EINDHOVEN UNIVERSITY OF TECHNOLOGY

Small-worlds, complex networks and random graphs

Remco van der Hofstad

Advances in Applied Probability
ICTS, Bengaluru,
August 5–17, 2019

Joint work with:

▷ G. Hooghiemstra (Delft)
▷ P. Van Mieghem (Delft)
▷ S. Bhamidi (North Carolina)
▷ J. Komjáthy (TU/e)
▷ D. Znamensky (Philips Research)
▷ H. van den Esker (Delft)
▷ S. Dommers (TU/e)

EURANDOM NETWORKS

Plan lectures

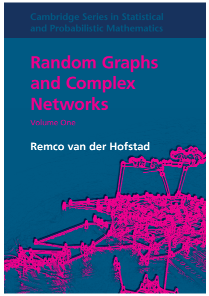
Lecture 1:
Real-world networks and random graphs

Lecture 2:
Small-world phenomena in random graphs

Lecture 3:
Information diffusion in random graphs

Material

▷ Intro random graphs:
Random Graphs and Complex Networks Volume 1
`http://www.win.tue.nl/~rhofstad/NotesRGCN.html`
Volume 2: in preparation on same site



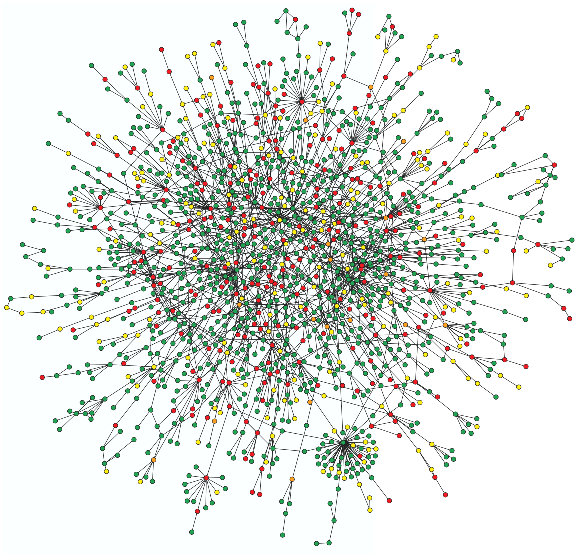
Treat selected parts of Chapters I.1, I.6–I.8 and II.2–II.7.

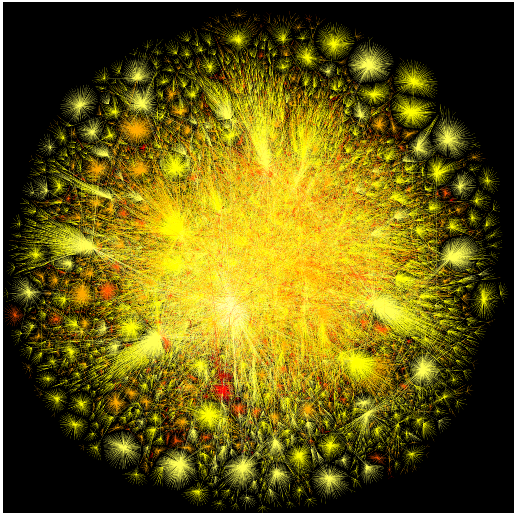
Argument are probabilistic, using
▷ first and second moment method;
▷ branching process approximations.

Lecture 1:

Real-world networks and random graphs

Complex networks





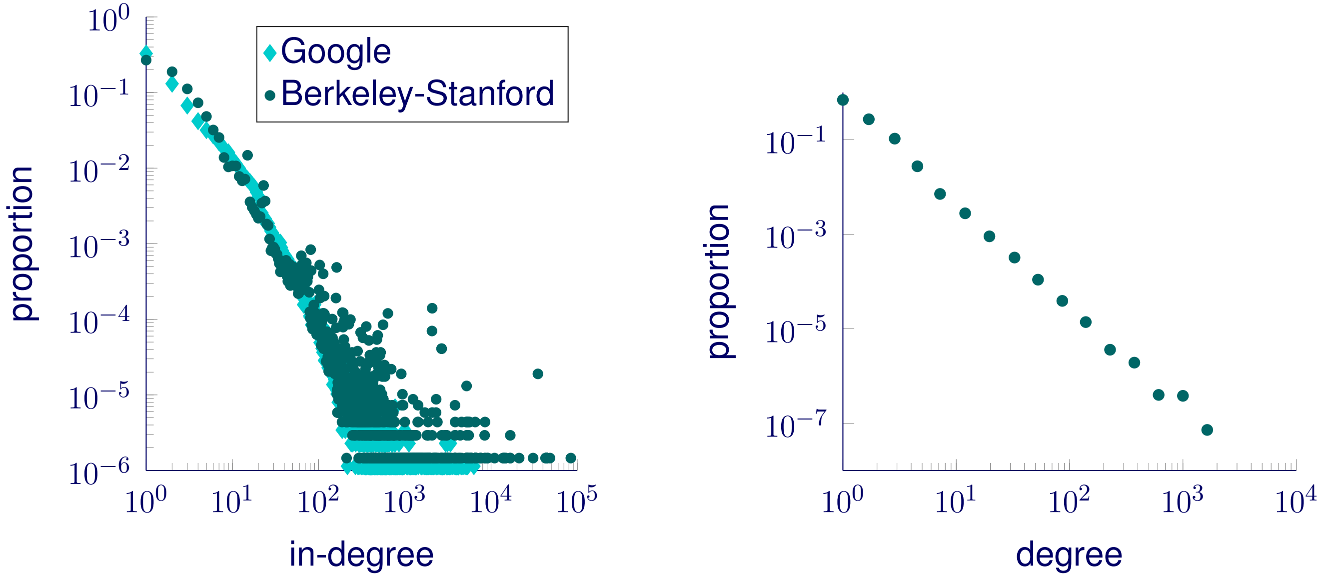
Yeast protein interaction network[a] Internet 2010[b]

Attention focussing on unexpected commonality.

[a] Barabási & Óltvai 2004

[b] Opte project `http://www.opte.org/the-internet`

Scale-free paradigm

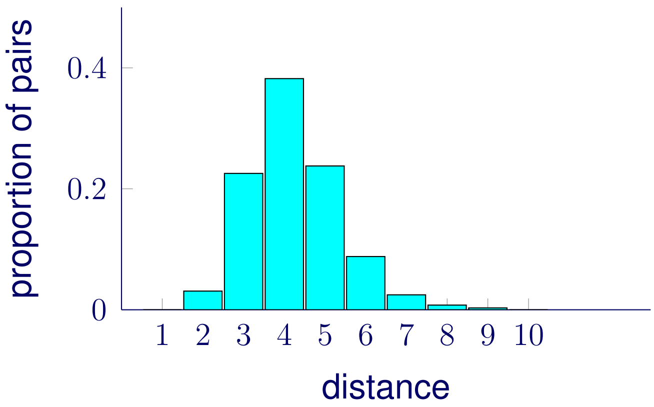


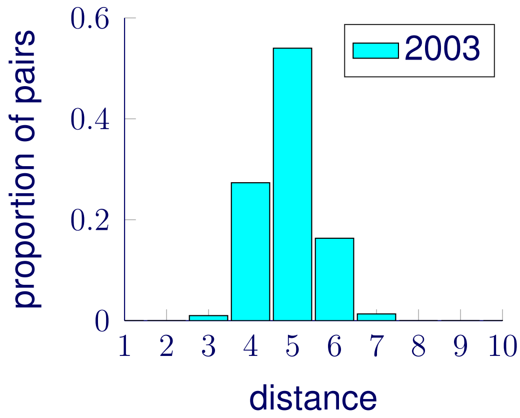
Loglog plot degree sequences WWW in-degree and Internet

▷ Straight line: proportion p_k of vertices of degree k satisfies $p_k = ck^{-\tau}$.

▷ Empirical evidence: Often $\tau \in (2, 3)$ reported.

Small-world paradigm





Distances in Strongly Connected Component WWW and IMDb.

Network science

▷ Complex networks modeled using

random graphs.

▷ Network functionality modeled by stochastic processes on them.

▷ A plethora of examples:

Disease spread	Synchronization
Information diffusion	Robustness to failures
Consensus reaching	Information retrieval
Percolation	Random walks...

▷ Also algorithms on networks important: PageRank, assortativity, community detection,...

▷ Prominent part of applied math for decades to come.

Models complex networks

▷ Inhomogeneous Random Graphs:
Static random graph, independent edges with inhomogeneous edge occupation probabilities, yielding scale-free graphs.
(Chapters I.6, II.2 and II.5)

[Extensions of Erdős-Rényi random graphs Chapters I.4 and I.5.]

▷ Configuration Model:
Static random graph with prescribed degree sequence.
(Chapters I.7, II.3 and II.6)

▷ Preferential Attachment Model:
Dynamic model, attachment proportional to degree plus constant.
(Chapters I.8, II.4 and II.7)

Universality??

Erdős-Rényi

Erdős-Rényi random graph is random subgraph of complete graph on $[n] := \{1, 2, \ldots, n\}$ where each of $\binom{n}{2}$ edges is occupied independently with prob. p.

Simplest imaginable model of a random graph.

▷ Attracted tremendous attention since introduction 1959, mainly in combinatorics community:

Probabilistic method (Spencer, Erdős et al.).

▷ Average degree equals $(n-1)p \approx np$, so choose $p = \lambda/n$ to have sparse graph.

▷ Egalitarian: Every vertex has equal connection probabilities. Misses hub-like structure of real networks.

Inhomogeneous random graphs

▷ Extensions of Erdős-Rényi random graph with different vertices.

▷ Chung-Lu: random graphs with prescribed expected degrees:
⋆ Connected component structure (2002)
⋆ Distance results (2002), PNAS
⋆ Book (2006)

▷ Most general:
⋆ Bollobas, Janson and Riordan (2007)
⋆ Söderberg (2007): Phys. Rev. E

We focus on

generalized random graph.

Generalized random graph

▷ Attach edge with probability p_{ij} between vertices i and j, where

$$p_{ij} = \frac{w_i w_j}{\ell_n + w_i w_j}, \qquad \text{with} \qquad \ell_n = \sum_{i\in[n]} w_i,$$

different edges being independent [Britton-Deijfen-Martin-Löf 05]

▷ Resulting graph is denoted by $\mathrm{GRG}_n(\boldsymbol{w})$.

Interpretation: w_i is close to expected degree vertex i.

⋆ Retrieve Erdős-Rényi RG with $p = \lambda/n$ when $w_i = n\lambda/(n-\lambda)$.

▷ Related models:

⋆ Chung-Lu model: $p_{ij} = w_i w_j/\ell_n \wedge 1$;

⋆ Norros-Reittu model: $p_{ij} = 1 - \mathrm{e}^{-w_i w_j/\ell_n}$.

⋆ Janson (2010): General conditions for asymptotic equivalence.

Regularity vertex weights

Condition I.6.3. Denote empirical distribution function weight by

$$F_n(x) = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{w_i \leq x\}}, \qquad x \geq 0.$$

(a) Weak convergence of vertex weight. There exists F s.t.

$$W_n \xrightarrow{d} W,$$

where W_n and W have distribution functions F_n and F.
(b) Convergence of average vertex weight.

$$\lim_{n \to \infty} \mathbb{E}[W_n] = \mathbb{E}[W] > 0.$$

(c) Convergence of second moment vertex weight.

$$\lim_{n \to \infty} \mathbb{E}[W_n^2] = \mathbb{E}[W^2].$$

Canonical choice weights

Aim: Proportion of vertices i with $d_i = k$ is close to

$$p_k = \mathbb{P}(D = k),$$

for some random variable D.

(A) Take $\boldsymbol{w} = (w_1, \ldots, w_n)$ as i.i.d. random variables with distribution function F.

(B) Take $\boldsymbol{w} = (w_1, \ldots, w_n)$ as

$$w_i = [1 - F]^{-1}(i/n).$$

Interpretation: Proportion of vertices i with $w_i \leq x$ is close to $F(x)$.

▷ Power-law example: $F(x) = [1 - (a/x)^{\tau-1}]\mathbb{1}_{\{x \geq a\}}$, for which

$$[1 - F]^{-1}(u) = a(1/u)^{-1/(\tau-1)}, \qquad \text{so that} \qquad w_j = a(n/j)^{1/(\tau-1)}.$$

Degree structure GRG

Denote proportion of vertices with degree k by

$$P_k^{(n)} = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{D_i = k\}},$$

where D_i is degree of $i \in [n]$. Then [Bollobás-Janson-Riordan (07)]

$$P_k^{(n)} \xrightarrow{\mathbb{P}} p_k = \mathbb{E}\Big[\mathrm{e}^{-W} \frac{W^k}{k!}\Big],$$

where W is a random variable having distribution function F. †

Recognize limit $(p_k)_{k \geq 0}$ as probability mass function of Poisson random variable with random parameter $W \sim F$.
In particular,

$$\sum_{l \geq k} p_l \sim c k^{-(\tau-1)} \quad \text{iff} \quad \mathbb{P}(W \geq k) \sim c k^{-(\tau-1)}.$$

Configuration model

▷ Invented by Bollobás (80) EJC

to study number of graphs with given degree sequence.

Inspired by Bender+Canfield (78) JCT(A)

Giant component: Molloy, Reed (95)

Popularized by Newman-Strogatz-Watts (01)

▷ In configuration model $\mathrm{CM}_n(\boldsymbol{d})$ degree sequence is prescribed:

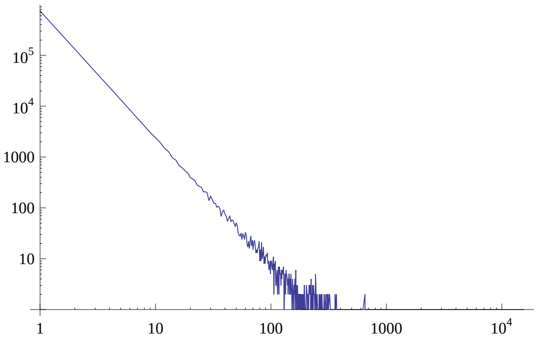
> ▷ n number of vertices;
> ▷ $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ sequence of degrees is given.

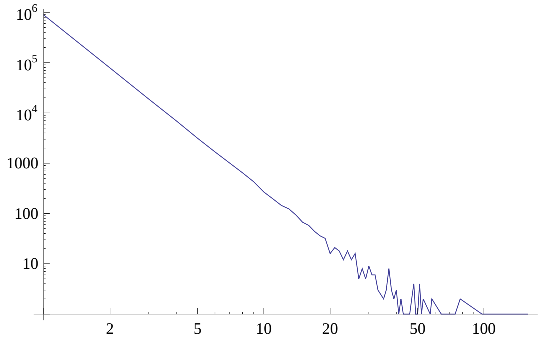
Often $(d_i)_{i\in[n]}$ taken to be i.i.d.

▷ Special attention to power-law degrees, i.e., for $\tau > 1$ and c_τ

$$\mathbb{P}(d_1 \geq k) = c_\tau k^{-\tau+1}(1 + o(1)).$$

Power laws CM





Loglog plot of degree sequence CM with i.i.d. degrees $n = 1,000,000$ and $\tau = 2.5$ and $\tau = 3.5$, respectively.

Graph construction CM

▷ Assign d_j half-edges to vertex j. Assume total degree

$$\ell_n = \sum_{i \in [n]} d_i$$

is even.

▷ Pair half-edges to create edges as follows:
Number half-edges from 1 to ℓ_n in any order.
First connect first half-edge at random with one of other $\ell_n - 1$ half-edges.

▷ Continue with second half-edge (when not connected to first) and so on, until all half-edges are connected.

▷ Resulting graph is denoted by $\mathrm{CM}_n(\boldsymbol{d})$.

Conclusion networks

Many real-world networks share important features:

scale-free and small-world paradigms.

Often, suggestion of infinite-variance degrees.

Models invented to describe properties:

Configuration model and generalized random graph.

Models are flexible in their degree structure.

Lecture 2:

Small-world phenomenon on random graphs

Graph construction CM

▷ Assign d_j half-edges to vertex j. Assume total degree

$$\ell_n = \sum_{i\in[n]} d_i$$

is even.

▷ Pair half-edges to create edges as follows:
Number half-edges from 1 to ℓ_n in any order.
First connect first half-edge at random with one of other $\ell_n - 1$ half-edges.

▷ Continue with second half-edge (when not connected to first) and so on, until all half-edges are connected.

▷ Resulting graph is denoted by $\mathrm{CM}_n(\boldsymbol{d})$.

Regularity vertex degrees

Condition I.7.5. Denote empirical distribution function degrees by

$$F_n(x) = \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{d_i\leq x\}}, \qquad x \geq 0.$$

(a) Weak convergence of vertex degrees. There exists F s.t.

$$D_n \xrightarrow{d} D,$$

where D_n and D have distribution functions F_n and F.
(b) Convergence of average vertex weight.

$$\lim_{n\to\infty} \mathbb{E}[D_n] = \mathbb{E}[D] > 0.$$

(c) Convergence of second moment vertex degrees.

$$\lim_{n\to\infty} \mathbb{E}[D_n^2] = \mathbb{E}[D^2] < \infty.$$

Canonical choice degrees

Aim: Proportion of vertices i with $d_i = k$ is close to

$$F(k) - F(k-1) = p_k = \mathbb{P}(D = k),$$

where D has distribution function F.

$\star$ Power-law degrees: precise structure of large degrees crucial.

(A) Take $\boldsymbol{d} = (d_1, \ldots, d_n)$ as i.i.d. rvs with distribution function F.

Double randomness!

(B) Take $\boldsymbol{d} = (d_1, \ldots, d_n)$ such that $d_i = [1 - F]^{-1}(i/n)$, with F distribution function on $\mathbb{N}$.

Power-law degrees:

$$[1 - F](k) \approx ck^{-(\tau-1)}, \quad \text{so that} \quad d_j \approx a(n/j)^{1/(\tau-1)}.$$

Simple CMs

Proposition I.7.7. Let $G = (x_{ij})_{i,j\in[n]}$ be multigraph on $[n]$ s.t.

$$d_i = x_{ii} + \sum_{j\in[n]} x_{ij}.$$

Then, with $\ell_n = \sum_{v\in[n]} d_v$,

$$\mathbb{P}(\mathrm{CM}_n(\boldsymbol{d}) = G) = \frac{1}{(\ell_n - 1)!!} \frac{\prod_{i\in[n]} d_i!}{\prod_{i\in[n]} 2^{x_{ii}} \prod_{1\leq i\leq j\leq n} x_{ij}!}.$$

Consequently, number of simple graphs with degrees $\boldsymbol{d}$ equals

$$N_n(\boldsymbol{d}) = \frac{(\ell_n - 1)!!}{\prod_{i\in[n]} d_i!} \mathbb{P}(\mathrm{CM}_n(\boldsymbol{d}) \text{ simple}),$$

and, conditionally on $\mathrm{CM}_n(\boldsymbol{d})$ simple,

$\mathrm{CM}_n(\boldsymbol{d})$ is uniform random graph with degrees $\boldsymbol{d}$.

Relation GRG and CM

Theorem I.6.15. The $\mathrm{GRG}_n(\boldsymbol{w})$ with edge probabilities $(p_{ij})_{1\leq i<j\leq n}$ given by

$$p_{ij} = \frac{w_i w_j}{\ell_n + w_i w_j},$$

conditioned on its degrees $\{d_i(X) = d_i \forall i \in [n]\}$ is uniform over all graphs with degree sequence $(d_i)_{i\in[n]}$.

Consequently, conditionally on degrees, $\mathrm{GRG}_n(\boldsymbol{w})$ has the same distribution as $\mathrm{CM}_n(\boldsymbol{d})$ conditioned on simplicity.

Allows properties of $\mathrm{GRG}_n(\boldsymbol{w})$ to be proved through $\mathrm{CM}_n(\boldsymbol{d})$ by showing that degrees $\mathrm{GRG}_n(\boldsymbol{w})$ satisfy right asymptotics.

Inspires Degree Regularity Condition.[†]

Self-loops + multi-edges

▷ CM can have cycles and multiple edges, but these are relatively scarce compared to the number of edges. [Theorem I.7.6]

▷ Let D_n denote degree of uniformly chosen vertex. Condition 7.5(a): D_n converges in distribution to limiting random variable D.

▷ When $\mathbb{E}[D_n^2] \to \mathbb{E}[D^2] < \infty$, then numbers of self-loops and multiple edges converge in distribution to two independent Poisson variables with parameters $\nu/2$ and $\nu^2/4$, respectively, where

$$\nu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]}.$$

[Theorem I.7.8, Prop. I.7.9]

▷ Proof: moment method (Bollobás 80, Janson 09) or Chen-Stein method (Angel-Holmgren-vdH 16).

Preferential attachment model

▷ Albert-Barabási (1999):
Emergence of scaling in random networks (Science).
34013 cit. (12-08-2019).

▷ Bollobás, Riordan, Spencer, Tusnády (2001):
The degree sequence of a scale-free random graph process (RSA)
852 cit. (12-08-2019).

[Yule (1925) and Simon (1955) already introduced similar models.]

In preferential attachment models, network is growing in time, in such a way that new vertices are more likely to be connected to vertices that already have high degree.

Rich-get-richer model.

Preferential attachment model

▷ Albert-Barabási (1999):
Emergence of scaling in random networks (Science).
34013 cit. (12-08-2019).
▷ Bollobás, Riordan, Spencer, Tusnády (2001):
The degree sequence of a scale-free random graph process (RSA)
852 cit. (12-08-2019).

[Yule (1925) and Simon (1955) already introduced similar models.]

In preferential attachment models, network is growing in time, in such a way that new vertices are more likely to be connected to vertices that already have high degree.

Old-get-richer model.

Preferential attachment

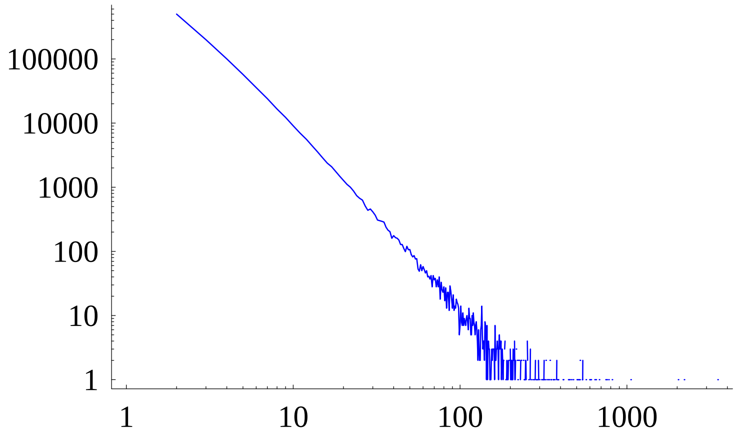
At time n, single vertex is added with m edges emanating from it. Probability that edge connects to ith vertex is proportional to

$$D_i(n-1)+\delta,$$

where $D_i(n)$ is degree vertex i at time n, $\delta > -m$ is parameter.

Yields power-law degree sequence with exponent $\tau = 3 + \delta/m > 2$.

Bol-Rio-Spe-Tus 01 $\delta = 0$, DvdEvdHH09,...



$m = 2, \delta = 0, \tau = 3, n = 10^6$

Albert-László Barabási

"...the scale-free topology is evidence of organizing principles acting at each stage of the network formation. (...) No matter how large and complex a network becomes, as long as preferential attachment and growth are present it will maintain its hub-dominated scale-free topology."

Degrees in PAM

Bollobás-Riordan-Spencer-Tusnády 01: First to give proof for $\delta = 0$. Tons of subsequent proofs, many of which follow same key steps:

▷ A clever Doob martingale:

$$M_n = \mathbb{E}[N_k(t) \mid \mathrm{PA}_n],$$

where $N_k(t)$ is number of vertices of degree k at time t, combined with Azuma-Hoeffding to prove concentration.

▷ Analysis of means: Identify asymptotics $\mathbb{E}[N_k(t)]$ and prove that

$$\frac{\mathbb{E}[N_k(t)]}{t} \to p_k.$$

Many different ways to do this. See Section I.8.4 for details.

[Alternatively, for $m = 1$, CTBP embeddings can be used, see work of e.g. K. Athreya.]

Network models I

▷ **Configuration model with clustering:**
Input per vertex i is number of simple edges, number of triangles, number of squares, etc. Then connect uniformly at random.

Result: Random graph with (roughly) specified degree, triangle, square, etc distribution over graph.

Application: Social networks?

▷ **Small-world model:**
Start with d-dimensional torus (=circle $d = 1,$ donut $d = 2,$ etc).
Put in nearest-neighbor edges. Add few edges between uniform vertices, either by rewiring or by simply adding.

Result: Spatial random graph with high clustering, but degree distribution with thin tails.

Application: None? Often used by neuroscientists.

Network models II

▷ **Random intersection graph:**
Specify collection of groups. Vertices choose group memberships. Put edge between any pairs of vertices in same group.

Result: Flexible collection of random graphs, with high clustering, communities by groups, tunable degree distribution.

Application: Collaboration graphs?

▷ **Spatial preferential attachment model:**
First give vertex uniform location. Let it connect to close by vertices with probability proportionally to degree.

Result: Spatial random graph with scale-free degrees and high clustering.

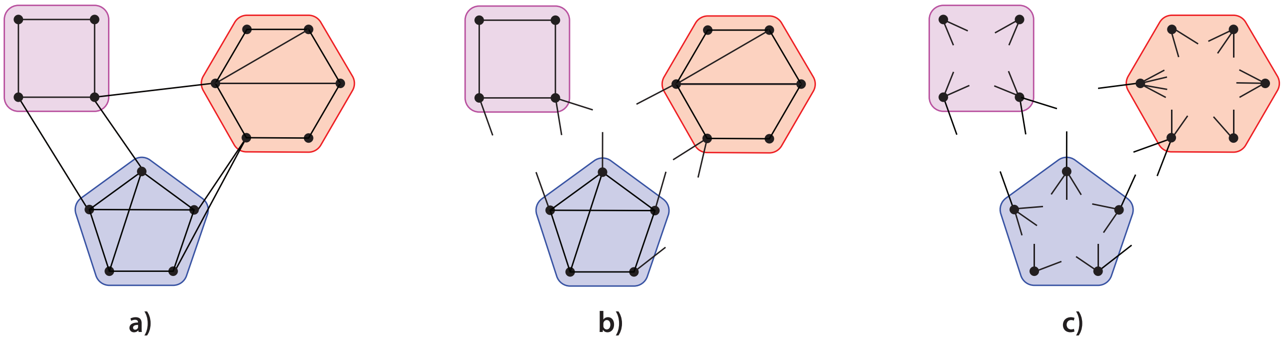
Application: Social networks, WWW?

Hierarchical CM

Vertex i is blown up to represent small community graph.
Connect inter-community half-edges uniformly at random.

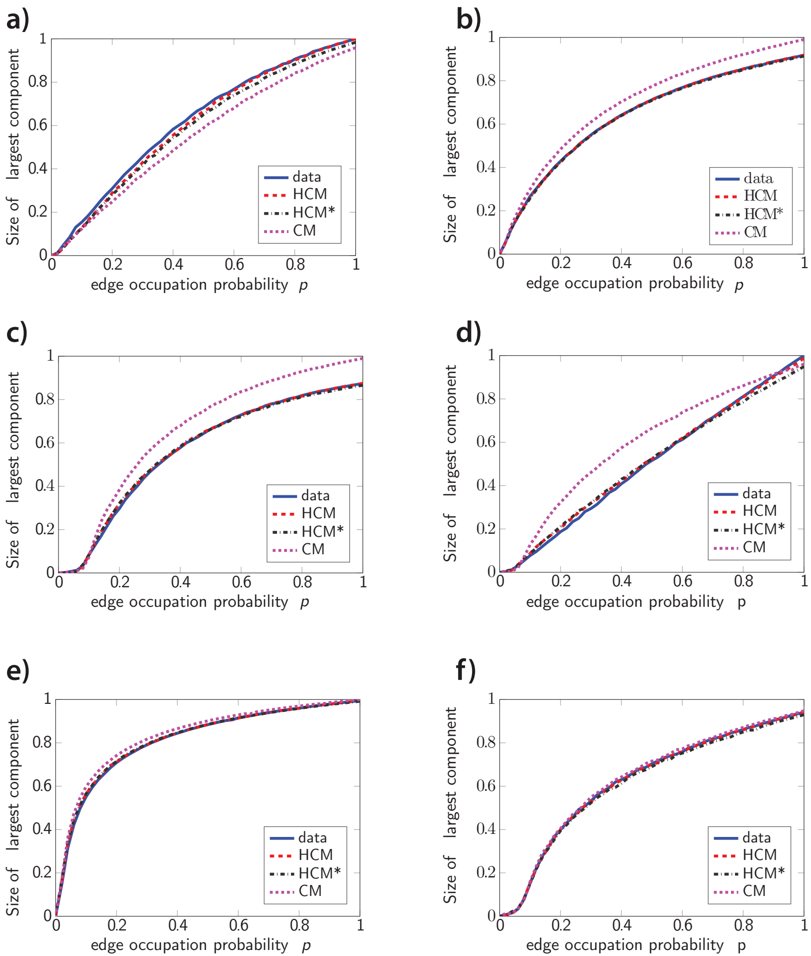
Result: Random graph with (roughly) specified communities.

Application: Many real-world networks on mesoscopic scale.
Stegehuis+vdH+vL16 Scientific Reports, Phys. Rev. E.



Network HCM HCM*

Percolation on HCM



Phase transition CM

Let $\mathcal{C}_{\max}$ denote largest connected component in $\mathrm{CM}_n(\boldsymbol{d})$.

Theorem 1. [Mol-Ree 95, Jan-Luc 07, Theorem II.3.4]. When Conditions I.7.5(a-b) hold,

$$\frac{1}{n}|\mathcal{C}_{\max}| \xrightarrow{\mathbb{P}} \zeta,$$

where $\zeta > 0$ precisely when $\nu > 1$ with $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D]$.

$\triangleright$ Note: $\zeta > 0$ always true when $\nu = \infty$: **Robustness!**

$\triangleright$ $d_{\min} = \min_{i\in[n]} d_i \geq 3 : \mathrm{CM}_n(\boldsymbol{d})$ with high probability connected. Wormald (81), Luczak (92).

$\triangleright$ $d_{\min} = \min_{i\in[n]} d_i \geq 2 : n - |\mathcal{C}_{\max}| \xrightarrow{d} X$ for non-trivial X. Luczak (92), Federico-vdH (17).

Phase transition for GRG

Let $\mathcal{C}_{\max}$ denote largest connected component in $\mathrm{GRG}_n(\boldsymbol{w})$.

Theorem 2. [Chu-Lu 03, Bol-Jan-Rio 07]. When Conditions I.6.3(a-b) hold, there exists $\zeta < 1$ such that

$$\frac{1}{n}|\mathcal{C}_{\max}| \xrightarrow{\mathbb{P}} \zeta,$$

where $\zeta > 0$ precisely when $\nu > 1$, where

$$\nu = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}.$$

$\triangleright$ Note: $\zeta > 0$ always true when $\nu = \infty$.

$\triangleright$ Bol-Jan-Rio 07 much more general.

Graph distances CM

H_n is graph distance between uniform pair of vertices in graph.

Theorem 3. [vdHHVM05, Theorem II.6.1]. When Conditions I.7.5(a-c) hold and $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] > 1$, conditionally on $H_n < \infty$,

$$\frac{H_n}{\log_\nu n} \xrightarrow{\mathbb{P}} 1.$$

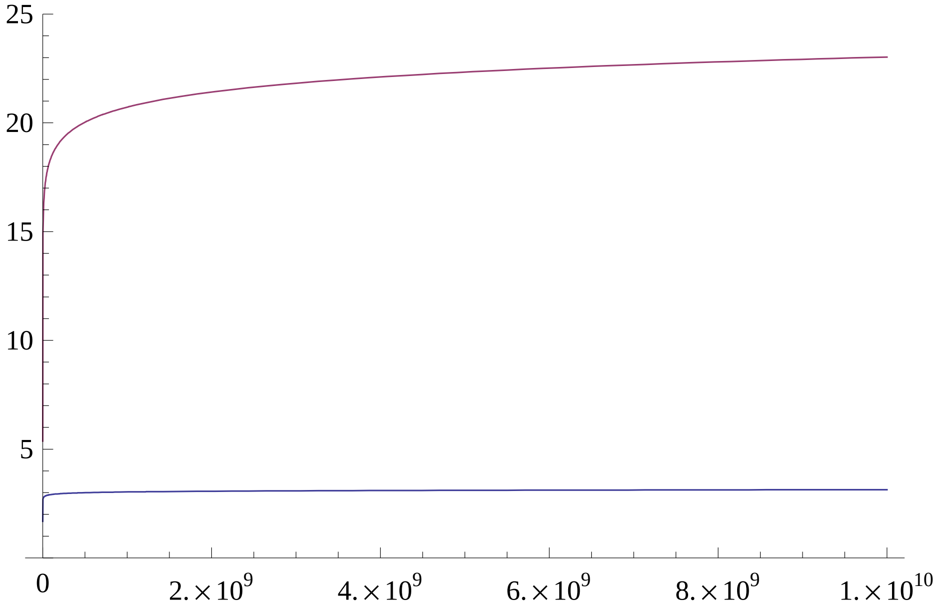
▷ For i.i.d. degrees having at most power-law tails, fluctuations are bounded.

Theorem 4. [vdHHZ07, Norros-Reittu 04, Theorem II.6.2]. Let Conditions I.7.5(a-b) hold. When $\tau \in (2,3)$, conditionally on $H_n < \infty$,

$$\frac{H_n}{\log\log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau-2)|}.$$

▷ vdH-Komjáthy16: For power-law tails, fluctuations are bounded and do not converge in distribution.

Six degrees of separation revisited



Plot of $x \mapsto \log x$ and $x \mapsto \log \log x$.

Diameter CM

Theorem 5. [Fernholz-Ramachandran 07, Theorem II.6.20]. Under Conditions I.7.5(a-b), there exists b s.t.

$$\frac{\mathrm{diam}(\mathrm{CM}_n(\boldsymbol{d}))}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{\log(\nu)} + 2b.$$

Here $b > 0$ precisely when $\mathbb{P}(D \leq 2) > 0$.

Theorem 6. [Caravenna-Garavaglia-vdH 17, Theorem II.6.21]. Under Conditions I.7.5(a-b), when $\tau \in (2,3)$ and $\mathbb{P}(D \geq 3) = 1$,

$$\frac{\mathrm{diam}(\mathrm{CM}_n(\boldsymbol{d}))}{\log \log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau-2)|} + \frac{2}{\log(d_{\min}-1)}.$$

Conclusion small-worlds

Many real-world networks share important features:

scale-free and small-world paradigms.

Often, suggestion of infinite-variance degrees.

Models invented to model/explain properties:

Configuration model, generalized random graph and preferential attachment.

Distances are remarkably similar across models.

Lecture 3:

Small worlds and Information diffusion on random graphs

Graph distances GRG

Theorem 7. [Chung-Lu 03, Bol-Jan-Rio 07, vdEvdHH08, Thm. II.5.2] When Conditions I.6.3(a-c) hold and $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] > 1$, conditionally on $H_n < \infty$,

$$\frac{H_n}{\log_\nu n} \xrightarrow{\mathbb{P}} 1.$$

Under somewhat stronger conditions, fluctuations are bounded.

Theorem 8. [Chung-Lu 03, Norros-Reittu 06, Theorem II.5.3]. When $\tau \in (2, 3)$, and Conditions I.6.3(a-b) hold, under certain further conditions on F_n, and conditionally on $H_n < \infty$,

$$\frac{H_n}{\log \log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau - 2)|}.$$

▷ Similar extensions for diameter as for CM (always logarithmic.) Again Bol-Jan-Rio 07 prove Theorem 7 in highly general setting.

Distances PA models

▷ Note that results CM and GRG are very alike, with CM having more general behavior (e.g., connectivity). Sign of the wished for universality.

> Non-rigorous physics literature predicts that scaling distances in preferential attachment models similar to the one in configuration model with equal
>
> power-law exponent degrees.

▷ In general, this question is still wide open, but certain indications are obtained.

▷ PAM tends to be much harder to analyze, due to

time dependence.

Distances PA models

Theorem 9 [Bol-Rio 04]. For all $m \geq 2$ and $\tau = 3$,

$$\mathrm{diam}(\mathrm{PA}_{m,0}(n)) = \frac{\log n}{\log\log n}(1 + o_{\mathbb{P}}(1)), \qquad H_n = \frac{\log n}{\log\log n}(1 + o_{\mathbb{P}}(1)).$$

Theorem 10 [Dommers-vdH-Hoo 10]. For all $m \geq 2$ and $\tau \in (3, \infty)$,

$$\mathrm{diam}(\mathrm{PA}_{m,\delta}(n)) = \Theta(\log n), \qquad H_n = \Theta(\log n).$$

Theorem 11 [Dommers-vdH-Hoo 10, Der-Mon-Mor 12, Car-Gar-vdH17]. For all $m \geq 2$ and $\tau \in (2, 3)$,

$$\frac{H_n}{\log\log n} \xrightarrow{\mathbb{P}} \frac{4}{|\log(\tau - 2)|}, \qquad \frac{\mathrm{diam}(\mathrm{PA}_{m,\delta}(n))}{\log\log n} \xrightarrow{\mathbb{P}} \frac{4}{|\log(\tau - 2)|} + \frac{2}{\log m}.$$

Conclusion small-worlds

Many real-world networks share important features:

scale-free and small-world paradigms.

Often, suggestion of infinite-variance degrees.

Models invented to model/explain properties:

Configuration model, generalized random graph and preferential attachment.

Distances are remarkably similar across models.

Graph distances CM

H_n is graph distance between uniform pair of vertices in graph.

Theorem 3. [vdHHVM05, Theorem II.6.1]. When Conditions I.7.5(a-c) hold and $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] > 1$, conditionally on $H_n < \infty$,

$$\frac{H_n}{\log_\nu n} \xrightarrow{\mathbb{P}} 1.$$

▷ For i.i.d. degrees having at most power-law tails, fluctuations are bounded.

Theorem 4. [vdHHZ07, Norros-Reittu 04, Theorem II.6.2]. Let Conditions I.7.5(a-b) hold. When $\tau \in (2,3)$, conditionally on $H_n < \infty$,

$$\frac{H_n}{\log\log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau-2)|}.$$

▷ vdH-Komjáthy16: For power-law tails, fluctuations are bounded and do not converge in distribution.

Proof CM: Neighborhoods

▷ Important ingredient in proof is description local neighborhood of uniform vertex $U_1 \in [n]$. Its degree has distribution $D_{U_1} \stackrel{d}{=} D$.

▷ Take any of D_{U_1} neighbors a of U_1. Law of number of forward neighbors of a, i.e., $B_a = D_a - 1$, is approximately

$$\mathbb{P}(B_a = k) \approx \frac{(k+1)}{\sum_{i\in[n]} d_i} \sum_{i\in[n]} \mathbb{1}_{\{d_i = k+1\}} \xrightarrow{\mathbb{P}} \frac{(k+1)}{\mathbb{E}[D]} \mathbb{P}(D = k+1).$$

Equals size-biased version of D minus 1. Denote this by $D^\star - 1$.

Local tree-structure CM

$\triangleright$ Forward neighbors of neighbors of U_1 are close to i.i.d. Also forward neighbors of forward neighbors have asymptotically same distribution...

$\triangleright$ Conclusion: Neighborhood looks like branching process with offspring distribution $D^\star - 1$ (except for root, which has offspring $D.$)

$\triangleright$ Tool to make this precise is

local weak convergence.

$\triangleright$ Here we will stick to informal explanation.

Structure local limit CM

$\triangleright$ $\mathbb{E}[D^2] < \infty$: Finite-mean BP, which has exponential growth of generation sizes:

$$\nu^{-k} Z_k \xrightarrow{a.s.} M \in (0, \infty),$$

on event of survival.

$\star$ Explains why distances random graph grow logarithmically.

$\triangleright$ $\tau \in (2, 3)$: Infinite-mean BP, which has double exponential growth of generation sizes:

$$(\tau - 2)^k \log(Z_k \vee 1) \xrightarrow{a.s.} Y \in (0, \infty),$$

on event of survival.

$\star$ Explains why distances grow doubly logarithmically.

Discussion small worlds

▷ Small worlds:
Results quantify small-world behavior random graphs. Random graphs are small worlds in general, ultra-small worlds when degrees have infinite variance.

▷ Many extensions to inhomogeneous random graphs, random intersection graphs, spatial scale-free graphs,...

Universality!

▷ Locally-tree like:
Random graphs studied here are locally tree-like. Much harder in general to move away from this.

Smallest-weight problems

▷ In many applications, edge weights represent cost structure graph, such as economic or congestion costs across edges.

▷ Time delay experienced by vertices in network is given by hop-count, which is number of edges on smallest-weight path.

How does weight structure influence
structure of smallest-weight paths?

▷ Assume that

edge weights are i.i.d. (continuous) random variables.

▷ Graph distances: weights = 1.

Choice of edge weights

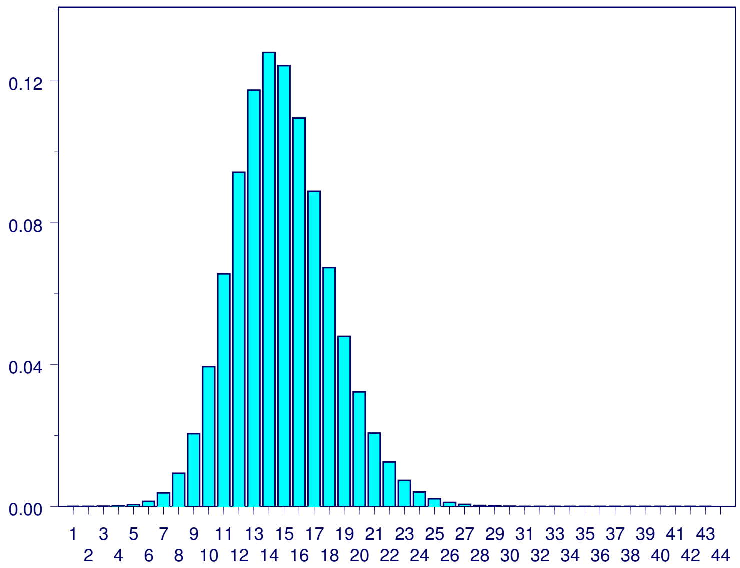
2000: CISCO recommended to use link weights that are proportional to inverse link capacity in Open Shortest Path First (OSPF). OSPF is interior routing protocol operating within a single autonomous system (AS).

CISCO recommendation: proportion $P_{\mathrm{Link}}[0, B]$ of link weights with value at most B is equal to proportion $P_{\mathrm{Cap}}[1/B, \infty)$ of links with capacity at least $1/B$.

$\triangleright$ Problem: No reliable data on empirical properties link capacities.

$\triangleright$ Solution: Use general continuous distribution link capacities. Thus, also edge weights have general continuous distribution.

Distances in IP graph



Poisson distribution??

Smallest-weight routing

▷ Smallest-weight routing problems fundamental for many related math and applied problems.

- Epidemic models;
- Rumor spread;
- Various randomized algorithms for communication (sensors);
- Competition processes,...

▷ One of the most basic information diffusion processes.

⋆ See also course of Ayalvadi Ganesh.

Setting

Graph denoted by $G = (V(G), E(G))$ with $|V(G)| = n$.
This talk: G configuration model. Complete graph A. Ganesh.

▷ Central objects of study: $\mathcal{C}_n$ is weight of smallest-weight path two uniform connected vertices:

$$\mathcal{C}_n = \min_{\pi\colon V_1 \to V_2} \sum_{e \in \pi} Y_e,$$

where π is path in G, while $(Y_e)_{e \in E(G)}$ are i.i.d. collection of weights with continuous law.

▷ Continuous weights: Optimal path π_n^* is a.s. unique. Then

$$H_n = |\pi_n^*|$$

denotes hopcount, i.e., number of edges in optimal path.

▷ Complete graph investigated in combinatorics (e.g., Janson 99) and theoretical physics (Havlin, Braunstein, Stanley, et al.).

Routing on sparse RGs

Theorem 12. [Bhamidi-vdH-Hooghiemstra AoP 17]. Let $\mathrm{CM}_n(\boldsymbol{d})$ satisfy Condition 7.5 (a-b), and

$$\lim_{n\to\infty} \mathbb{E}[D_n^2 \log(D_n \vee 1)] = \mathbb{E}[D^2 \log(D \vee 1)].$$

Let weights be i.i.d. with general continuous distribution. Then, there exist $\alpha_n, \alpha, \beta, \gamma_n, \gamma > 0$ with $\alpha_n \to \alpha, \gamma_n \to \gamma$ s.t.

$$\frac{H_n - \alpha_n \log n}{\sqrt{\beta \log n}} \xrightarrow{d} Z, \qquad \mathcal{C}_n - \gamma_n \log n \xrightarrow{d} \mathcal{C}_\infty,$$

where Z is standard normal, $\mathcal{C}_\infty$ is some limiting random variable.

Universality!

Role hubs

Theorem 13. [Bhamidi-vdH-Hooghiemstra AoAP10]. Let degrees in $\mathrm{CM}_n(\boldsymbol{d})$ be i.i.d. with $\mathbb{P}(D \geq 2) = 1$ and power-law distribution with $\tau \in (2, 3)$. Let weights be i.i.d. exponential r.v.'s. Then

$$\frac{H_n - \alpha \log n}{\sqrt{\alpha \log n}} \xrightarrow{d} Z, \qquad \mathcal{C}_n \xrightarrow{d} \mathcal{C}_\infty,$$

for some limiting random variable $\mathcal{C}_\infty$, where Z is standard normal and $\alpha = 2(\tau - 2)/(\tau - 1) \in (0, 1)$.

▷ Hopcount not order $\log n$: Weights $(1 + E_e)_{e \in \mathcal{E}_n}$, where E_e i.i.d. exponential, and $\tau \in (2, 3)$ [Baroni, vdH, Komjáthy (19)]

$$W_n, H_n = 2 \log\log n / |\log(\tau - 2)| \text{ plus tight.}$$

Weights $(1 + X_e)_{e \in \mathcal{E}_n}$: know exactly when above tightness holds

▷ Baroni, vdH, Komjáthy (17): $\mathcal{C}_n \xrightarrow{d} \mathcal{C}_\infty$ in explosive CTBP case.

Check work of Komjáthy for various extensions.

Discussion

▷ Random weights have marked effect on optimal flow problem.

Surprisingly universal behavior for FPP on CM.
Even limiting random variables display large amount of universality.

▷ Universality is leading paradigm in statistical physics.
Few examples where universality can be made rigorous.

▷ Proofs rely on coupling to continuous-time branching processes. arising as FPP on the local weak limit.

Proofs

Adding weights to branching process gives rise to

age-dependent branching process.

Is particular type of continuous-time branching process.

$\triangleright$ Let Z_t be number of alive individuals.

$\triangleright \mathbb{E}[D^2] < \infty$: CTBP is Malthusian: $\mathrm{e}^{-\alpha t} Z_t \xrightarrow{a.s.} W$ for some $W > 0$;

$$\mathcal{C}_n \approx \log n / \alpha ...$$

$\triangleright \tau \in (2,3)$: CTBP can be explosive: $Z_t = \infty$ for some $t > 0$.
True for most weights...

$$\mathcal{C}_\infty = T_1 + T_2,$$

the sum of two i.i.d. explosion times.

Winner takes it all!

FPP serves as a tool in many models:

Theorem 14. [Deijfen-vdH AoAP (2016)]
Consider competition model, where species compete for territory at unequal rates. For $\tau \in (2,3)$, under conditions Theorem 13, each of species wins majority vertices with positive probability. Number of vertices for losing species converges in distribution.

$\triangleright$ Antunovic, Dekel, Mossel, and Peres (2011): First passage percolation as competition model on random regular graphs.

$\triangleright$ Baroni, vdH, Komjáthy (2015): Extension to deterministic unequal weights

$\triangleright$ Komjáthy (2016): Deterministic equal weights: coexistence.

$\triangleright$ Alberg, Deijfen, Janson (2017): Extension to $\mathbb{E}[D^2] < \in$ and exponential edge weights.

Conclusion routing on CM

▷ Many results on FPP on random graphs:

Results show high amount of universality when degrees finite-variance. Unclear what universality classes are for infinite-variance degrees.

▷ Difficulty:

hubs $\mathrm{CM}_n(\boldsymbol{d})$ highly dominant when $\tau \in (2,3)$.

▷ What are fluctuations diameter FPP?

Extension Theorem 6 Amini, Draief, Lelarge (2011)

▷ Infinite variance degrees?

▷ General edge weights?

Literature CM

[1] Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *Eur. J. Combin.* **1**(4): 311–316, (1980).

[2] van der Hofstad, Hooghiemstra and Van Mieghem. Distances in random graphs with finite variance degrees. *RSA* **27**(1): 1042–9832, (2005).

[3] Fernholz and Ramachandran. The diameter of sparse random graphs. *RSA* **31**(4): 482–516, (2007).

[4] van der Hofstad, Hooghiemstra and Znamenskyi. Distances in random graphs with finite mean and infinite variance degrees. *Elec. J. Prob.* **12**(25): 703–766, (2007).

[5] Molloy and Reed. A critical point for random graphs with a given degree sequence. *RSA* **6**: 161–179, (1995).

[6] Newman, Strogatz and Watts. Random graphs with arbitrary degree distribution and their application. *Phys. Rev. E* **64**: 026118, (2000).

Literature IRG

[1] Bollobás, Janson and Riordan. The phase transition in inhomogeneous random graphs. *RSA* **31**(1): 3–122, (2007).

[2] Britton, Deijfen and Martin-Löf. Generating simple random graphs with prescribed degree distribution. *JSP* **124**(6): 1377–1397, (2006).

[3] Chung and Lu. The average distance in a random graph with given expected degrees. *Internet Math.* **1**(1): 91–113, (2003).

[4] Chung and Lu. Complex graphs and networks. CBMS Regional Conference Series in Mathematics. (2006).

[5] van den Esker, van der Hofstad and Hooghiemstra. Universality for the distance in finite variance random graphs. *J. Stat. Phys.* **133**(1): 169–202, (2006).

[6] Norros and Reittu. On a conditionally Poissonian graph process. *Adv. Appl. Probab.* **38**(1): 59–75, (2006).

Literature PAM

[1] Barabási and Albert. Emergence of scaling in random networks. *Science* **286**(2): 509–512, (1999).

[2] Bollobás and Riordan. The diameter of a scale-free random graph. *Combinatorica* **1**(2): 215–225, (2004).

[3] Bollobás, Riordan, Spencer and Tusnády. The degree sequence of a scale-free random graph process. *RSA* **18**(3): 279–290, (2001).

[4] Deijfen, van den Esker, van der Hofstad and Hooghiemstra. A preferential attachment model with random initial degrees. *Ark. Mat.* **47**(1): 41–72, (2009).

[5] Dereich, Mönch and Mörters. Typical distances in ultrasmall random networks. *Journ. Stat. Phys.* **139**(1): 72–107, (2010).

[6] Dommers, van der Hofstad and Hooghiemstra. Diameters in preferential attachment graphs. *Adv. Appl. Probab.* **44**(2): 583–601, (2012).

Literature FPP on graphs

[1] Bhamidi, Hooghiemstra and van der Hofstad. First passage percolation on random graphs with finite mean degrees. *AAP* **20**(5): 1907–1965, (2010).

[2] Bhamidi, Hooghiemstra and van der Hofstad. First passage percolation on sparse random graphs. *AoP* **45**(4): 2568–2630, (2017).

[3] Baroni, van der Hofstad and Komjáthy. Nonuniversality of weighted random graphs with infinite variance degree. *Journal of Applied Probability,* **54**(1): 146–164, (2017).

[4] Deijfen and van der Hofstad. The winner takes it all. *AAP,* **26**(4): 2419–2453, (2016).

[5] Janson. One, two and three times $\log n/n$ for paths in a complete graph with random weights. *CPC* **8**(04):347–361, (1999).