Preventing instability : Perturbation to a Magnetic Neutron Stars with Shear modulus

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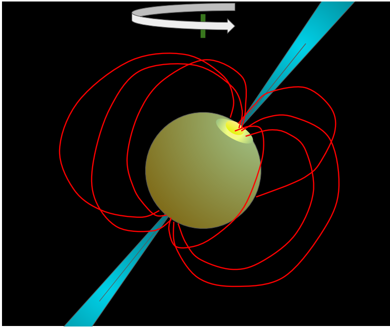
Feb 7th, 2020
ICTS, Bengaluru

Outline of this talk

- Degenerate stars
- Magnetic degenerate stars
- Equilibrium magnetic structure
- Stability of magnetic equilibrium structures
- Effect of shear stress in the stability
- Summary

Degenerate star : implications

- Strong gravity, rotation and magnetic energy play important role in the observational characteristics.
- NS Mass 1-2 $M_\odot$ & Radius 10-20 km $\rightarrow$ binary accretor, GW emitter
- NS can rotate a few hundred times in a second $\rightarrow$ pulsar
- NS magnetic field : up to $\sim 10^{12}$ T ($=10^{16}$ G) $\rightarrow$ gamma ray burst



Magnetic degenerate star : Equilibrium structure

stellar structure eq.s (Newtonian)

- The hydrostatic force balance eq. :
  $\frac{1}{\rho}\nabla P = -\nabla\Phi_g + \frac{1}{\rho}(\boldsymbol{j} \times \boldsymbol{B})$
- Poisson's equation:
  $\nabla^2\Phi_g = 4\pi G\rho$
- Maxwell's equation (with $\sigma \to \infty$):
  $\nabla.\boldsymbol{B} = 0$
  and $\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{j}$

EoS:

$P = P(\rho)$

virial condition:

$3\Pi + W + \mathscr{M} = 0$

where, $\Pi = \int PdV$; $W = \frac{1}{2}\int \rho\Phi_g dV$ *and* $\mathscr{M} = \int \frac{B^2}{2\mu_0} dV$

Assumptions and method

- stationary ($\frac{\partial}{\partial t} \to 0$)
- axisymmetric, i.e. $\frac{\partial}{\partial \phi} \to 0$.
- non-rotating.
- The source of the magnetic field (i.e. the current distribution) is confined within the surface of degenerate star.
- $\sigma \to \infty$

HSCF : integral formalism (Hachisu, 1986; Tomimura & Eriguchi, 2005)

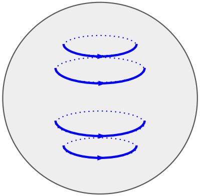
$$\frac{1}{\mu_e m_H} E_F + \Phi_g = \mathrm{M}(u) + C$$

here, $u = A_\phi . r \sin\theta$, $\Phi_g(\boldsymbol{r}) = -G \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}'$,

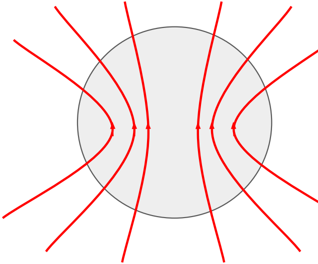
$A_\phi(\boldsymbol{r}) \sin\phi = \frac{\mu_0}{4\pi} \int \frac{j_\phi(\boldsymbol{r}') \sin\phi'}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}'$.

B Field components : Poloidal & Toroidal

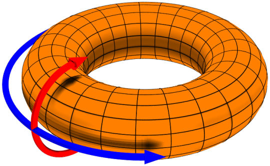
$$\mathbf{B} = B_\phi \hat{\phi} + B_r \hat{r} + B_\theta \hat{\theta}$$



Toroidal

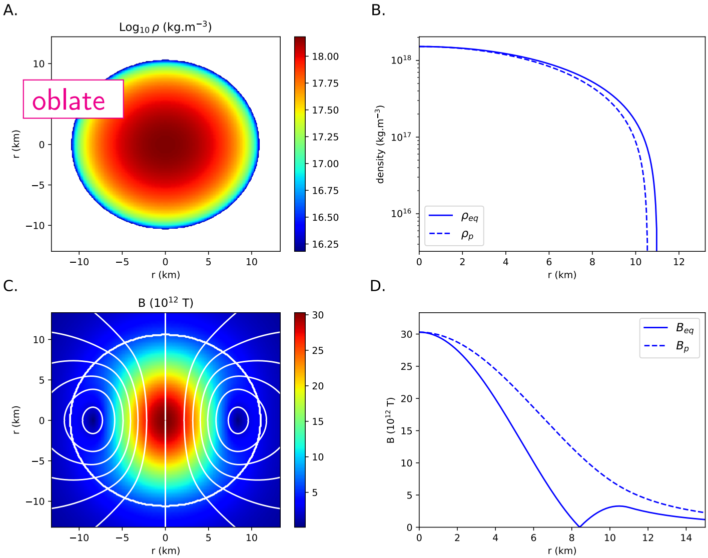


Poloidal

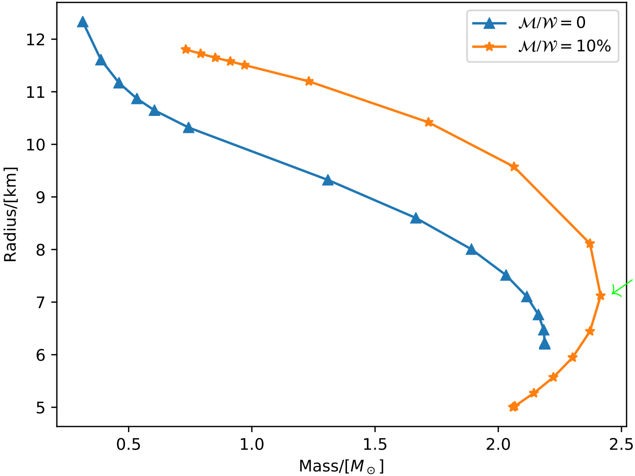


Wikipedia

Mass distribution & Poloidal field



EoS : APR (1998)
Magnetic field geometry: Poloidal
Used code : Lorene

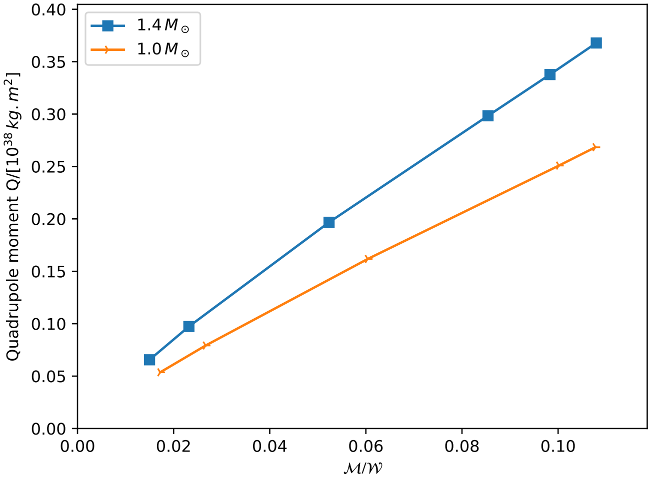


Mass : $2.41\ M_\odot$

B_{pole} : 1.6×10^{17} T

Deformation of magnetic Neutron Star

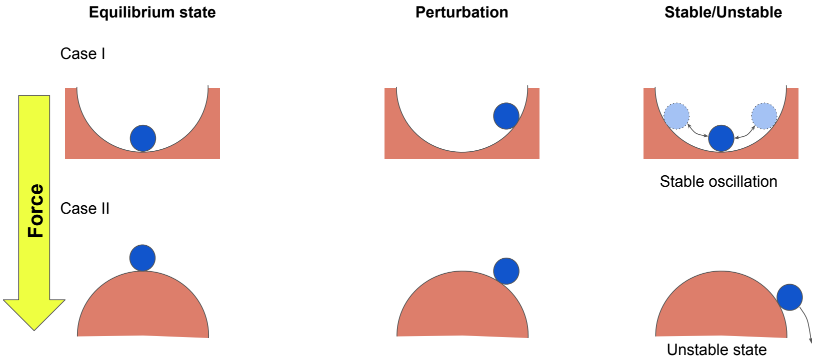
$|h_0| = \frac{6G}{c^4} Q \frac{\Omega^2}{r}$ [Bonazzola & Gourgoulhon 1996]



Stability ?

- Typical time scales (for a poloidal field configuration):
  - $\tau_{sound} \equiv \frac{R}{<v_s>} \sim 10^{-4}$ s
  - $\tau_{Alfven} \equiv \frac{R}{<v_A>} \sim 10^{-3}$ s
  - $\tau_{viscous} \equiv \frac{<\rho>R^2}{\eta_{v\,eff}} \sim 10^{10}$ s [Cutler & Lindblom 1987]
  - $\tau_{ohmic} \equiv \frac{R^2}{\eta_{m\,eff}} \sim 10^{13}$ s [Baym et al. 1969]

Stable & Unstable system



Magneto-hydrodynamic Eqs

- Momentum equation : $\rho[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla P - \rho \nabla \Phi_g + (\mathbf{j} \times \mathbf{B})$
- Continuity equation : $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$
- Faraday equation : $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$
- Adiabatic condition: $\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)(P\rho^{-\gamma}) = 0$
- Poisson equation : $\nabla^2 \Phi_g = 4\pi G \rho$
- EoS : $P = P(\rho)$
  - **Not considered** : viscosity, heating, conduction, radiation etc.

Uniform Radial Perturbation

Conserved Values : Total Mass ($\rho\times$ Volume) and Magnetic Flux ($B\times$ Area)

Length scale $L_0 \equiv V_0^{\frac{1}{3}} \rightarrow L = \left(\frac{V}{V_0}\right)^{\frac{1}{3}} L_0 = (1+\epsilon)L_0$

$\Longrightarrow \epsilon$: expansion coefficient

Equilibrium Condition : $\nabla_0 P_0 = -\rho\nabla_0\Phi_0 + (\mathbf{j_0}\times\mathbf{B_0})$

Equation of State (EoS) : $P \propto \rho^\Gamma$

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P - \rho\nabla\Phi + (\mathbf{j}\times\mathbf{B})$$

$$\Longrightarrow \quad \rho\frac{d^2\epsilon}{dt^2} = -\epsilon\,[3\Gamma-4]\,|\nabla_0 P_0|$$

- $\boxed{\Gamma > 4/3 \Longrightarrow \text{Stable oscillation}}$

MHD equations : Linear expansion

$P = P_0 + \delta P$, $\rho = \rho_0 + \delta\rho$, $\mathbf{B} = \mathbf{B_0} + \delta\mathbf{B}$, $\mathbf{J} = \nabla\times\mathbf{B}$, $\Phi_g = \Phi_{g0}$

Consider, $\mathbf{f} = \rho_0\mathbf{v}$ and $\boldsymbol{\beta} = \rho_0\delta\mathbf{B}$.

$$\frac{\partial\mathbf{f}}{\partial t} = \underbrace{\left[-\nabla\delta P + \frac{\nabla P_0}{\rho_0}\delta\rho\right]}_{\text{term : } -\frac{1}{\rho}\nabla P} - \frac{1}{\rho_0}\left(\mathbf{J_0}\times\mathbf{B_0}\right)\delta\rho + \frac{1}{\rho_0}\left(\mathbf{J_0}\times\boldsymbol{\beta}\right)$$

$$+ \underbrace{\left[\frac{1}{\rho_0}\left(\nabla\times\boldsymbol{\beta}\right)\times\mathbf{B_0} - \frac{1}{\rho_0^2}\left(\nabla\rho_0\times\boldsymbol{\beta}\right)\times\mathbf{B_0}\right]}_{\text{perturbation of } \mathbf{J}\times\mathbf{B}}$$

$$\frac{\partial\delta\rho}{\partial t} = -\nabla\cdot\mathbf{f}; \qquad \delta P = c_s^2\delta\rho$$

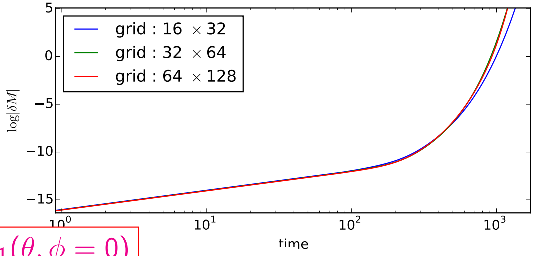
$$\frac{\partial\boldsymbol{\beta}}{\partial t} = \nabla\times\left(\mathbf{f}\times\mathbf{B_0}\right) - \frac{\nabla\rho_0}{\rho_0}\times\left(\mathbf{f}\times\mathbf{B_0}\right) - \rho_0\nabla\psi$$

$$\frac{\partial\psi}{\partial t} = \frac{\alpha\sqrt{c_s^2 + c_A^2}}{\Delta r}\psi - \left(c_s^2 + c_A^2\right)\nabla\cdot\mathbf{B}$$

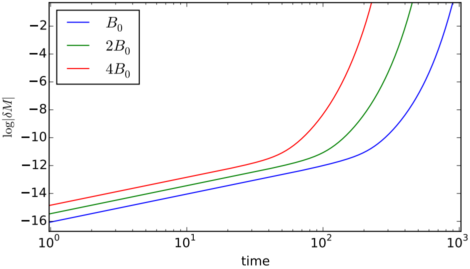
Tayler Instability (m = 1)

δM : total perturbed magnetic energy

$$\tau_{Alfven} \sim \frac{R}{\langle c_A \rangle} = 200$$

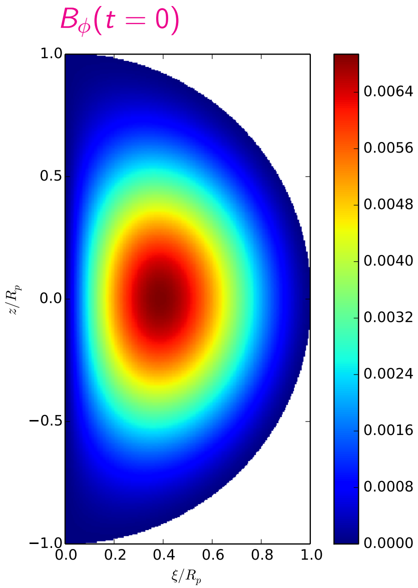


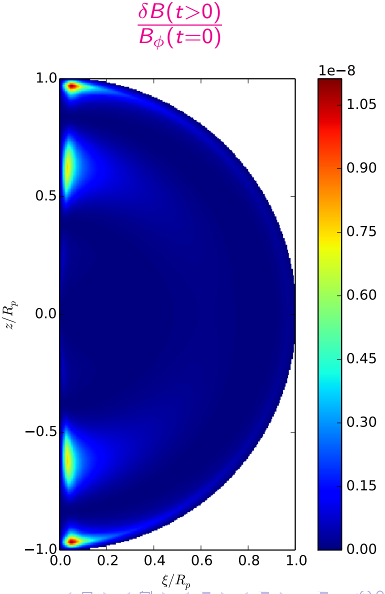
$$\mathbf{v}_{t=0} = f(r)\hat{r} \times \nabla Y_{11}(\theta, \phi = 0)$$



$\tau_{Alfven} \sim$ 200, 100, 50

Linear Stability : Toroidal field



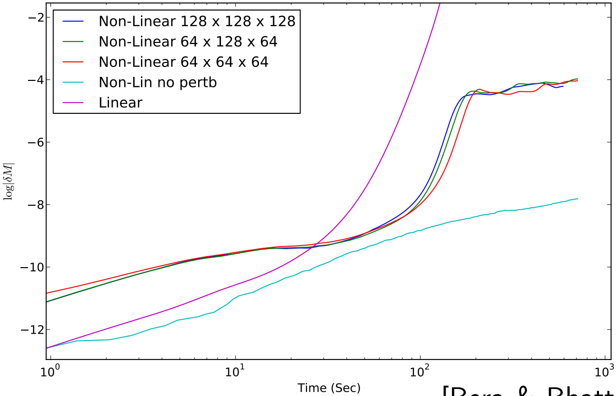


Polytropic star : $\Gamma = 5/3$

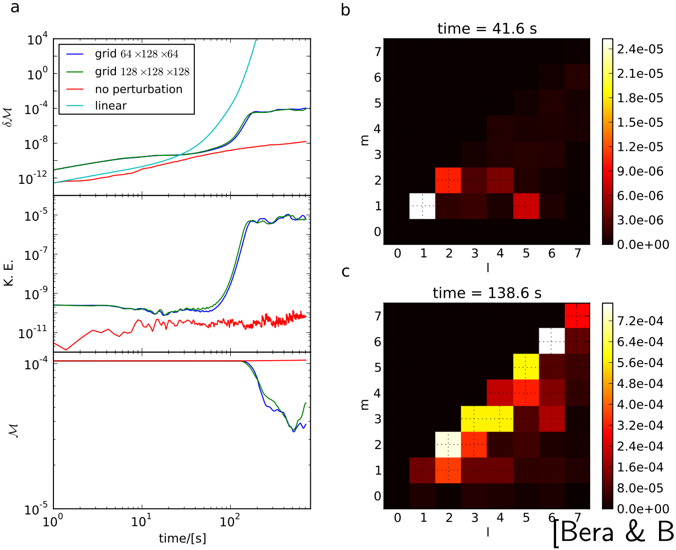
WD Mass $= 0.8\ M_\odot$
Radius $= 9716$ km
$\mathcal{M}/W = 8.92 \times 10^{-3}$

$\tau_{Alfven} \sim 32$ seconds

$$\boxed{\mathbf{v}_{t=0} = f(r)\hat{r} \times \nabla Y_{11}(\theta, \phi)}$$



[Bera & Bhattacharya 2017]



[Bera & Bhattacharya, 2017]

Why instability?

Flowers & Ruderman 1977

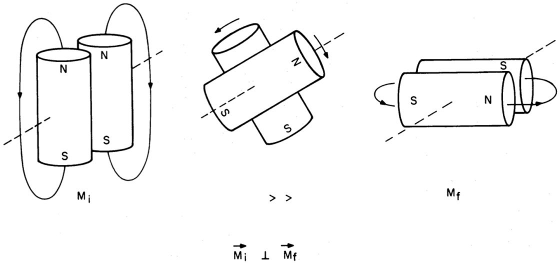
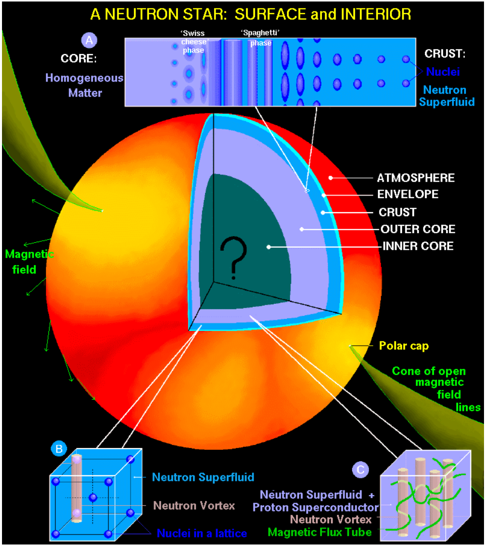


FIG. 1.—Initially parallel bar magnets which are free to rotate about a common axis perpendicular to $\boldsymbol{M}$ are unstable. Oppositely directed motions of the two magnets will bring N and S poles of the different magnets together, reducing the exterior magnetic field. In general the final total $\boldsymbol{M}$ will be much less than the initial $\boldsymbol{M}$ and approximately perpendicular to it.

Neutron Star : structure

Image credit : Dany Page [http://www.astroscu.unam.mx/]



$$n \rightarrow p + e^- + \bar{\nu}_e$$
$$p + e^- \rightarrow n + \nu_e$$

Shear stress in neutron star

- The matter density within NS gradially decreases towards the stellar surface from the core region.
- At the outer part of NS, ionized nuclei form crystalized structure.
- Perturbed ionic lattice structure from its minimum energy configuration stores electrostatic energy and, hence the restoring force becomes active.
- Highly densed matter at the core of NS does not get an opportunity to form lattice. Therefore the shear effect is mainly localized to the NS crust.

Formulating Shear

- Assume an applied perturbation at $\mathbf{x}$ displaces $\mathbf{x} \to \mathbf{x} + \xi(\mathbf{x})$.
- Non-uniform ξ over the space deform the system.
- Spacial gradient of ξ measures strain : $\mathbf{S} = \nabla\xi$ or, $S_{ij} = \xi_{i;j}$
- Trace-less body-shear component $\Sigma_{ij} = \frac{1}{2}\left(S_{ij} + S_{ji}\right) - \frac{1}{3}g_{ij}S_{kk}$
- In curvilinear coordinate $S_{ik} = \xi_{i;k} = \xi_{i,k} + \Gamma_{ijk}\xi_j$
- For example, in spherical polar coordinate
$$\Sigma_{rr} = \frac{2}{3}\frac{\partial\xi_r}{\partial r} - \frac{2}{3r}\xi_r - \frac{\cot\theta}{3r}\xi_\theta - \frac{1}{3r}\frac{\partial\xi_\theta}{\partial\theta} - \frac{1}{3r\sin\theta}\frac{\partial\xi_\phi}{\partial\phi}$$
- Stress due to $\mathbf{\Sigma}$ is : $\mathbf{T} = -2\mu\mathbf{\Sigma} \Rightarrow \boldsymbol{f}_{shear} = -\nabla\boldsymbol{T} = 2\nabla(\mu\mathbf{\Sigma})$
Here, μ is the shear modulus.

Shear Modulus

- Shear modulus depends on the characteristics of the sate of matter.

- The studies on the crystal structure at the NS crust show [e.g. Strohmayer et al 1991 ApJ 375, 679]

$$\frac{\mu_{eff}}{n(Ze)^2/a} = \frac{0.1194}{1 + 1.781 \times (100/\Gamma)^2}$$

here, $n = 1/(4\pi/3)a^3$ is the ion number density, a is the radius of the Wigner-Seitz cell containing one single ion of charge Ze. For $^{56}Fe_{26}$, $a > 2.2 \times 10^{-16}$ cm.

- $\mu_{eff} \sim 0.1 * n(Ze)^2/a = 3.7 \times 10^{29} \mathrm{Jule} \cdot \mathrm{m}^{-3} = 3.7 \times 10^{30} \mathrm{erg} \cdot \mathrm{cm}^{-3}$

Evolutionary set up including Shear

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{v} \tag{1}$$

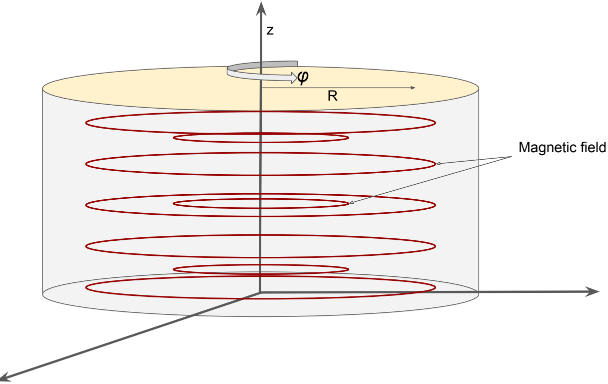
$$\begin{aligned}\frac{\partial f_i}{\partial t} &= \left[-\nabla \delta P + \frac{\nabla P_0}{\rho_0}\delta\rho\right]_i \\ &\quad \left[-\frac{1}{\rho_0}\left(\mathbf{J_0} \times \mathbf{B_0}\right)\delta\rho + \frac{1}{\rho_0}\left(\mathbf{J_0} \times \boldsymbol{\beta}\right)\right]_i \\ &\quad + \left[\frac{1}{\rho_0}\left(\nabla \times \boldsymbol{\beta}\right) \times \mathbf{B_0} - \frac{1}{\rho_0^2}\left(\nabla\rho_0 \times \boldsymbol{\beta}\right) \times \mathbf{B_0}\right]_i \\ &\quad + S_{ijkl}\frac{\partial^2 \xi_k}{\partial x_j \partial x_l}\end{aligned} \tag{2}$$

$$\frac{\partial \delta\rho}{\partial t} = -\nabla \cdot \mathbf{f} \tag{3}$$

$$\frac{\partial \boldsymbol{\beta}}{\partial t} = \nabla \times \left(\mathbf{f} \times \mathbf{B_0}\right) - \frac{\nabla\rho_0}{\rho_0} \times \left(\mathbf{f} \times \mathbf{B_0}\right) \tag{4}$$

Local magnetic field evolution

- Assume a cylindrical volume with incompressable plasma having toroidal magnetic field $B_{0\phi} \sim R^{\alpha}$ in equilibrium.
- Here the current density $\Rightarrow \mathbf{j_0} = \frac{\alpha+1}{\mu_0 R} B_{0\phi}\hat{z}$. To have a realistic current density $\alpha > 1$.



$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} - \nabla p$$
$$\nabla \mathbf{v} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

- Expantion of the perturbed variables as $e^{im\phi+ik_z z+\omega t}$
-

$$\rho\omega v_R + j_{1z}B_{0\phi} + j_{0z}B_{1\phi} = \frac{\partial}{\partial R}\left\{\frac{R\rho\omega}{m^2+k_z^2R^2}\left(R\partial_R v_R + v_R\right)\right\} - \frac{\partial}{\partial R}\left[\frac{m^2}{m^2+k_z^2R^2}\frac{\alpha+1}{\mu_0 R\omega}{B_0}_\phi^2 v_R\right]$$

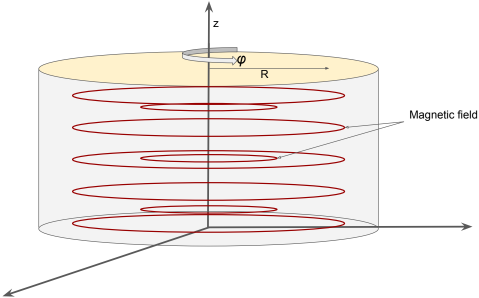
- In the limit $k_z \to 0$, v_R will have a bounded solution and $\omega^2 < 0$ if $\frac{{B_0}_\phi^2}{\mu_0 R^2} < 0$. But it is not possible.

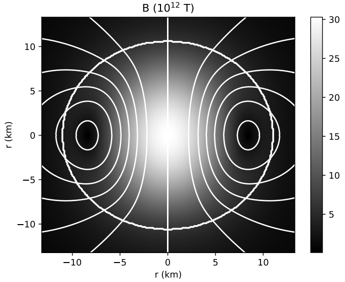
- Hence the pure toroidal field geometry becomes unstable.

Local magnetic field evolution : effect of shear modulus

- The force balance equation including shear:
  $\rho \frac{dv_i}{dt} = [\mathbf{j} \times \mathbf{B}]_i - \nabla_i p + S_{ijkl}\partial_j \partial_l \xi_k$
- $\frac{\partial \xi_i}{\partial t} = v_i$
- In the linear order approximation having the variables with the expansion $e^{im\phi + ik_z z + \omega t}$, $\omega \xi_i = v_i$.
- Now assuming a characteristic wavenumber k_λ in the expansion of $\xi \sim \exp(i\boldsymbol{k}_\lambda \cdot \boldsymbol{x})$, the shear term becomes $-2\mu_{eff} k_\lambda^2 v_i/\omega$
-
  $$\rho \omega v_R = -j_{1z} B_{0\phi} - j_{0z} B_{1\phi} + \frac{\partial}{\partial R}\left\{ \frac{R\rho\omega}{m^2 + k_z^2 R^2} \left(R\partial_R v_R + v_R \right) \right\} - \frac{\partial}{\partial R}\left[\frac{m^2}{m^2 + k_z^2 R^2} \frac{\alpha + 1}{\mu_0 R \omega} B_{0\phi}^2 v_R \right] - 2\mu_{eff} k_\lambda^2 v_i/\omega$$
- Now the stability condition becomes $\frac{B_{0\phi}^2}{\mu_0 R^2} - 2\mu_{eff} k_\lambda^2 < 0$.

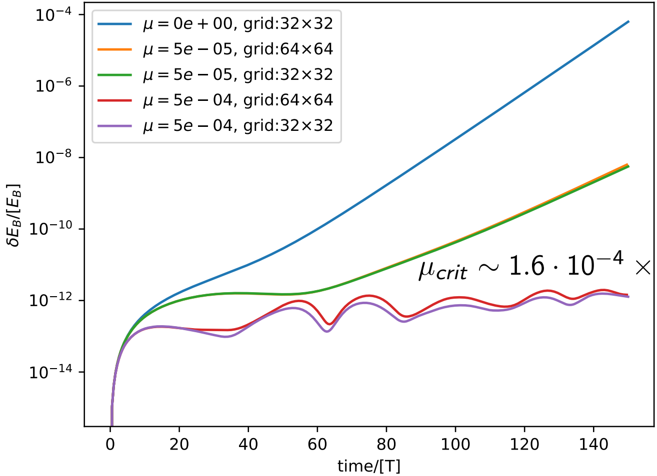
- matter density varies over space
- magnetic field and electric current density may be complex
- compressability (defined by EoS) can effect





1.3 $M_\odot$ neutron star with 11 km radius and pure toroidal magnetic field. $\mathcal{M}/\mathcal{W} = 8 \times 10^{-3}$
$< B >\sim 10^{12}$ T=10^{16} G

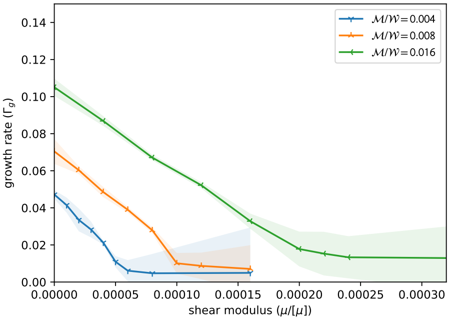
Toroidal B; ϕ mode m=1

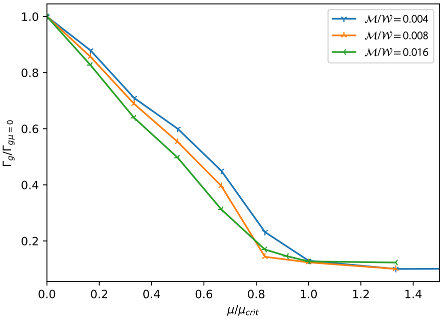


$\mu_{crit} \sim 1.6 \cdot 10^{-4} \times [2.37 \cdot 10^{35}]\mathrm{Jule} \cdot \mathrm{m}^{-3}$

Growth rate $\Gamma_g \equiv \frac{\partial \ln \delta E_B}{\partial t}$

Toroidal B; ϕ mode m=1



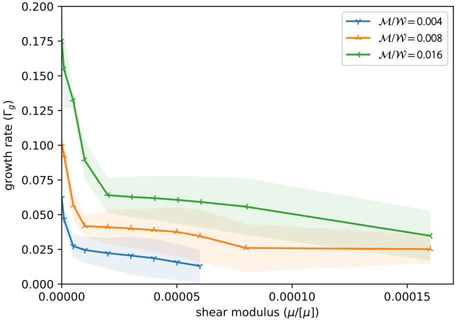


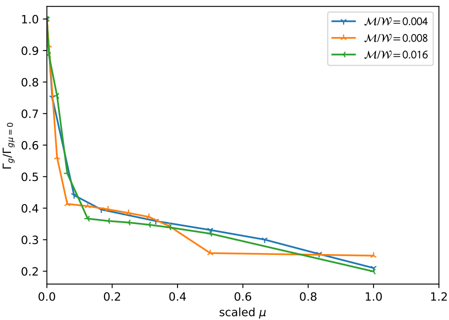
The scaling relation is closely followed.

$\mu_{eff}\big|_{crit} \propto \frac{B_0^2}{\mu_0}$

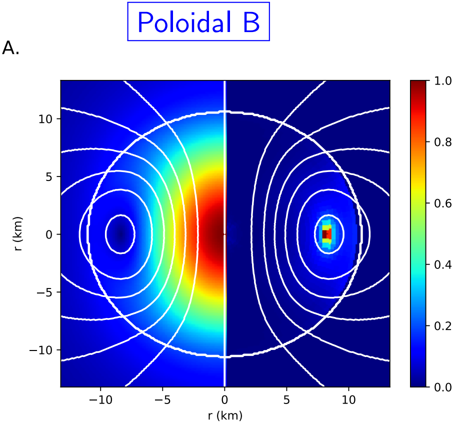
Uniform shear modulus : Growth rate

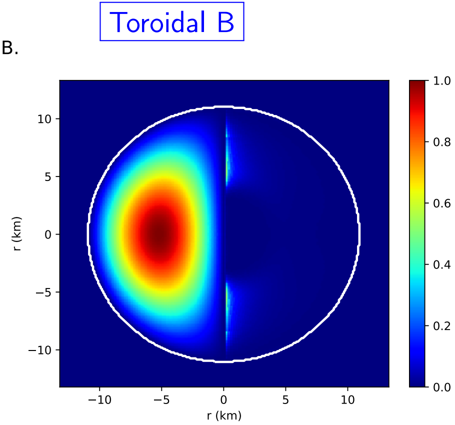
Poloidal B; ϕ mode m=2



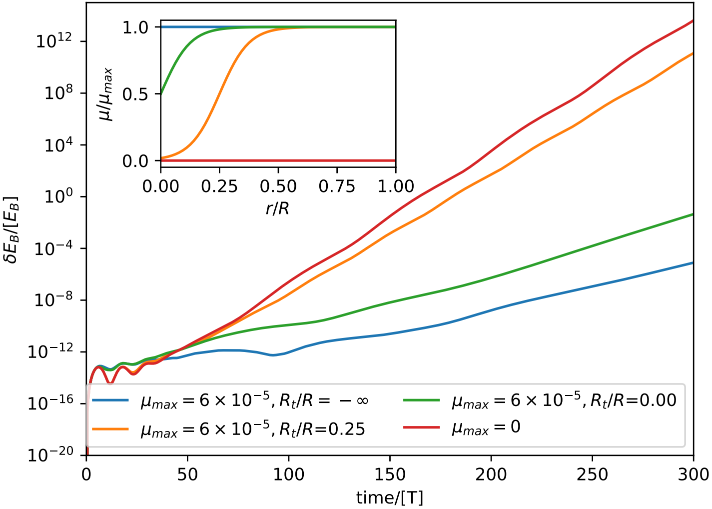


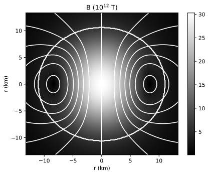
Field geometry : equilibrium & perturbed components

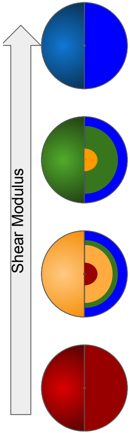




Crustal shear modulus







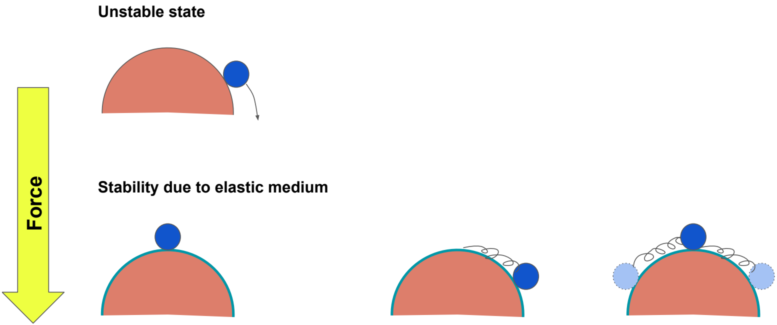
Astrophysical importance

- Stability condition :: $\mu > \mu_{crit}$ depends on field geometry
- short dynamical time $\Rightarrow$ magnetic field stagnation at crust
- long NS evolutionary time $\Rightarrow$ magneto-thermal evolution may bring instability in magnetic field geometry
- Source of Event : Failure of NS crust for local B $\rightarrow$ observable as GRB/FRB [Thompson & Duncan 1995]

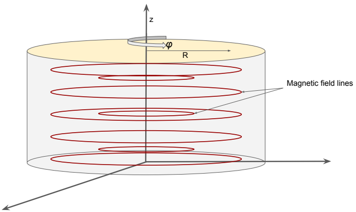
Summary

- Coaxial magnetic field lines, produced by a free plasma current, are unstable against perturbation.
- The instability time scale due to the axisymmetric magnetic field in a degenerate star is about a few Alfvén times.
- The inclusion of the effects due to Shear Modulus reduces the instability growth rate and it can make the system stable upto a limit.

Stability : Elastic medium



Outlook : stability condition



- Sear modulus : $\left(2\mu - \frac{2\alpha^2+2\alpha+1}{k_\lambda^2}\frac{B_{0\phi}^2}{\mu_0}\right) > 0$

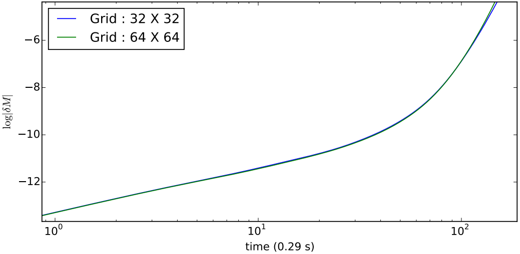
- Compressible fluid ($|\omega\tau_s| \gg 1$):
$$\frac{\partial}{\partial r}\left[\frac{r^2}{\Gamma p_0}\left|\frac{\partial p_0}{\partial r}\right|\frac{\Gamma_1}{\Gamma}\frac{1}{|\omega|^2\tau_s^2}\right] - \frac{2\alpha^2+2\alpha+1}{\rho r^2|\omega|^2}\frac{B_{0\phi}^2}{\mu_0} > 0$$

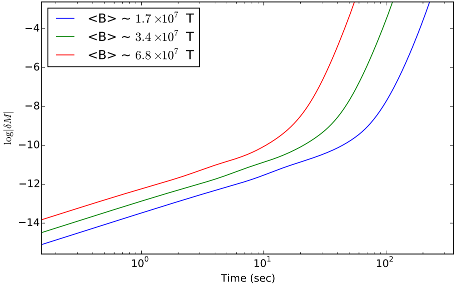
- Dynamical motion : ??

δM : total perturbed magnetic energy

WD Mass $= 0.89\ M_\odot$
Radius $= 6643$ km
$\mathcal{M}/W = 8.76 \times 10^{-3}$

$\tau_{Alfven} \sim \frac{R}{\langle c_A \rangle}$
$= 17$ seconds



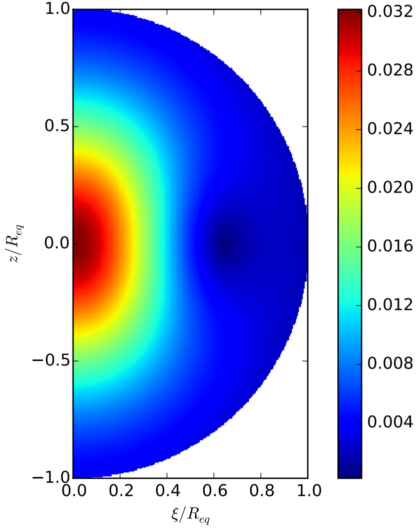


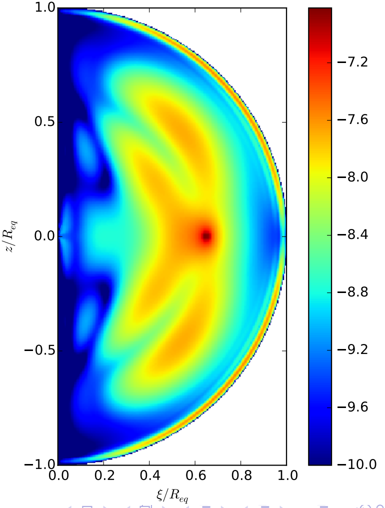
$\tau_{Alfven} \sim$ 68, 34, 17 seconds

Linear Stability : Poloidal field

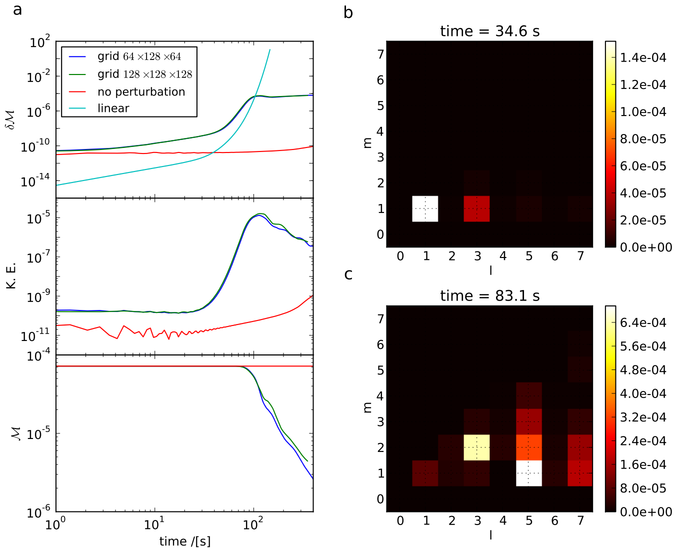
t $= 0$: $B_0(unit = 3.03 \times 10^9\ T)$

$\log[\frac{\delta B(t=0.24sec)}{B_0}]$





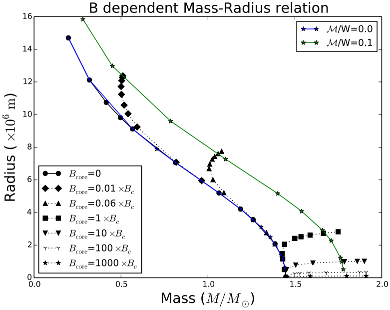
Non-linear evolution : axisymmetric toroidal configuration

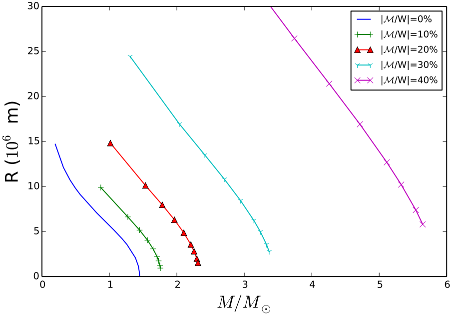


Mass-radius relation of magnetized WDs

pure poloidal field

pure toroidal field





[Bera & Bhattacharya, 2014, 2016]

ϕ-decomposition :
$\delta\rho(t,r,\theta,\phi) = \sum_{m=0}^{\infty} [\delta\rho^{+}(t,r,\theta)\cos m\phi + \delta\rho^{-}(t,r,\theta)\sin m\phi]$.

- **Initial condition** (t=0) : Equilibrium solution + Perturbation [function of Y_{lm}]
- **Spatial Boundary** : $\boldsymbol{\beta} \rightarrow 0$, $\delta\rho \rightarrow 0$ and $\mathbf{f} \rightarrow 0$ as $r \rightarrow 0, R_{surf}$. θ-boundary depends on symmetry.
- **Time evolution** : 2-step (predictor-corrector) MacCormack method (1969)
- **Divergence cleaning** : Hyperbolic-parabolic cleaning [Dedner et.al. (2002)]
- **Numerical dissipations** : resolution dependent i) fourth order Kreiss-Oliger dissipation and ii) electrical resistivity for long term evolution

PLUTO : A MHD code, based on Godunov type shock-capturing schemes, solves conservative form of equations using finite volume formalism [Mignone et. al; 2007]

- **Grid** :: Spherical polar (r = [0.1, 0.82], θ = [0.05, 3.1], ϕ = [0, 2π])
- **Initial condition** :: Equilibrium Solution + Perturbation
- **Time evolution** :: hll
- **Boundary** :: outer radial : Fixed gradients, inner radial and axial directions : fixed equilibrium values, ϕ : periodic
- **Gravitational potential** :: Assumed to be fixed (Φ_0)

Uniform shear and poloidal field