Continued fractions, the Chen-Stein method and extreme value theory

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$\dagger$ supported by IFCPAR Grant

Regular Continued Fractions:

A Crash Course

Regular continued fractions

$$\frac{7}{24}$$

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Therefore by Euclidean Algorithm (of computing gcd), any rational number

$$\omega = p/q \in (0, 1)$$

(with $gcd(p, q) = 1$) will have a *terminating (regular) continued fraction expansion*.

Conversely . . .

Whenever $A_1, A_2, A_3, A_4 \in \mathbb{N}$,

$$[A_1, A_2, A_3, A_4] := \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \cfrac{1}{A_4}}}} \in (0, 1)$$

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More generally, by induction on n,

$$\omega = [A_1, A_2, \ldots A_n]$$

(with $A_1, A_2, \ldots A_n \in \mathbb{N}$) is a rational number in $(0, 1)$.

Non-terminating continued fraction expansion

Theorem

A number $\omega \in (0,1)$ has a unique non-terminating continued fraction expansion

$$\omega = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \cdots}}} =: [A_1, A_2, A_3, \ldots]$$

(with each $A_i \in \mathbb{N}$) if and only if $\omega \notin \mathbb{Q}$.

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Examples: $\pi \approx \frac{22}{7}$ and $\pi \approx \frac{355}{113}$.

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See, for example, Khintchine (1964).

Gauss Dynamical System

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Quick Observation: T, A_1 measurable $\Rightarrow$ each A_n measurable.

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Extremes of Continued Fractions

and the

Melancholic Life of Wolfgang Doeblin

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- Since then: Philipp (1976), Samur (1989), Nakada and Natsui (2003), Pollicott (2009), Tyran-Kamińska (2010), Bazarova, Berkes and Horváth (2016), Chang and Ma (2017).

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- See also Wolfgang Doeblin: A mathematician rediscovered.

Doeblin-Iosifescu Asymptotics

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A reformulation of Gauss's theorem

Exercise (in *Probability Theory II*): Suppose X is a random variable having probability density function

$$f_X(x) = \frac{1}{(1+x)\log 2}, \quad x \in (0,1).$$

Then show that $\{1/X\} \stackrel{\mathcal{L}}{=} X$.

A stationary process

Take $\Omega = (0, 1)$, $\mathcal{A} = \mathcal{B}_{(0,1)}$, $P(dx) = ((1 + x) \log 2)^{-1} \, dx$.

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$A_n = n^{th}$ digit in the regular continued fraction expansion of a random number $\omega \in (0,1)$ chosen according to the law P.

T preserves $P \Rightarrow \{A_n\}$ is a strictly stationary process. In particular, $A_1, A_2, A_3, \ldots$ are identically distributed.

Two easy observations

- **Direct Computation:** For all $m \in \mathbb{N}$,

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$$P\left(\frac{A_1 \log 2}{n} > u\right) = P\left(A_1 \geq \left\lceil \frac{un}{\log 2} \right\rceil\right) \sim \frac{1}{un}$$

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as $n \to \infty$. In particular,

$$nP\left(\frac{A_1 \log 2}{n} > u\right) \to u^{-1}$$

(A_1 is regularly varying with index 1).

If $A_1, A_2, A_3, \ldots$ were independent

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$$\mathbb{1}_{(A_1 \log 2 > un)},\ \mathbb{1}_{(A_2 \log 2 > un)},\ \mathbb{1}_{(A_3 \log 2 > un)},\ \ldots \stackrel{iid}{\sim} Ber(p_n),$$

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Let $M_n^{(1)} := \max\{A_i \log 2 : 1 \leq 1 \leq n\}$, $n \in \mathbb{N}$. *Then for all* $u > 0$,

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Our Contribution

The main question

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Question

What is the rate of convergence in (DI)?

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- Rate of convergence of the scaled maxima for the geodesic flow on the modular surface.

The main result

Theorem (Ghosh, Kirsebom, R. (2019))

There exists $\kappa > 0$ *and a sequence* $1 \ll \ell_n \ll \log n$ *such that for all* $u > 0$ *and for all* $n \in \mathbb{N}$,

$$d_{TV}(\mathcal{E}_n^u, \mathcal{E}_\infty^u) := \sup_{A \subseteq \mathbb{N} \cup \{0\}} \left| P(\mathcal{E}_n^u \in A) - P(\mathcal{E}_\infty^u \in A) \right| \leq \frac{\kappa}{\min\{u, u^2\}} \frac{\ell_n}{n}.$$

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Corollary

Suppose $M_n^{(k)} := k^{th}$ *maximum of* $\{A_i \log 2 : 1 \leq i \leq n\}$. *For all* $u > 0$ *and for all* $n \in \mathbb{N}$,

$$\sup_{k \in \mathbb{N}} \left| P\left(\frac{M_n^{(k)}}{n} \leq u\right) - e^{-u^{-1}} \sum_{i=0}^{k-1} \frac{u^{-i}}{i!} \right| \leq \frac{\kappa}{\min\{u, u^2\}} \frac{\ell_n}{n}.$$

Comparison with existing results

- Resnick and de Haan (1989): If $A_1, A_2, \ldots$ were independent, then

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Sketch of Proof of the Main Result:

Ergodic Theory + Hard Analysis + Applied Probability

Sketch of proof

Recall $\mathcal{E}_n^u = \sum_{j=1}^n \mathbb{1}_{(A_j \log 2 > un)} \overset{approx}{\sim} Bin\big(n, p_n = P(A_1 \log 2 > un)\big)$.

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- Estimate $d_{TV}(\tilde{\mathcal{E}}_n^u, \mathcal{E}_\infty^u)$ using *second order regular variation.*

How to estimate $d_{TV}(\tilde{\mathcal{E}}_n^u, \mathcal{E}_\infty^u)$?

Recall $\tilde{\mathcal{E}}_n^u \sim Poi(np_n)$ and $\mathcal{E}_\infty^u \sim Poi(u^{-1})$.

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Theorem (Philipp (1970); see also Galambos (1972))

There exists $C > 0$ and $\theta > 1$ such that for all $m, n \in \mathbb{N}$, for all $F \in \sigma(A_1, A_2, \ldots, A_m)$, and for all $H \in \sigma(A_{m+n}, A_{m+n+1}, \ldots)$,

$$|P(F \cap H) - P(F)P(H)| \leq C\theta^{-n} P(F)P(H).$$

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- Overall, application of Theorem 1 of Smith (1988), instead of Theorem 2 of Arratia, Goldstein and Gordon (1989), will result in an argument of similar length.
- However, we have not compared the rates obtained by these two results in our setup.

Consequences and Future Work

The central limit theorem

Theorem 3.1 of Davis and Hsing (1995) + extremal point process convergence yields

$$\frac{A_1 + A_2 + \cdots + A_n - E\big(A_1 \mathbb{1}_{(A_1 \le n)}\big)}{n} \xrightarrow{\mathcal{L}} S,$$

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This recovers a result of Samur (1989), who proved this using "direct frontal attack" with the help of exponential mixing.

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P_n (a seq of r.v.s) = the proportion of numbers of the form $7k+2$ ($k \in \mathbb{N} \cup \{0\}$) among first n RCF digits of a number chosen unifromly at random from $(0,1)$.

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Regular continued fraction is egalitarian towards all (residue) classes.

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$$PSL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} \Big/ \left\{ \pm \right\}$$

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Our work yields the rate of convergence in Pollicott's result.

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- Extremes in the context of number theory and geometry.

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Thank You Very Much