Connecting the Random Connection Model

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ICTS
Aug. 10, 2019

The Poisson Random Connection Model

- The RCM is a random graph.
- Vertex set is a Poisson Point Process $\mathcal{P}_\lambda$ of intensity $\lambda > 0$ in $\mathbb{R}^d$, $d \in \mathbb{N}$.
  1. For $A \subset \mathbb{R}^d$, the number of points in A follows a Poisson distribution with mean $\lambda|A|$:

$$P(\text{The number of points in } A = k) = e^{-\lambda|A|}\frac{(\lambda|A|)^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

  2. For $A, B \subset \mathbb{R}^d$ the number of points in A and B are independent.
- Connection Function: $g : \mathbb{R}^d \to [0, 1]$.
- Edges: There is an edge between $x, y \in \mathcal{P}_\lambda$ with probability $g(x - y)$ independently of everything else.

Applications

- Physics: Bonds in Particle Systems
- Epidemiology: Infected herd at location x infecting another at location y.
- Telecommunication: Communication between two transmitters.
- Biology: Sensing between two cells.

Related Models

- Random Geometric Graph: $g(x) = 1$ if $|x| \leq 2r$ and zero otherwise.
- SINR Graph: Edge between x_i, x_j if the signal-to-noise-plus-interference-ratio

$$SINR = \frac{PE_{ij}\ell(x_i, x_j)}{N + \gamma \sum_{k \neq i,j} PE_{kj}\ell(x_k, x_j)} > T.$$

- Inhomogenous RCM: Heavy-tailed degree distribution.

$$g(x) = 1 - \exp\left(-\eta \frac{W_x W_y}{|x|^{\alpha}}\right).$$

Independent Weights. $P(W_x > w) = w^{-\beta}, \qquad w \geq 1, \beta > 0.$

- RCM with preferential attachment: (Jacob and Morters (2015))

Phase Transition

- Phase Transition: A small change in a local parameter results in an abrupt change in the global behaviour.
- The phase transition of interest is the size of the largest component: multi-hop transmission, conductivity, spread of epidemics
- Infinite System: Transition from finite connected components to an infinite component.
- Finite System: Components logarithmic in size to a giant component which covers a non-trivial fraction of the nodes.

Percolation

- Assume there is a point at the origin O. Let C be the component containing the origin.

- Percolation Probability: $\theta(\lambda) = P(|C| = \infty)$.
- The RCM is said to percolate if $\theta(\lambda) > 0$.

- If the RCM percolates then there is w.p. 1 a unique infinite component.

- Critical Intensity: $\lambda_c := \inf\{\lambda > 0 : \theta(\lambda) > 0\}$.
- $\theta(\lambda)$ is a monotonic function.
- Non-trivial Phase Transition: $0 < \lambda_c < \infty$.

Non-trivial Phase Transition in the RCM

- The points connected to O form an inhomogenous Poisson point process of intensity $\lambda g(x)$.
- Expected Degree $= \lambda \int_{\mathbb{R}^d} g(x)dx$.
- No non-trivial phase transition if $\int_{\mathbb{R}^d} g(x)dx$ equals zero or ∞.

Theorem (Penrose 1991)

Suppose $0 < \int_{\mathbb{R}^d} g(x)dx < \infty$. *Then there exists a* $\lambda_c \in (0, \infty)$ *such that* $\theta(\lambda) = 0$ *for* $\lambda < \lambda_c$ *and* $\theta(\lambda) > 0$ *for* $\lambda > \lambda_c$.

A Inhomogeneous Random Connection Model

$$g(x) = 1 - \exp\left(-\eta \frac{W_x W_y}{|x|^\alpha}\right).$$

Independent Weights: $P(W_x > w) = w^{-\beta}, \qquad w \geq 1, \beta > 0.$

Theorem (Deijfen, Hofstad and Hooghiemstra 2013, Deprez and Wüthrich 2018)

1. *A non-trivial phase transition occurs for*
   - i) $d = 1$ *only if* $\alpha\beta > 2$ *and* $1 < \alpha < 2$
   - ii) $d \geq 2$ *only if* $\alpha > d$ *and* $\alpha\beta > 2d$.
2. *For all* $d \geq 1$ *the percolation function has been shown to be continuous if* $\alpha\beta > 2d$ *and* $\alpha \in (d, 2d)$.

For $d \geq 2$ the case when $\min\{\alpha, \alpha\beta\} \geq 2d$ is open.

Isolated Nodes in the RGG; Towards Connectivity

- U be the d-dimensional unit torus.
- $\mathcal{P}_n$ is a homogeneous Poisson point process with intensity n on U.
- $G(\mathcal{P}_n, r)$: The RGG with vertex set $\mathcal{P}_n$ and x, y share an edge if $d(x, y) \leq 2r$.
- Expected Number of isolated nodes $(E[W_n])$ $= ne^{-n2^d\theta_d r^d}$ where θ_d denotes the volume of the unit ball in $\mathbb{R}^d$.
- So if we take $r_n^d = \frac{\log n + c}{2^d \theta_d n}$ then $E[W_n] \to e^{-c}$.
- $W_n \to Poi(e^{-c})$ in distribution.

The RGG: Connectivity Regime

- Penrose (2003): Sharp Phase Transition
- For any sequence $w(n) \to \infty$, the following holds :

$$\mathbb{P}\left\{G(\mathcal{P}_n, r_n) \text{ is connected}\right\} \to \begin{cases} 0 & \text{if } n2^d\theta_d r_n^d = \log n - w(n) \\ 1 & \text{if } n2^d\theta_d r_n^d = \log n + w(n). \end{cases}$$

- $\mathbb{P}\left\{G(\mathcal{P}_n, r_n) \text{ is not connected}, W_n = 0\right\} \to 0$ as $n \to \infty$.

The RCM: Connectivity Regime

Proposition (Mao and Anderson 2011)

Let $g(r) = o(r^{-c})$, $c > d$ *and set* $r_n(\xi)^d = \frac{\log n + \xi}{n\alpha}$ *where* $\alpha = \int_{\mathbb{R}^d} g(|x|)dx$. *Then for the scaled connection function* $g_n(r) = g(r/r_n)$ *we have*

$$W_n \xrightarrow{d} Poi(e^{-\xi}).$$

- Penrose [2016]: g exponential decay, bounds on the rate of convergence.

Strong Laws for Isolated Nodes

- Couple the Poisson point processes such that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Let $\{U_{ij}\}_{i,j\geq 1}$ be independent uniform random variables on $(0,1)$.
- Draw an edge between $X_i, X_j \in \mathcal{P}_n$ if $U_{ij} \leq g\left(\frac{|X_i - X_j|}{r}\right)$. Call this graph $G_n(r)$.
- $W_n(r) =$ Number of isolated nodes in $G_n(r)$.
- Define the analog of the nearest neighbour distance

$$M_n := \inf\{r > 0 : W_n(r) = 0\}.$$

Strong Laws for Isolated Nodes

Theorem (I. 2018)

(i) Suppose that $g(r) = o(r^{-c})$ as $r \to \infty$ for some $c > d$. Then almost surely

$$\limsup_{n\to\infty} \frac{\alpha n M_n^d}{\log n} \leq 1. \tag{1}$$

(ii) If $g(r) = o(r^{-c})$ as $r \to \infty$ for some $c > 3d$, then almost surely

$$\liminf_{n\to\infty} \frac{\alpha n M_n^d}{\log n} \geq \frac{c-3d}{c-d}. \tag{2}$$

Corollary

Suppose that $g(r) = o(r^{-c})$ as $r \to \infty$ for every $c > 0$. Then almost surely

$$\lim_{n\to\infty} \frac{\alpha n M_n^d}{\log n} = 1. \tag{3}$$

Sufficient Condition for Connectivity in the RCM

Theorem (I. 2018)

Let $g(r) = o(r^{-c})$, $c > d$ *and set* $\hat{r}_n(\gamma)^d = \frac{\gamma \log n}{n\alpha}$. $\alpha = \int_{\mathbb{R}^d} g(|x|)dx$. *Define*

$$\beta = \inf\left\{\gamma > 0 : \gamma g\left(1 + \left(\frac{\alpha}{\gamma\theta}\right)^{\frac{1}{d}}\right) > 1\right\}.$$

Then for any $\gamma > \beta$ *the graph with the scaled connection function* $g_n(r) = g(r/\hat{r}_n(\gamma))$ *is connected w.h.p. as* $n \to \infty$.

Inhomogeneous Random Graph [Penrose, 2018]

- Vertex set is a inhomogenous PPP with intensity $n\mu$ where μ is a probability measure.
- Connection function $g_n(x,y) = g_n(y,x)$ satisfies
  min expected vertex degree $\geq \epsilon$ max expected vertex degree and
  $g_n < 1 - \epsilon$.
- D_j be the number of vertices of degree j.
- If $E[D_j] \to c$ then $D_j \xrightarrow{d} Po(c)$.
- Does not include the simple RGG.

Inhomogeneous RCM [joint work with Sanjoy Kumar Jhawar]

$$g(x) = 1 - \exp\left(-\eta \frac{W_x W_y}{|x|^\alpha}\right).$$

Independent Weights: $\quad P(W_x > w) = w^{-\beta}, \qquad w \geq 1, \beta > 0.$

$$r_n(\xi)^d = \frac{\log n - \log\log n + \xi + \log\left(\frac{\alpha\beta}{d}\right)}{c_0 n},$$

$$c_0 = \frac{2\pi\alpha\beta}{d(\alpha\beta - d)}\eta^{\frac{d}{\alpha}}\Gamma\left(1 - \frac{d}{\alpha}\right).$$

A Inhomogeneous Random Connection Model

$$r_n(\xi)^d = \frac{\log n - \log\log n + \xi + \log\left(\frac{\alpha\beta}{d}\right)}{c_0 n}.$$

Theorem (I. and Sanjoy Jhawar)

Consider the random graph with vertex set a homogeneous PPP in the unit torus with intensity n and connection function $g(\cdot/r_n(\xi))$. Suppose $\alpha > d$ and $\beta > 1$.

(i) *Then $E[D_0] \to e^{-\xi}$ as $n \to \infty$.*

(ii) *If in addition we have $\alpha\beta > \min\{4d, 11(\alpha - d)\}$, then $D_0 \xrightarrow{L} Po(e^{-\xi})$ as $n \to \infty$.*

Connecting this Inhomogeneous Random Connection Model

$$\hat{r}_n(\gamma) := \frac{\gamma \log n}{\kappa n}, \qquad \gamma > 0,$$

$$\kappa := \theta E^{W_0}\left[1 - \exp\left(-\eta W_0\right)\right] < \infty \qquad \text{provided } \beta > 1.$$

$$T(\gamma) := 1 - \exp\left\{-\eta\left(1 + \left(\frac{\kappa}{\gamma\theta}\right)^{\frac{1}{d}}\right)^{-\alpha}\right\}.$$

$$\rho := \inf\left\{\gamma > 0 : \gamma T(\gamma) > 1\right\}.$$

Theorem (I. and Sanjoy Jhawar)

Consider the random graph with vertex set a homogeneous PPP in the unit torus with intensity n and connection function $g(\cdot/\hat{r}_n(\gamma))$. Suppose $\beta > 1$ and $\alpha > d$. Then the graph is connected w.h.p. for all $\gamma > \rho$.