endomorphisms** (E) -
 - Semi-simple algebra B over \mathbb{Q}_p with a maximal order $\mathcal{O}_B \hookrightarrow \text{End}_k(H_0)$. Then $B \curvearrowright V = M_{0,\mathbb{Q}}$ (linear action) commuting with Frobenius, and \mathcal{O}_B stabilizes the lattice $\Lambda = M_0$.
 - Fix B -stable decomposition $V = V_0 \oplus V_1$, $\dim V_0 = d$, $\dim V_1 = h - d$, $\Lambda \cap V_1$ reducing modulo p to $\omega_{H_0^V/k} \subset MH_0(k)$ and $\Lambda \cap V_0$ mapping onto $\text{Lie}(H_0)$.
- **Polarization** (P) - a quasi-isogeny $\lambda_0 : H_0 \xrightarrow{\sim} H_0^V$ inducing
 - a non-degenerate alternating $(,): V \times V \rightarrow \mathbb{Q}_p$.
 - a (“Rosati”) involution $*$ in B such that $(bv, w) = (v, b^*w)$.
- *Remark*: we have simplified the set-up a little for the exposition.

PEL deformation functor: Let $S \in \text{Nilp}_W$.

- $\mathcal{D}(S) = \{(G, \iota, \lambda)\} / \simeq$ where:
 - (G, ι) as before, with a compatible action $\mathcal{O}_B \hookrightarrow \text{End}_S(G)$.
 $MG(S)$ should be, locally on S , isomorphic as an \mathcal{O}_B -module to $\Lambda \otimes_W S$.
 - Kottwitz' condition: $\forall a \in \mathcal{O}_B \det_S(a; \text{Lie}(G)) = \det(a; V_0)$.
 - A polarization condition, dropped when the data (P) is missing.

Theorem (Rapoport-Zink)

The functor \mathcal{D} is "representable" by a formal scheme over W whose ideal of definition is locally finitely generated. Every irreducible component of its (reduced) special fiber is proper over k .

Goal: The Drinfeld p -adic symmetric domain as an example of such a moduli space.

- Let $B = D$, the division algebra of invariant $1/h$ over \mathbb{Q}_p .

The Drinfeld moduli problem

Definition. A special formal \mathcal{O}_D -module over $S \in \text{Nilp}_W$ is a formal group \mathcal{G} over S , of height h^2 and dimension h , equipped with $\mathcal{O}_D \hookrightarrow \text{End}_S(\mathcal{G})$, such that the induced representation of $\mathbb{Z}_p^h \subset \mathcal{O}_D$ on $\text{Lie}(\mathcal{G})$ is the regular representation (note $\mathbb{Z}_p^h \subset W \rightarrow S$).

- Fix H_0/k a special formal \mathcal{O}_D -module (all isogenous). Explicitly:

$$H_0 = H_0 \times H_0^\sigma \times \cdots \times H_0^{\sigma^{h-1}}$$

where $H_0^{\sigma^i}$ is “the H_0 ” of the Lubin-Tate moduli problem of dim 1 and ht h with the \mathcal{O}_D -action defined over \mathbb{F}_p (see Exercise) and twisted by σ^i .

- Besides the action of \mathcal{O}_D , note a (commuting) action of $GL_h(\mathbb{Q}_p)$ on H_0 by quasi-isogenies.
- $V \simeq D \otimes_{\mathbb{Z}_p} W$, $F = \Pi_r \otimes \sigma, \dots$ EL moduli problem $\mathcal{D}^{\text{Drin}}$.
- $GL_h(\mathbb{Q}_p) \curvearrowright \mathcal{D}^{\text{Drin}}$.

The moduli problem $\mathcal{D}^{\text{Drin}}$ had been considered by Drinfeld. It is the moduli problem of deformations of special formal \mathcal{O}_D -modules.

Theorem (Drinfeld)

The formal scheme \mathcal{X} representing $\mathcal{D}^{\text{Drin}}$ is such that $\mathcal{X}^{\text{an}} \simeq \mathfrak{X}$.

In fact, the formal scheme structure on \mathfrak{X} can be “read” from a reduction map

$$r : \mathfrak{X}(C) \rightarrow |\mathcal{BT}|$$

to the Bruhat-Tits building of $PGL_h(\mathbb{Q}_p)$.

- When $h = 2$ the special fiber of \mathcal{X} is a tree of \mathbb{P}^1 's, each intersecting transversally $p + 1$ others at the \mathbb{F}_p -rational points. $|\mathcal{BT}|$ is the $p + 1$ -regular tree; $r_v^{-1}(v)$, for a vertex v , is an affinoid isomorphic to the affinoid obtained from \mathbb{P}^1 upon removal of the $p + 1$ \mathbb{Q}_p -rational residue disks, and $r^{-1}(\varepsilon)$, for an edge ε , is an open annulus.

- 1 The Fargues-Fontaine curve X^{FF}
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A review of some of Fontaine's rings

- $F = C^b$, $\mathcal{O}_F = \varprojlim_{\times p} \mathcal{O}_C/p$, complete alg. closed (in particular *perfect*) non-arch. field, $\text{char.} F = p$.
 - $G_{\mathbb{Q}_p} = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on F .
 - Fix $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$, $p^b = (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathcal{O}_F$.
 - If $x = (x^{(0)}, x^{(1)}, x^{(2)}, \dots) \in \mathcal{O}_F$ then $x^\sharp = \lim (\tilde{x}^{(m)})^{p^m} \in \mathcal{O}_C$ exists. But note that the definition of x^\sharp is *not intrinsic to F* : it presumes the knowledge of F as the tilt of C !
- $A_{\text{inf}} = W(\mathcal{O}_F) \xrightarrow{\Theta} \mathcal{O}_C$, $\Theta(\sum_{n=0}^{\infty} p^n [x_n]) = \sum_{n=0}^{\infty} p^n x_n^\sharp$, a homomorphism!
 - $\ker(\Theta) = (\xi)$, $\xi = p - [p^b]$ "primitive element of degree 1".
 - $G_{\mathbb{Q}_p}$ acts by functoriality, compatible with Θ .
 - Frobenius φ acts on A_{inf} , does not preserve $\ker(\Theta)$.

- $A_{cris} = p$ -adic completion of $A_{inf}[\xi^n/n!] \subset A_{inf}[1/p]$ (initial object in the category of p -adic pd-thickenings of \mathcal{O}_C).
 - $G_{\mathbb{Q}_p}$ and φ actions extend to A_{cris} .
 - $t = \log[\varepsilon] \in A_{cris}$, $\sigma(t) = \chi_{cyc}(\sigma)t$, $\varphi(t) = pt$, t unique up to \mathbb{Z}_p^\times , $\Theta(t) = 0$.
 - $B_{cris}^+ = A_{cris}[1/p] \xrightarrow{\Theta} C$, $B_{cris} = B_{cris}^+[1/t]$.

Lemma

The ring $B_e = B_{cris}^{\varphi=1} = \bigcup_{n=0}^{\infty} t^{-n} B_{cris}^{+, \varphi=p^n}$ (increasing union) is a PID and $B_e \cap B_{cris}^+ = \mathbb{Q}_p$. Moreover, $B_e^\times = \mathbb{Q}_p^\times$.

Remark. This came as a big surprise and was the discovery that led to the Fargues-Fontaine curve. See Colmez' introduction to Astérisque 406. "A tale of a train ride".

- $B_{dR}^+ = \lim_{\leftarrow} A_{inf}[1/p]/(\xi^n) \supset B_{cris}^+$, but much cruder.
 - a CDVR, t a uniformizer, $B_{dR} = B_{dR}^+[1/t]$ a CDVF, v_{dR} the corresponding normalized valuation.
 - $\Theta: B_{dR}^+/tB_{dR}^+ \simeq C$ (residue field).
 - $G_{\mathbb{Q}_p}$ action (but not φ) extends to B_{dR} .

Theorem (Fundamental exact sequence of p -adic Hodge theory)

The following sequence is exact:

$$(FES) \quad 0 \rightarrow \mathbb{Q}_p \rightarrow B_{cris}^{\varphi=1} \rightarrow B_{dR}/B_{dR}^+ \rightarrow 0.$$

Remarks. (i) It is instructive to view (FES) as the analogue of

$$0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}[z] \rightarrow \mathbb{C}((1/z))/\mathbb{C}[[1/z]] \rightarrow 0,$$

$$B_{cris}^+ \xleftrightarrow{\sim} \mathbb{C}[x,y], \quad B_{cris}^{+, \varphi=p^n} \xleftrightarrow{\sim} \mathbb{C}[x,y]^{\text{hom.deg. } n}, \quad z = x/y, \quad t \xleftrightarrow{\sim} y.$$

(ii) Works for any *perfectoid pair* (R, R^+) replacing (C, \mathcal{O}_C) .

The Fargues-Fontaine curve

Definition

$X = X^{FF} = \text{Proj}(P)$, where $P = \bigoplus_{n=0}^{\infty} B_{\text{cris}}^{+, \varphi=p^n}$.

Theorem

- (a) X is an integral, noetherian, regular, 1-dim scheme (a “curve”).
- (b) $\Theta \rightsquigarrow$ a closed point $\infty \in X$ with residue field C .
- (c) $H^0(X - \{\infty\}, \mathcal{O}_X) \simeq B_e$.
- (d) $\widehat{\mathcal{O}}_{X, \infty} \simeq B_{dR}^+$.
- (e) (FES) $\Rightarrow H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$.

Although X is not of finite type over $P_0 = \mathbb{Q}_p$ (e) is an indication that X is “complete”. We shall see that it has a *theory of divisors* and behaves as if it had genus 0 (taken with a grain of salt...).

Remark. The closed points $|X|$ are in bijection with “Frobenius orbits of **untilts** of F ”. More precisely:

- An untilt of F is a pair (C', ι') where C' is a complete algebraically closed non-archimedean field containing \mathbb{Q}_p and $\iota' : (C')^\flat \simeq F$. Let $|Y|$ be the set of untilts, up to equivalence. (Kedlaya and Temkin have shown that untilts need not be isomorphic to C even as abstract topological fields, ignoring ι).
- Frobenius acts: $\varphi(C', \iota') = (C', \varphi \circ \iota')$, and $|X| \simeq |Y|/\varphi^{\mathbb{Z}}$.
- $|Y|$ is the set of closed points of an analytic (adic) space \mathfrak{Y} and $\mathfrak{Y} \rightarrow \mathfrak{X} = X^{an}$ is étale.
- If $(C', \iota') = y \in |Y|$, $\exists \Theta_y : \mathcal{O}_{\mathfrak{Y}, y} \twoheadrightarrow C'$ just like $\Theta = \Theta_\infty$.
- If $y \mapsto x \in |X|$ then (*) $\widehat{\mathcal{O}}_{X, x} \simeq \widehat{\mathcal{O}}_{\mathfrak{X}, x} \simeq \widehat{\mathcal{O}}_{\mathfrak{Y}, y}$.
- The choice of C as a distinguished untilt dictates $\infty \in \mathfrak{Y}$, above $\infty \in \mathfrak{X}$. The element t is a uniformizer at $\infty \in \mathfrak{Y}$, but by (*) becomes a uniformizer at $\infty \in X$.

Line bundles and divisors on X

If $f \in \text{Frac}(B_e)$, define the divisor $\text{div}(f) = \sum_{x \in |X|} \text{ord}_x(f) \cdot [x]$ as usual. Such a divisor is called principal.

Theorem

A divisor is principal if and only if it is of degree 0.

For $d \in \mathbb{Z}$ let $\mathcal{O}(d)$ be the line bundle on X associated with the graded module $P(d) = \bigoplus_{n \in \mathbb{Z}} B_{\text{cris}}^{+, \varphi = p^{n+d}}$. An equivalent formulation is:

Theorem

Every line bundle on X is isomorphic to a unique $\mathcal{O}(d)$. Thus $\text{deg} : \text{Pic}(X) \simeq \mathbb{Z}$.

Vector bundles: On $\mathbb{P}_{\mathbb{C}}^1$ every vector bundle is a direct sum of line bundles (Grothendieck). Here the analogy between X and $\mathbb{P}_{\mathbb{C}}^1$ *breaks down* for the first time. Let $\lambda = d/h \in \mathbb{Q}$ (reduced, $h > 0$) and let $\mathcal{O}(\lambda)$ be the vector bundle associated with the graded module

$$P(\lambda) = \bigoplus_{n \in \mathbb{Z}} (N_{-\lambda} \otimes_{W[1/p]} B_{\text{cris}}^+) \varphi = p^n.$$

($N_{-\lambda}$ is the standard isocrystal of slope $-\lambda$). Recall that the degree of a vector bundle \mathcal{V} is the degree of the line bundle $\det(\mathcal{V})$.

Theorem

- (i) $\mathcal{O}(\lambda)$ is a vector bundle of rank h , degree d and Harder - Narasimhan slope λ .
- (ii) Every vector bundle on X is $\bigoplus_{\lambda} \mathcal{O}(\lambda)^{m_{\lambda}}$ for unique $m_{\lambda} \in \mathbb{N}$.
- (iii) $\text{End}(\mathcal{O}(\lambda)) \simeq D_{\lambda}$, the division algebra over \mathbb{Q}_p of invariant λ .

Theorem says that the functor $\mathcal{E} : \text{Flisocrys}_k \rightsquigarrow \text{VecBun}_X$,

$$\mathcal{E}(N, \varphi) = \left(\bigoplus_{n \in \mathbb{Z}} (N \otimes_{W[1/p]} B_{\text{cris}}^+)^{\varphi = p^n} \right) \sim,$$

is essentially surjective. But it is far from being an equivalence!

Corollary

$$\pi_1^{\text{ét}}(X) \simeq \text{Gal}(\bar{\mathbb{Q}}_p / \mathbb{Q}_p).$$

Key step: Theorem \Rightarrow if $f : X' \rightarrow X$ is finite étale, $f_* \mathcal{O}_{X'} \simeq \mathcal{O}_X^{\text{deg}(f)}$.

- *Alternative description of vector bundles over $\mathbb{P}_{\mathbb{C}}^1$ (Beauville - Laszlo gluing)*: a rk r vector bundle $\mathcal{V} \iff$ a finite free $\mathbb{C}[z]$ -module V , a finite free $\mathbb{C}[[1/z]]$ -module V_{∞} , and

$$\rho : V \otimes_{\mathbb{C}[z]} \mathbb{C}((1/z)) \simeq V_{\infty} \otimes_{\mathbb{C}[[1/z]]} \mathbb{C}((1/z)).$$

- If $\{e_i\}$ is a basis of V , e'_i a basis of V_{∞} and $e_i = \sum a_{ij} e'_j$,

$$\text{deg}(\mathcal{V}) = v_{\infty}(\det(a_{ij})).$$

(B, \mathfrak{v}) -pairs and vector bundles

Definition (Berger)

A (B, \mathfrak{v}) -pair is $\underline{M} = (M_e, M_{dR}^+, \rho)$ where M_e is a finite free B_e -module, M_{dR}^+ a finite free B_{dR}^+ -module and

$$\rho : M_e \otimes_{B_e} B_{dR} \simeq M_{dR}^+ \otimes_{B_{dR}^+} B_{dR} =: M_{dR}.$$

The degree $\deg(\underline{M})$ is defined by the same procedure as above, replacing \mathfrak{v}_∞ by \mathfrak{v}_{dR} .

Proposition

The category VecBun_X of vector bundles over X is equivalent to the category of (B, \mathfrak{v}) -pairs. The map is

$$\mathcal{V} \mapsto (H^0(X - \{\infty\}, \mathcal{V}), \mathcal{V} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{X, \infty}),$$

and in the opposite direction by “Beauville-Laszlo gluing”. The correspondence respects ranks and degrees, hence slopes.

Remark. Let $N \in \text{FIsocryst}_k$. The vector bundle $\mathcal{E}(N, \varphi)$ corresponds to the pair

$$((N \otimes B_{\text{cris}})^{\varphi=1}, N \otimes B_{dR}^+).$$

However, the Harder-Narasimhan slope of $\mathcal{E}(N, \varphi)$ is the *negative* of the Frobenius slope of N .

Corollary

Canonically, $H^0(X, \mathcal{E}(N, \varphi)) \simeq (N \otimes B_{\text{cris}}^+)^{\varphi=1}$.

Proof.

$H^0(X, \mathcal{E}(N, \varphi)) = (N \otimes B_{\text{cris}})^{\varphi=1} \cap N \otimes B_{dR}^+ = (N \otimes B_{\text{cris}}^+)^{\varphi=1}$ (the last equality needs justification, even if N is trivial). \square

Recall: The category “ p -div gps over k up to isogeny” is equivalent to the full subcategory of F -isocrystals whose slopes lie in $[0, 1]$ (Dieudonné-Manin). We examine the same category, but over \mathcal{O}_C/p . Its “objects up to isomorphism” are in bijection with those of the same category over k (a consequence of the *isotriviality theorem*) but the category is much richer, and far from semi-simple!

- If G is a p -div gp over \mathcal{O}_C/p let

$$M_{cris}(G) = MG(A_{cris} \twoheadrightarrow \mathcal{O}_C/p)$$

(“**crystalline Dieudonné module**”). Then $M_{cris}(G)[1/p]$ is a finite projective B_{cris}^+ -module. Let $\mathcal{E}(G)$ be the vector-bundle associated to the graded P -module

$$\bigoplus_{n=0}^{\infty} (M_{cris}(G)[1/p])^{\varphi=p^{n+1}}.$$

- Let H_0 be a p -div gp over k such that $G \xrightarrow{q.i.} H_0 \times_k \mathcal{O}_C/p$. The given quasi-isogeny determines an isomorphism $M_{cris}(G) \simeq A_{cris} \otimes_W M_0$, hence

$$\mathcal{E}(G) \simeq \mathcal{E}(M_0(1), \varphi).$$

However, the *functorial dependence of $\mathcal{E}(G)$ on G can not be read from H_0 alone!*

- Example:* Lubin-Tate case: H_0 unique formal p -div gp of ht h , dim 1: Then $M_0 \simeq N_{(h-1)/h}$, $M_0(1) \simeq N_{-1/h}$ so $\mathcal{E}(G) \simeq \mathcal{O}(1/h)$ (the isom. depending on the q.i. above).

Theorem (Full-faithfulness, Scholze-Weinstein)

(i) *The functor $M_{cris}(-)$ is fully faithful, i.e.*

$$\text{Hom}_{\mathcal{O}_C/p}(G, G') \simeq \text{Hom}_{A_{cris}, \varphi}(M_{cris}(G), M_{cris}(G')).$$

(ii) *The functor $\mathcal{E}(-)$ is an equivalence between the category of “ p -div gps over \mathcal{O}_C/p up to isogeny” and the full subcategory of vector bundles over X all of whose slopes lie in the interval $[0, 1]$.*

Corollary (Universal covering as global sections)

Canonically,

$$\tilde{G}(\mathcal{O}_C/p) \simeq H^0(X, \mathcal{E}(G)), \quad MG(\mathcal{O}_C \rightarrow \mathcal{O}_C/p)[1/p] \simeq i_\infty^* \mathcal{E}(G)(-1).$$

Proof.

We have $\tilde{G}(\mathcal{O}_C/p) \simeq \text{Hom}_{\mathcal{O}_C/p}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, G)[1/p]$. By “full-faithfulness” (in the isogeny category is enough!) this is

$$\text{Hom}_{B_{\text{cris}}^+, \varphi}(M_{\text{cris}}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})[1/p], M_{\text{cris}}(G)[1/p]).$$

But $M(\underline{\mathbb{Q}_p/\mathbb{Z}_p})[1/p] = D(\mu_{p^\infty})[1/p] \simeq N_1$ so we get

$$\begin{aligned} \tilde{G}(\mathcal{O}_C/p) &\simeq \text{Hom}_{B_{\text{cris}}^+, \varphi}(B_{\text{cris}}^+ \otimes N_1, M_{\text{cris}}(G)[1/p]) \\ &= M_{\text{cris}}(G)[1/p]^{\varphi=P} = H^0(X, \mathcal{E}(G)). \end{aligned}$$



Lemma

Let \mathcal{V} be a vector bundle on X . Then $H^i(X, \mathcal{V}) = 0$ for $i \geq 2$, and if (M_e, M_{dR}^+, ρ) is the associated (B, \mathfrak{v}) -pair there is a "Mayer-Vietoris" exact sequence

$$0 \rightarrow H^0(X, \mathcal{V}) \rightarrow M_e \oplus M_{dR}^+ \rightarrow M_{dR} \rightarrow H^1(X, \mathcal{V}) \rightarrow 0.$$

This enables one to calculate the cohomology. We have already seen (i) of the following theorem.

Theorem

- (i) Let $\lambda = d/h$ (reduced, $h > 0$). Then $H^0(X, \mathcal{O}(\lambda)) = 0$ if $\lambda < 0$ and is equal to $(B_{cris}^+)^{\varphi^h = p^d}$ otherwise.
- (ii) $H^1(X, \mathcal{O}(\lambda)) = 0$ iff $\lambda \geq 0$.

Remarks. (i) Once again, $H^1(X, \mathcal{O}_X) = 0$ may be counted as an indication of “genus 0”, but note the *second* time the analogy with $\mathbb{P}_{\mathbb{C}}^1$ breaks down: $H^1(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{P}}(-1)) = 0$.

(ii) The spaces $H^0(X, \mathcal{O}(\lambda))$ are \mathbb{Q}_p -vector spaces, but for $\lambda > 0$ they are *never* finite dimensional. In fact they belong to a very interesting category of “Banach-Colmez vector spaces”. More to come soon, when we relate them to the (LOG) exact sequence.

- **Extensions.** The vector bundles $\mathcal{O}(\lambda)$ are indecomposable, but not irreducible. In general, if

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$$

is a short exact sequence of vector bundles, both rk and deg are “Euler-Poincaré characteristics” so the slope $\mu = deg/rk$ satisfies the usual Harder-Narasimhan formalism

$$\mu(\mathcal{E}) = \frac{rk(\mathcal{E}')}{rk(\mathcal{E})} \mu(\mathcal{E}') + \frac{rk(\mathcal{E}'')}{rk(\mathcal{E})} \mu(\mathcal{E}'').$$

- Recall that \mathcal{E} is called *semistable* if whenever $\mathcal{E}' \subset \mathcal{E}$ is a sub-bundle we have $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$.

Proposition

A vector bundle on the Fargues-Fontaine curve is semistable if and only if it is isoclinic (has only one slope).

Example. For $n \leq 0$, $\text{Ext}^1(\mathcal{O}(1-n), \mathcal{O}(n)) \simeq H^1(X, \mathcal{O}(2n-1)) \neq 0$, so there is a non-split extension

$$(*) \quad 0 \rightarrow \mathcal{O}(n) \rightarrow \mathcal{O}(1/2) \rightarrow \mathcal{O}(1-n) \rightarrow 0.$$

Take $n=0$, fix $\mathcal{E}(\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathcal{O}$, $\mathcal{E}(G) \simeq \mathcal{O}(1/2)$, $\mathcal{E}(\mu_{p^\infty}) \simeq \mathcal{O}(1)$.
By the equivalence of categories, $(*) \leftrightarrow$ a unique

$$\underline{\mathbb{Q}_p/\mathbb{Z}_p} \xrightarrow{\alpha} G \xrightarrow{\beta} \mu_{p^\infty}.$$

Note that α and β are only quasi-homomorphisms and that modulo \mathfrak{m}_C we have $\alpha \equiv \beta \equiv 0$.

Filtered F -isocrystals

Goal: Upgrade “ p -div gps over \mathcal{O}_C/p up to isogeny” to the same category over \mathcal{O}_C . *Insight* (Grothendieck): should add a “Hodge filtration”. We have already seen the following definition when we discussed the weakly admissible period domain \mathfrak{F}^{wa} . Let $K_0 = W(k)[1/p] \subset K$ be a finite ext'n.

Definition

A K -filtered F -isocrystal (over k) is $\underline{D} = (D, \varphi, \text{Fil}^\bullet)$ where (D, φ) is an F -isocrystal and Fil^\bullet is a separated exhaustive descending filtration on D_K . Define the *slope* μ by

$$t_{\text{Newton}}(\underline{D}) = v_p(\det(\varphi)), \quad t_{\text{Hodge}}(\underline{D}) = \sum i \dim \text{gr}_{\text{Fil}^\bullet}^i D_K,$$

$$\text{deg}(\underline{D}) = t_{\text{Hodge}}(\underline{D}) - t_{\text{Newton}}(\underline{D}), \quad \mu(\underline{D}) = \text{deg}(\underline{D})/\text{rk}(\underline{D}).$$

Call \underline{D} *semistable* if for any strict sub-object \underline{D}' (*strict* means that the filtration on D'_K is induced by that of \underline{D}) $\mu(\underline{D}') \leq \mu(\underline{D})$, and *weakly admissible* \Leftrightarrow s.st. of slope 0.

Modifications of vector bundles

\underline{D} a K -filtered F -isocrystal, $\mathcal{E}(\underline{D}) = \mathcal{E}(D, \varphi, \text{Fil}^\bullet)$ v.b. associated with the (B, ν) -pair $((D \otimes_{K_0} B_{\text{cris}})^{\varphi=1}, \text{Fil}^0(D_K \otimes_K B_{dR}))$.

- The degree (slope) of $\mathcal{E}(\underline{D})$ are the *same* as those of \underline{D} .
- $\text{Fil}^1 D = 0 \Rightarrow \text{Fil}^0(D_K \otimes_K B_{dR}) \subset D \otimes_{K_0} B_{dR}^+ \rightsquigarrow$ exact sequence

$$0 \rightarrow \mathcal{E}(D, \varphi, \text{Fil}^\bullet) \rightarrow \mathcal{E}(D, \varphi) \rightarrow i_{\infty,*}(D \otimes_{K_0} B_{dR}^+ / \text{Fil}^0) \rightarrow 0,$$

last term a finite length “skyscraper sheaf” supported at ∞ .

- Gives a *modification* of vector bundles at ∞ . Similarly define modifications with “legs” at several points. Relax “ $\text{Fil}^1 D = 0$ ” by allowing the first arrow to go backwards. Notion (similar to Drinfeld’s “shtukas”) is key to the geometrization of LLC (Scholze and Fargues).

Lecture V: Applications: Classification over \mathcal{O}_C , Galois representations and duality

- ① p -divisible groups over \mathcal{O}_C up to isogeny
 - ① The big diagram revisited
 - ② A classification
- ② Applications to Galois representations of G_K
 - ① Crystalline Galois representations
 - ② Weakly admissible is admissible
- ③ Duality between the Lubin-Tate and Drinfeld towers
 - ① The Drinfeld tower
 - ② A simple proof of a theorem of Faltings

p -div gps over \mathcal{O}_C up to isogeny

- G - p -div gp over \mathcal{O}_C , $G_0 = G \times_{\mathcal{O}_C} \mathcal{O}_C/p$, and as in RZ
 $\iota : G_0 \xrightarrow{q.i.} H_0 \times_k \mathcal{O}_C/p$, $h = ht$, $d = \dim$.
- $M_0 = D(H_0^\vee)$ covariant Dieudonné module of H_0 . Then

$$\mathcal{E}(G_0) \xrightarrow{L^*} \mathcal{E}(M_0(1)_{\mathbb{Q}}, \varphi) := \mathcal{E}.$$

Define the trivial vector bundle

$$\mathcal{F} = V_p G(\mathcal{O}_C) \otimes_{\mathbb{Q}_p} \mathcal{O}_X.$$

Theorem (Scholze-Weinstein)

(i) There is a natural modification of vector bundles associated with (G, ι)

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow i_{\infty,*}(\text{Lie } G_C) \rightarrow 0.$$

Furthermore, if G is defined over \mathcal{O}_K for a finite K/K_0 ,
 $\mathcal{F} \simeq \mathcal{E}(D, \varphi, \text{Fil}^\bullet)$ where $D = M_0(1)_{\mathbb{Q}}$, $\text{Fil}^{-1} = D_K$, Fil^0 is of rank $h - d$, $\text{Fil}^1 = 0$, and $(D, \varphi, \text{Fil}^\bullet)$ is weakly admissible.

Theorem (continued)

(ii) The global sections of the exact sequence in (i) are identified canonically with the exact sequence

$$\text{(LOG)} \quad 0 \rightarrow V_p G(\mathcal{O}_C) \rightarrow \tilde{G}(\mathcal{O}_C) \xrightarrow{\theta} \text{Lie } G_C \rightarrow 0.$$

(iii) The fiber at ∞ of the exact sequence in (i) (i.e. taking $- \otimes_{\mathcal{O}_{X,\infty}} C$) is

$$V_p G(\mathcal{O}_C) \otimes_{\mathbb{Q}_p} C \rightarrow MG(\mathcal{O}_C)[1/p] \rightarrow \text{Lie } G_C \rightarrow 0.$$

The first arrow factors through the Hodge-Tate map

$$V_p G(\mathcal{O}_C) \otimes_{\mathbb{Q}_p} C \xrightarrow{\alpha_G} \omega_{G^v/C} \hookrightarrow MG(\mathcal{O}_C)[1/p].$$

(iv) The map “global sections to fiber at ∞ ” is “qlog”.

A classification

Example. Back to Lubin-Tate, $(G, \iota) \in \mathcal{M}(\mathcal{O}_C)$. As already seen,

$$\tilde{G}(\mathcal{O}_C) = \tilde{G}_0(\mathcal{O}_C/p) \simeq H^0(X, \mathcal{E}) = B_{\text{cris}}^{+, \varphi^h = p},$$

and 1-dim'l *Lie* G_C may be identified with C (choice of a parameter). The map θ is then identified with

$$\Theta : B_{\text{cris}}^{+, \varphi^h = p} \rightarrow C.$$

*The relation between “ p -divisible groups” and “modifications” allows to give a **complete classification of p -div gps over \mathcal{O}_C up to isogeny.***

- Let \mathcal{C} be the category of modifications

$$(*) \quad 0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow i_{\infty, *}(W) \rightarrow 0$$

where (i) \mathcal{F} and \mathcal{E} are vector bundles over X (ii) $\mathcal{F} \simeq \mathcal{O}_X^h$ for some $h \in \mathbb{N}$ (iii) W is a f. dim'l C -vector space (B_{dR}^+ -module killed by t , modification is *minuscule*).

- Let \mathcal{C}' be the category of pairs (V, W) where V is a \mathbb{Q}_p -v.sp. and $W \subset V_C$ a sub C -v.sp. (no extra structure!).

Theorem (Scholze-Weinstein, Fargues-Fontaine)

The categories of p -divisible groups over \mathcal{O}_C up to isogeny, \mathcal{C} and \mathcal{C}' are all naturally equivalent.

sketch.

To pass from \mathcal{C} to \mathcal{C}' let $V = H^0(X, \mathcal{F})$. To go backwards let $\mathcal{F} = V \otimes \mathcal{O}_X$. In both directions, we relate the extension $(*)$ to $W \hookrightarrow V_C$ as follows. A basic computation shows that $\text{Ext}(i_{\infty,*}C, \mathcal{O}_X) \simeq C$, hence $\text{Ext}(i_{\infty,*}(W), \mathcal{F}) \simeq \text{Hom}_C(W, V_C)$. Here, the extension $(*)$ associated to a homomorphism $u: W \rightarrow V_C$ is the pull-back of

$$0 \rightarrow V \otimes \mathcal{O}_X \rightarrow V \otimes \mathcal{O}_X(1) \rightarrow V \otimes i_{\infty,*}(C) \rightarrow 0$$

under $i_{\infty,*}u$. Note \mathcal{E} is locally free $\Leftrightarrow u$ is injective. □

cont'd.

The above construction also shows that in any extension like (*) we have

$$\mathcal{O}_X^h \subset \mathcal{E} \subset \mathcal{O}_X(1)^h$$

so all the slopes of \mathcal{E} lie in $[0, 1]$, by semistability of isoclinic vector bundles. Thus $\mathcal{E} = \mathcal{E}(G_0)$ for a p -div gp G_0 over \mathcal{O}_C/p , unique up to isogeny. We wish to upgrade the equivalence $G_0 \rightsquigarrow \mathcal{E}(G_0)$ to an equivalence between p -div gps (up to isogeny) over \mathcal{O}_C and the category of modifications \mathcal{C} .

We have already seen how to associate with a p -div gp G over \mathcal{O}_C a modification in \mathcal{C} with $V = V_p G(\mathcal{O}_C)$, $W = \text{Lie}(G_C)$, $\mathcal{F} = V \otimes \mathcal{O}_X$, $\mathcal{E} = \mathcal{E}(G_0)$. This is functorial, and the key steps are to prove (i) that it is fully faithful (ii) that it is essentially surjective. For the details, see [S-W], §5.2. □

cont'd.

We only remark that one works first at the generic fiber of the adic spaces, building G_η^{ad} out of the multiplicative group $G' = T_p G(-1) \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$ mimicking the construction of the modification (*) out of $W \hookrightarrow V_C$. Namely, one defines G_η^{ad} as the fiber product

$$\begin{array}{ccc} G_\eta^{ad} & \rightarrow & W \otimes \mathbb{G}_a \\ \downarrow & & \downarrow \\ (G')_\eta^{ad} & \rightarrow & V_C \otimes \mathbb{G}_a \end{array} .$$

The special features of C are involved in the reconstruction of the formal group G from its generic fiber G_η^{ad} , which is pretty delicate. □

Remark. A remarkable feature of the classification over \mathcal{O}_C is that it is in terms of linear algebra alone, and not semi-linear algebra as Dieudonné theory over k .

- Given G , the pair $W \hookrightarrow V_C$ is identified with the Hodge-Tate map $\alpha_{G_V}^V : \text{Lie}(G_C) \hookrightarrow V_p G(\mathcal{O}_C) \otimes_{\mathbb{Q}_p} C$ (we've ignored Tate twists).
- **Example: Assume $h = 2, d = 1$.** The only possibilities for $\mathcal{O}_X^2 \subset \mathcal{E} \subset \mathcal{O}_X(1)^2$ are $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)$ or $\mathcal{E} \simeq \mathcal{O}(1/2)$. As we have seen before, the first occurs if $W = L_C$ for a \mathbb{Q}_p -rat'l line $L \subset V$ and the second otherwise.
- **Exercise:** Write all the possibilities when $h = 5, d = 2$ (there are 7 such) and the corresponding Newton polygons.

Crystalline Galois representations

- $k = \bar{\mathbb{F}}_p$, $K_0 = W(k)[1/p]$, $K_0 \subset K \subset C$ a finite extension.
- V - h -dim'l continuous \mathbb{Q}_p -rep'n of $G_K = \text{Gal}(\bar{K}/K)$.
- Define

$$D_{\text{cris}}(V) = (V \otimes B_{\text{cris}})^{G_K}, \quad D_{\text{dR}}(V) = (V \otimes B_{\text{dR}})^{G_K}.$$

The first is a K_0 -v.sp. and inherits an action of φ , the second is a K -v.sp. and inherits a filtration Fil^\bullet .

- $\dim_{K_0} D_{\text{cris}}(V) \leq h$, $\dim_K D_{\text{dR}}(V) \leq h$ and V is called **crystalline** (resp. de-Rham) if equalities hold. We have $D_{\text{cris}}(V)_K \subset D_{\text{dR}}(V)$ (with equality if V is crystalline), so with the induced filtration $\underline{D}(V) = (D_{\text{cris}}(V), \varphi, \text{Fil}^\bullet)$ becomes a K -filtered φ -module.
- If V is crystalline, $D = D_{\text{cris}}(V)$, one recovers (Fontaine)

$$V = (D \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \text{Fil}^0(D_K \otimes B_{\text{dR}}).$$

- Interpretation via the Fargues-Fontaine curve: Assume V is crystalline.

$$V \rightsquigarrow \mathcal{E}(V) := \mathcal{E}(D_{\text{cris}}(V), \varphi, \text{Fil}^\bullet), \quad V = H^0(X, \mathcal{E}(V)).$$

Lemma

V crystalline $\Rightarrow \underline{D}(V)$ weakly admissible.

Proof.

Write $\underline{D} = \underline{D}(V)$ and $\mathcal{E}(V) \simeq \bigoplus \mathcal{O}(\lambda_i)$. $\dim_{\mathbb{Q}_p} H^0(X, \mathcal{E}(V)) < \infty$ implies all $\lambda_i \leq 0$. Since this dimension is exactly $h = \text{rk} \mathcal{E}(V)$, all $\lambda_i = 0$ and $\mathcal{E}(V)$ is trivial. This implies $\mu(\underline{D}) = 0$.

Suppose $\underline{D}' \subset \underline{D}$ is a *strict* sub-object. Then $\mathcal{E}(\underline{D}') \subset \mathcal{E}(\underline{D}) \simeq \mathcal{O}_X^h$ is a *sub-bundle*. But $\mathcal{E}(\underline{D})$ isoclinic \Rightarrow semi-stable, so $\mu(\underline{D}') = \mu(\mathcal{E}(\underline{D}')) \leq 0$, showing that \underline{D} is semi-stable. \square

- The converse is the celebrated “weakly admissible = admissible” theorem.

Weakly admissible is admissible

Theorem (Colmez-Fontaine)

Every weakly admissible K -filtered φ -module is $\underline{D}(V)$ for a crystalline representation V .

Proof.

Suppose \underline{D} is weakly admissible. Then $\mathcal{E}(\underline{D})$ is semistable of slope 0, so by the classification of vector bundles on X must be trivial, i.e. isomorphic to \mathcal{O}_X^h . This means (using the language of (B, \mathbf{v}_{dR}) -pairs) that

$$V = H^0(X, \mathcal{E}(D)) \simeq (D \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \text{Fil}^0(D_K \otimes_K B_{dR})$$

is h -dimensional. But it is known that this equality of dimensions forces $D = D_{\text{cris}}(V)$, hence V is crystalline. \square

The Drinfeld tower

- Change notation: $M_n^{LT} = M_{n,\eta} = \mathcal{M}_{n,\eta}^{ad}$ the generic fiber of the Lubin-Tate tower, similarly $M_\infty^{LT} = \mathcal{M}_{\infty,\eta}^{ad}$, a perfectoid space over $\mathrm{Spa}(W[1/p], W)$.
- $M_0^{Drin} = (\mathcal{D}^{Drin})_\eta^{ad}$ the generic fiber of the formal scheme representing the Drinfeld moduli problem, over $\mathrm{Spa}(W[1/p], W)$.
- As in the Lubin-Tate case, there is a *finite étale cover* $M_n^{Drin} \rightarrow M_0^{Drin}$ (in the category of adic spaces) representing triples (H, ι, α_n) where $(H, \iota) \in M_0^{Drin}$ and $\alpha_n: \mathcal{O}_D/p^n \rightarrow H[p^n]$ is an “analytic \mathcal{O}_D level p^n structure” (meaning that at any geometric point it induces an isomorphism onto the p^n -torsion, compatible with the action of \mathcal{O}_D).

- The role of the groups D^\times and GL_h is interchanged: $GL_h(\mathbb{Q}_p)$ acts on each M_n^{Drin} . The Galois group of $M_n^{Drin} \rightarrow M_0^{Drin}$ is $(\mathcal{O}_D/p^n)^\times$.
- *Fact:* $\exists M_\infty^{Drin}$, adic space over $Spa(W[1/p], W)$, representing the functor on complete affinoid $(W[1/p], W)$ algebras (R, R^+)

$$M_\infty^{Drin}(R, R^+) = \{(H, \iota, \alpha) \mid (H, \iota) \in M_0^{Drin}(R, R^+), \alpha \dots\} / \simeq$$

where $\alpha : \mathcal{O}_D \rightarrow T_p H_\eta^{ad} \sim \lim_{\leftarrow} H[p^n]$ is \mathcal{O}_D -compatible and induces an isomorphism on any geometric point of $Spa(R, R^+)$.

- *Fact:* M_∞^{Drin} is (pre)perfectoid, $M_\infty^{Drin} \sim \lim_{\leftarrow} M_n^{Drin}$.
- There are analytic maps defined as in LT case

$$\pi_{GM}^{Drin} : M_\infty^{Drin} \rightarrow M_0^{Drin} = \bigsqcup_{i \in \mathbb{Z}} \mathfrak{X} \rightarrow \mathfrak{X} \simeq M_\infty^{Drin} / D^\times$$

$$\pi_{HT}^{Drin} : M_\infty^{Drin} \rightarrow (\mathbb{P}_W^{h-1})_\eta^{ad} \simeq M_\infty^{Drin} / GL_{h-1}(\mathbb{Q}_p).$$

The duality theorem

Theorem (Faltings, Fargues, Scholze-Weinstein)

There is a canonical isomorphism of adic spaces $M_\infty^{LT} \simeq M_\infty^{Drin}$, compatible with the action of $GL_h(\mathbb{Q}_p) \times D^\times$, under which the period maps are interchanged:

$$\pi_{GM}^{Drin} = \pi_{HT}^{LT}, \quad \pi_{GM}^{LT} = \pi_{HT}^{Drin}.$$

The original proof was difficult, partly because of missing language. [S-W], Theorem 7.2.3 and [F-F] 8.3.5 gave a conceptual proof using the equivalence of the moduli problems represented by the towers with categories of modifications of vector bundles on the FF curve. We shall outline the main construction at the level of (C, \mathcal{O}_C) -points, as usual.

- Let $(G, \iota, \alpha_\infty) \in M_\infty^{LT}(C, \mathcal{O}_C)$. This gives a *trivialized* modification

$$0 \rightarrow \mathcal{O}_X^h \rightarrow \mathcal{E} \rightarrow i_{\infty,*} W \rightarrow 0$$

where $W = \text{Lie}(G_C)$, $\mathcal{E} = \mathcal{E}(H_0) = \mathcal{O}(1/h)$ and $\mathcal{E}(G_0)$ is identified with \mathcal{E} using ι . The *trivialization* is the identification of the kernel of the map to the Lie algebra, canonically given as $V_p G(\mathcal{O}_C) \otimes \mathcal{O}_X$, with \mathcal{O}_X^h . It uses α_∞ .

- The group $GL_h(\mathbb{Q}_p) \simeq \text{Aut}(\mathcal{O}_X^h)$ acts on such a trivialized modification by push-out of the first factor. It does not change the modification class, but only its trivialization. The group D^\times acts by changing the identification of \mathcal{E} with $\mathcal{E}(G_0)$. This action yields a new modification.
- Apply the sheaf-hom functor $\mathcal{R}\mathcal{H}om(-, \mathcal{O}(1/h))$. Get an exact sequence

$$0 \rightarrow \mathcal{E}nd(\mathcal{O}(1/h)) \rightarrow \mathcal{O}(1/h)^h \rightarrow \mathcal{E}xt^1(i_{\infty,*} W, \mathcal{O}(1/h)) \rightarrow 0.$$

- We used $\mathcal{E}xt^1(\mathcal{O}(1/h), \mathcal{O}(1/h)) = 0$ (easy).
- The first factor is canonically $D \otimes \mathcal{O}_X$, D acting naturally.
- D acts on the second factor via $D \simeq \text{End}(\mathcal{O}(1/h))$.
- The last factor is a skyscraper sheaf at ∞ . Since W is 1-dimensional it is $i_{\infty,*} W'$ where

$$W' \simeq \text{Ext}_{B_{dR}^+}^1(C, (B_{dR}^+)^h) \simeq C^h$$

and $\mathbb{Q}_{p^h} \subset D$ acts with h distinct characters, each with multiplicity 1.

- \rightsquigarrow a “special trivialized modification of vector bundles with D -action”, which (by an analogue of the main theorem with PEL structure) corresponds to a triple $(G', \mathfrak{t}', \alpha'_\infty) \in M_\infty^{\text{Drin}}(C, \mathcal{O}_C)$.

- The D action described above was obtained from the action of D on $\mathcal{O}(1/h)$ when we took $\mathcal{R}\mathcal{H}om(-, \mathcal{O}(1/h))$ and therefore did not change the new trivialized modification, but *enhanced* it to correspond to a “moduli problem with endomorphisms in D ”. It was an *algebra action*.
- We still have the $D^\times \times GL_h(\mathbb{Q}_p)$ *group action* on the *set* of all “special trivialized modifications with D -action” and one checks that the functor $\mathcal{R}\mathcal{H}om(-, \mathcal{O}(1/h))$ between the two categories of modifications preserves these actions.
- Finally, one constructs in a similar way a quasi-inverse, establishing the duality between $M_\infty^{Drin}(C, \mathcal{O}_C)$ and $M_\infty^{LT}(C, \mathcal{O}_C)$.