Strict monotonicity of principal eigenvalues of elliptic operators in $\mathbb{R}^d$ and risk-sensitive control

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Definitions and Motivation

- Consider the second order elliptic operator $\mathcal{L}^f$ of the form

$$\mathcal{L}^f \varphi \;=\; \sum_{i,j=1}^{d} a^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial \varphi}{\partial x_i} + f\,\varphi\,. \tag{0.1}$$

- When D is a smooth bounded domain, and a, b, f are regular enough, existence of a principal eigenvalue and corresponding eigenfunction under a Dirichlet boundary condition can be obtained by an application of Krein-Rutman theory. This eigenvalue is the bottom of the spectrum of $-\mathcal{L}^f$ with Dirichlet boundary condition.

- For unbounded domains, principal eigenvalue problems have been recently considered by Berestycki and Rossi.

• Not surprisingly, certain properties of the principal eigenvalue which hold in bounded domains may not be true for unbounded ones. For instance, when D is smooth and bounded it is well known that for the Dirichlet boundary value problem, the principal eigenvalue is simple, and the associated principal eigenfunction is positive. Moreover, it is the unique eigenvalue with a positive eigenfunction.

• But if D is unbounded and smooth, then there exists a constant $\lambda^* = \lambda^*(f)$ such that any $\lambda \in [\lambda^*, \infty)$ is an eigenvalue of $\mathcal{L}^f$ with a positive eigenfunction. The lowest such value λ^* serves as a definition of the principal eigenvalue when D is not bounded.

- The principal eigenvalue is known to be strictly monotone as a function of the bounded domain D (the latter ordered with respect to set inclusion), and also strictly monotone in the coefficient f when the domain is bounded.

- These properties fail to hold in unbounded domains. Strict monotonicity of $f \mapsto \lambda^*(f)$ and its implications are a central theme in today's talk.

Assumptions

(A1) *Local Lipschitz continuity:* The function $\sigma \;=\; \left[\sigma^{ij}\right] \,:\, \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is locally Lipschitz in x with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, with $\|\sigma\| \doteq \sqrt{\operatorname{trace}(\sigma\sigma^{\mathsf{T}})}$, we have

$$\|\sigma(x) - \sigma(y)\| \;\le\; C_R\,|x-y| \qquad \forall\, x, y \in B_R\,.$$

We also assume that $b \;=\; \left[b^1, \dots, b^d\right]^{\mathsf{T}} \,:, \mathbb{R}^d \to \mathbb{R}^d$ is locally bounded and measurable.

(A2) *Affine growth condition:* b and σ satisfy a global growth condition of the form

$$\langle b(x), x\rangle^{+} + \|\sigma(x)\|^2 \;\le\; C_0\bigl(1+|x|^2\bigr) \qquad \forall\, x \in \mathbb{R}^d,$$

for some constant $C_0 > 0$.

(A3) *Nondegeneracy:* For each $R > 0$, it holds that

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \;\geq\; C_R^{-1}|\xi|^2 \qquad \forall\, x \in B_R\,,$$

and for all $\xi = (\xi_1, \dots, \xi_d)^{\mathsf{T}} \in \mathbb{R}^d$, where, $a = \frac{1}{2}\sigma\sigma^{\mathsf{T}}$.

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$ be a given filtered probability space with a complete, right continuous filtration $\{\mathfrak{F}_t\}$. Let W be a standard Brownian motion adapted to $\{\mathfrak{F}_t\}$. Consider the stochastic differential equation

$$X_t \;=\; X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s\,. \tag{0.2}$$

It is well known that under (A1)–(A3), there exists a unique solution of (0.2).

• Suppose that $\tau(D)$ denotes the first exit time of the process X from a domain D. The process X is said be *recurrent* if for any bounded domain D we have $\mathbb{P}_x(\tau(D^c) < \infty) = 1$ for all $x \in \bar{D}^c$. Otherwise the process is called transient. A recurrent process is said to be *positive recurrent* if $\mathbb{E}_x[\tau(D^c)] < \infty$ for all $x \in \bar{D}^c$. It is known that for a non-degenerate diffusion the property of recurrence (or positive recurrence) is independent of D and x, i.e., if it holds for some domain D and $x \in \bar{D}^c$, then it also holds for every domain D, and all points $x \in \bar{D}^c$.

The generator associated to (0.2) is given by $\mathcal{L} : C^2(\mathbb{R}^d) \to L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$

$$\mathcal{L}g(x) \doteq a^{ij}(x)\,\partial_{ij}g(x) + b^i(x)\,\partial_i g(x)\,.$$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally bounded, Borel measurable function, which is bounded from below in $\mathbb{R}^d$, i.e., $\inf_{\mathbb{R}^d} f > -\infty$. We refer to a function f with these properties as a *potential*, and $\mathcal{L}^f = \mathcal{L} + f$.

• By an *eigenpair* of $\mathcal{L}^f$ we mean a pair (Ψ, λ), with Ψ a positive function in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, for all $p \in [1, \infty)$, and $\lambda \in \mathbb{R}$, that satisfies

$$\mathcal{L}^f \Psi = a^{ij}\partial_{ij}\Psi + b^i\partial_i\Psi + f\,\Psi = \lambda\Psi\,. \tag{0.3}$$

We refer to λ as the *eigenvalue*, and to Ψ as the *eigenfunction*.

Given an eigenpair (Ψ, λ), the associated *twisted diffusion* Y is an Itô process as in (0.2), but with the drift b replaced by $b + 2a\nabla(\log\Psi)$. When $\lambda = \lambda^*(f)$, the eigenfunction is denoted as Ψ^* and is called the *ground state*. The corresponding twisted diffusion, denoted by Y^*, is referred to as the *ground-state diffusion*.

Lemma

For each $r \in (0, \infty)$ there exists a unique pair $(\widehat{\Psi}_r, \hat{\lambda}_r) \in \bigl(\mathcal{W}^{2,p}_{\mathrm{loc}}(B_r) \cap C(\bar{B}_r)\bigr) \times \mathbb{R}$, for any $p \in [1, \infty)$, satisfying $\widehat{\Psi}_r > 0$ on B_r, $\widehat{\Psi}_r = 0$ on ∂B_r, and $\widehat{\Psi}_r(0) = 1$, which solves

$$\mathcal{L}\widehat{\Psi}_r(x) + f(x)\,\widehat{\Psi}_r(x) \;=\; \hat{\lambda}_r\,\widehat{\Psi}_r(x) \qquad a.e.\ x \in B_r\,. \tag{0.4}$$

Moreover, $\hat{\lambda}_r$ has the following properties:

(a) *The map $r \mapsto \hat{\lambda}_r$ is continuous and strictly increasing.*

(b) *In its dependence on the function f, $\hat{\lambda}_r$ is nondecreasing, convex, and Lipschitz continuous (with respect to the L^∞ norm), with Lipschitz constant 1. In addition, if $f \lneqq f'$, then $\hat{\lambda}_r(f) < \hat{\lambda}_r(f')$.*

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Definition

Let f be a potential. The principal eigenvalue $\lambda^*(f)$ on $\mathbb{R}^d$ of the operator $\mathcal{L}^f$ is defined as $\lambda^*(f) \doteq \lim_{r\to\infty} \hat{\lambda}_r(f)$.

The following definition of the principal eigenvalue is commonly used in the pde literature.

Definition

$$\hat{\Lambda}(f) \;=\; \inf\Big\{\lambda \in \mathbb{R} \;:\; \exists\, \varphi \in \mathcal{W}^{2,d}(\mathbb{R}^d),\, \varphi > 0, \\ \mathcal{L}\varphi + (f-\lambda)\varphi \le 0, \text{ a.e. in } \mathbb{R}^d\Big\}.$$

Lemma

The following hold

(i) *For any $r > 0$, the Dirichlet eigensolutions $(\widehat{\Psi}_n, \hat{\lambda}_n)$ have the following stochastic representation*

$$\widehat{\Psi}_n(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}_r}[f(X_t)-\hat{\lambda}_n]\,dt}\,\widehat{\Psi}_n(X_{\breve{\tau}_r})\,1_{\{\breve{\tau}_r<\tau_n\}}\Bigr] \qquad \forall\, x \in B_n \setminus \overline{B}_r\,,$$

for all large enough $n \in \mathbb{N}$.

(ii) *It holds that $\lambda^*(f) = \hat{\Lambda}(f)$.*

Strict Monotonicity

Let $C_{\mathrm{o}}^{+}(\mathbb{R}^d)$ denote the collection of all non-trivial, nonnegative, continuous functions which vanish at infinity. We consider the following two properties of $\lambda^*(f)$.

(P1) Strict monotonicity at f. For some $h \in C_{\mathrm{o}}^{+}(\mathbb{R}^d)$ we have $\lambda^*(f-h) < \lambda^*(f)$.

(P2) Strict monotonicity at f on the right. For all $h \in C_{\mathrm{o}}^{+}(\mathbb{R}^d)$ we have $\lambda^*(f) < \lambda^*(f+h)$.

It follows by the convexity of $f \mapsto \lambda^*(f)$ that (P1) implies (P2).

Recall that the twisted process corresponding to an eigenpair (Ψ, λ) of $\mathcal{L}^f$ is defined by the SDE

$$dY_s \;=\; b(Y_s)ds + 2a(Y_s)\nabla\psi(Y_s)\,ds + \sigma(Y_s)\,dW_s\,,$$

where $\psi = \log(\Psi)$. A twisted process corresponding to a principal eigenpair is called a *ground state process*, and the eigenfunction Ψ^* is called a *ground state*.

Definition (exponential ergodicity)

The process X governed by (0.2) is said to be exponentially ergodic if for some compact set $\mathcal{B}$ and $\delta > 0$ we have $\mathbb{E}_x\bigl[\mathrm{e}^{\delta\,\tau(\mathcal{B}^c)}\bigr] < \infty$, for all $x \in \mathcal{B}^c$.

Theorem (Arapostathis, Biswas, S)

The following hold.

(a) *A ground state process is recurrent if and only if $\lambda^*(f)$ is strictly monotone at f on the right, in which case the principal eigenvalue $\lambda^*(f)$ is also simple, and the ground state Ψ^* satisfies*

$$\Psi^*(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}_r}[f(X_s)-\lambda^*(f)]\,ds}\,\Psi^*(X_{\breve{\tau}})\,1_{\{\breve{\tau}_r<\infty\}}\Bigr]$$
$$\forall\, x\in\bar{B}_r^c\,,\quad \forall\, r>0\,.$$

(b) *The ground state process is exponentially ergodic if and only if $\lambda^*(f)$ is strictly monotone at f.*

(c) *If $\lambda>\lambda^*(f)$, the twisted process is transient.*

We define the Green's measure G_λ, $\lambda \in \mathbb{R}$, by

$$G_\lambda(g) \doteq \mathbb{E}_0\left[\int_0^\infty \mathrm{e}^{\int_0^t [f(X_s)-\lambda]\, ds}\, g(X_t)\, dt\right] \quad \text{for all } g \in C_{\mathrm{c}}^{+}(\mathbb{R}^d)\,.$$

The density of the Green's measure with respect to the Lebesgue measure is called the Green's function. Existence of a Green's function (and Green's measure) is used by [Pinsky, 1995, Chapter 4.3] in his definition of the generalized principal eigenvalue of $\mathcal{L}^f$. A number $\lambda \in \mathbb{R}$ is said to be *subcritical* if G_λ possesses a density, *critical* if it is not subcritical and $\mathcal{L}^{f-\lambda} V = 0$ has a positive solution V, and *supercritical* if it is neither subcritical nor critical.

The following lemma generalizes a result of Pinsky, where, under a regularity assumption on the coefficients, it is shown that a critical eigenvalue λ is always simple.

Lemma

The following are equivalent.

(i) *The twisted process Y corresponding to the eigenpair (Ψ, λ) is recurrent.*

(ii) *$G_\lambda(g)$ is infinite for some $g \in C_c^+(\mathbb{R}^d)$.*

(iii) *For some open ball $\mathcal{B}$, and with $\breve{\tau} = \breve{\tau}(\mathcal{B})$, we have*

$$\Psi(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve{\tau}}[f(X_s)-\lambda]\,ds}\,\Psi(X_{\breve{\tau}})\,1_{\{\breve{\tau}<\infty\}}\Big]\,, \quad x \in \bar{\mathcal{B}}^c\,,$$

where Ψ is an eigenfunction corresponding to the eigenvalue λ.

In addition, in (ii)–(iii) "some" may be replaced by "all", and if any one of (i)–(iii) holds, then λ is a simple.

Sufficient Conditions for Strict Monotonicity

For a potential f we define

$$\mathcal{E}_x(f) \doteq \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x\left[\mathrm{e}^{\int_0^T f(X_s)\,ds}\right], \quad \text{and} \quad \mathcal{E}(f) \doteq \inf_{x\in\mathbb{R}^d} \mathcal{E}_x(f).$$

We refer to $\mathcal{E}(f)$ as the *risk-sensitive* average of f.
We introduce the following hypothesis.

(H1) There exists a lower-semicontinuous, inf-compact function $\ell : \mathbb{R}^d \to [0,\infty)$ such that $\mathcal{E}(\ell) < \infty$.

Theorem

Assume (H1), *and suppose that* f *is a potential such that* $\ell - f$ *is inf-compact. Then for any continuous* $h \in C_{\mathrm{o}}^{+}(\mathbb{R}^d)$ *we have*

$$\lambda^*(f - h) \;<\; \lambda^*(f) \;=\; \mathcal{E}_x(f) \quad \forall\, x \in \mathbb{R}^d\,.$$

Theorem

Let $\mathcal{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ *such that* $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, *satisfying*

$$\mathcal{L}\mathcal{V} \;\le\; \kappa_0 1_{\mathcal{K}} - \gamma\mathcal{V} \quad \text{on } \mathbb{R}^d\,,$$

for some compact set $\mathcal{K}$ *and positive constants* κ_0 *and* γ. *Let* f *be a nonnegative bounded measurable function with* $\limsup_{x\to\infty} f(x) < \gamma$. *Then for any* $h \in C_{\mathrm{o}}^{+}(\mathbb{R}^d)$, *we have* $\lambda^*(f - h) < \lambda^*(f) = \mathcal{E}_x(f)$ *for all* $x \in \mathbb{R}^d$.

Risk-sensitive control

Consider the controlled diffusion on $\mathbb{R}^d$:

$$dX_t \;=\; b(X_t, U_t)\,dt + \sigma(X_t)\,dW_t\,.$$

The control process U takes values in a compact, metrizable set $\mathbb{U}$. The set $\mathfrak{U}$ of *admissible controls* consists of the control processes U that are *non-anticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \;\doteq\; \text{the completion of } \cap_{y>s}\sigma\{X_0, U_r, W_r,\; r \le y\} \text{ relative to } (\mathfrak{F}, \mathbb{P})\,.$$

Let $\mathfrak{C}$ denote the class of functions $c(x,u)$ in $C(\mathbb{R}^d \times \mathbb{U}, \mathbb{R}_+)$ that are locally Lipschitz in x uniformly with respect to $u \in \mathbb{U}$. Let $c \in \mathfrak{C}$ denote the *running cost* function.

For any admissible control $U \in \mathfrak{U}$, define the risk-sensitive objective function $\mathcal{E}_x^U(c)$ by

$$\mathcal{E}_x^U(c) \doteq \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x\left[\mathrm{e}^{\int_0^T c(X_s,U_s)\,ds}\right].$$

We also define $\Lambda_x^* \doteq \inf_{U\in\mathfrak{U}} \mathcal{E}_x^U(c)$.

A stationary Markov control is a measurable map from $\mathbb{R}^d$ to $\mathbb{U}$. Let $\mathfrak{U}_{\mathrm{SM}}$ denote the class of all such stationary Markov controls. We say that a stationary Markov control $v \in \mathfrak{U}_{\mathrm{SM}}$ is optimal (for the risk-sensitive criterion) if $\mathcal{E}_x^v(c_v) = \Lambda_x^*$ for all $x \in \mathbb{R}^d$, and we let $\mathfrak{U}_{\mathrm{SM}}^*$ denote the class of these controls.

For any admissible control $U \in \mathfrak{U}$, define the risk-sensitive objective function $\mathcal{E}_x^U(c)$ by

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Assumption 1[Uniform exponential ergodicity] There exists an inf-compact function $\ell \in C(\mathbb{R}^d)$ and a positive function $\mathcal{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, satisfying $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, such that

$$\sup_{u \in \mathbb{U}} \mathcal{L}_u \mathcal{V} \;\le\; \bar{\kappa}\, 1_{\mathcal{K}} - \ell \mathcal{V} \quad \text{a.e. on } \mathbb{R}^d\,,$$

for some constant $\bar{\kappa}$, and a compact set $\mathcal{K}$.

Example

Let σ be bounded and $b : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d$ be such that

$$\langle b(x,u) - b(0,u), x \rangle \;\le\; -\kappa\, |x|^{\alpha}, \quad \text{for some } \alpha \in (1,2] \quad (x,u) \in \mathbb{R}^d \times \mathbb{U}\,.$$

Then $\mathcal{V}(x) = \exp(\theta\, |x|^{\alpha})$, for $|x| \ge 1$, satisfies the above assumption for sufficiently small $\theta > 0$, and $\ell(x) \sim |x|^{2\alpha - 2}$.

We introduce the class of running costs $\mathcal{C}_\ell$ defined by

$$\mathcal{C}_\ell \doteq \Bigl\{c \in \mathfrak{C} : \ \ell(\cdot) - \max_{u\in\mathbb{U}} c(\cdot, u) \text{ is inf-compact}\Bigr\}\,.$$

Theorem (Arapostathis, Biswas, S.)

Suppose the assumption holds, and $c \in \mathcal{C}_\ell$. Then $\Lambda^* = \Lambda^*_x$ does not depend on x, and there exists a positive solution $V \in C^2(\mathbb{R}^d)$ satisfying

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}_u V + c(\cdot, u)V\bigr] \;=\; \Lambda^* V \quad \textit{on } \mathbb{R}^d\,, \quad \textit{and } V(0) = 1\,. \qquad (0.5)$$

In addition, if $\overline{\mathfrak{U}}_{\mathrm{SM}} \subset \mathfrak{U}_{\mathrm{SM}}$ denotes the class of Markov controls v which satisfy

$$\mathcal{L}_v V + c_v V \;=\; \min_{u\in\mathbb{U}} \bigl[\mathcal{L}_u V + c(\cdot, u)V\bigr] \qquad \textit{a.e. in } \mathbb{R}^d\,,$$

then the following hold.

Theorem Contd.

(a) $\overline{\mathfrak{U}}_{\mathrm{SM}} \subset \mathfrak{U}^*_{\mathrm{SM}}$, and it holds that $\lambda^*_v(c_v) = \Lambda^*$ for all $v \in \overline{\mathfrak{U}}_{\mathrm{SM}}$;

(b) $\mathfrak{U}^*_{\mathrm{SM}} \subset \overline{\mathfrak{U}}_{\mathrm{SM}}$;

(c) (0.5) has a unique positive solution in $C^2(\mathbb{R}^d)$ (up to a multiplicative constant).

The existence of an inf-compact ℓ as in **Assumption 1** is not possible when a and b are bounded. So we consider the following alternative assumption.

Assumption 2 There exists a function $\mathcal{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, such that $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, a compact set $\mathcal{K}$, and positive constants κ_0 and γ, satisfying

$$\max_{u\in\mathbb{U}} \mathcal{L}_u\mathcal{V}(x) \;\le\; \kappa_0 1_{\mathcal{K}}(x) - \gamma\mathcal{V}(x), \quad x \in \mathbb{R}^d,$$

$$\limsup_{|x|\to\infty} \sup_{u\in\mathbb{U}} c(x,u) \;<\; \gamma .$$

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$$\max_{u \in \mathbb{U}} \mathcal{L}_u \mathcal{V}(x) \;\le\; \kappa_0 1_{\mathcal{K}}(x) - \gamma \mathcal{V}(x), \quad x \in \mathbb{R}^d\,,$$

$$\limsup_{|x| \to \infty} \sup_{u \in \mathbb{U}} c(x,u) \;<\; \gamma\,.$$

Theorem

Under **Assumption 2**, *there exists a positive solution* $V \in C^2(\mathbb{R}^d)$ *satisfying*

$$\min_{u \in \mathbb{U}} \left[\mathcal{L}_u V + c(\cdot, u) V\right] = \Lambda^* V. \tag{0.6}$$

Let $\overline{\mathfrak{U}}_{\mathrm{SM}} \subset \mathfrak{U}_{\mathrm{SM}}$ *be as in previous Theorem. Then (a) and (b) of previous Theorem hold, and (0.6) has a unique positive solution in* $C^2(\mathbb{R}^d)$ *up to a multiplicative constant.*

Reference

Strict Monotonicity of Principal Eigenvalues of Elliptic Operators on $\mathbb{R}^d$ and Risk-Sensitve Control (with A. Arapostathis and A. Biswas), Journal de Mathmatiques Pures et Appliques, vol. 124, 2019, 169-219.

Thank You