Aperiodically driven integrable systems and their emergent steady states

Arnab Sen

Department of Theoretical Physics,

Indian Association for the Cultivation of Science, Kolkata

Collaborators: Sourav Nandy (IACS), Diptiman Sen (IISc)

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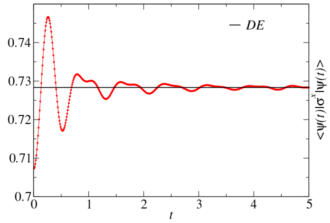
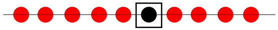
Plan of the talk

- Non-equilibrium steady states of driven many-body systems ($i\frac{d\rho}{dt} = [H(t), \rho]$)—no external bath attached
- Case of periodically driven (Floquet) systems $[H(t) = H(t + nT)]$
- Perturbed Floquet integrable systems–periodic structure in time broken (s.t. drive $H(t)$ stays periodic on average in time) $\Rightarrow$ *new steady states that are not possible with periodic drives*
- Case of (any typical realization of) random noise
- Case of scale-invariant noise
- Conclusions and future directions

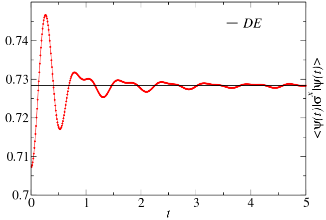
Driven systems: Steady state

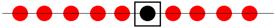




- Does an ensemble description exist for steady states of driven quantum systems? Guiding principles?
- Lots of progress in recent years for quenches ($H_i \to H_f$) [Rigol, Dunjko, Olshanii (2008)] and periodically driven systems ($H(t) = H(t + nT)$) [Lazarides, Das, Moessner (2014,2015), D'Alessio, Rigol (2014), Bukov, D'Alessio, Polkovnikov (2015)]
- Steady state description for generic driving protocols still an open issue.

Guiding principle





- Conserved quantities during the dynamics play crucial role. E.g. $\langle\psi(t)|H_f|\psi(t)\rangle$ independent of t for quench.
- Write maximum entropy statistical description consistent with conservations (as we do in Statistical Mechanics, e.g. Jaynes (1957))
- Remarkably, **local properties** of the resulting pure state at late times indistinguishable from the max. ent. result in thermodynamic limit.

Eigenstate Thermalization Hypothesis

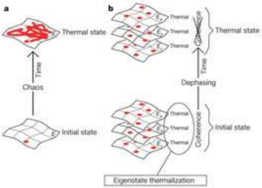
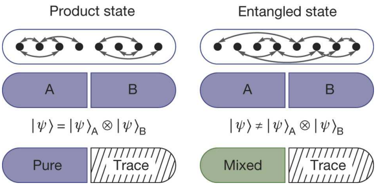


Figure 2 | Thermalization in classical versus quantum mechanics. a, In classical mechanics, time evolution constructs the thermal state from an initial state that generally bears no resemblance to the former. **b,** In quantum mechanics, according to the ETH, every eigenstate of the hamiltonian always implicitly contains a thermal state. The coherence between the eigenstates initially hides it, but time dynamics reveals it through dephasing.



R. Islam et al, Nature (2015)

- Let $|\psi(0)\rangle = \sum_i c_i |\mathcal{E}_i\rangle$ where $|\mathcal{E}_i\rangle$ denote the post-quench eigenstates of H_f.
- $\langle\psi(t)|O|\psi(t)\rangle = \sum_i |c_i|^2 \langle\mathcal{E}_i|O|\mathcal{E}_i\rangle + \sum_{i\neq j} c_i c_j^* \exp(-i(E_i - E_j)t)\langle\mathcal{E}_j|O|\mathcal{E}_i\rangle$
- High-energy eigenstates in a generic system expected to follow ETH (Deutsch, Srednicki, Rigol+Dunjko+Olshanii, Kim+Ikeda+Huse)

Periodically driven systems

- Synchronization of local properties with the drive frequency at late time allows for a *periodic ensemble* $\Rightarrow$ can lead to novel nonequilibrium states like *Floquet time crystals* [Else, Bauer, Nayak (2016), Khemani, Lazarides, Moessner, Sondhi (2016)]
- $U(T) = \exp(-iH_F T)$ where $U(T)$ is the Floquet operator and H_F the Floquet Hamiltonian.
- Thus, stroboscopic propagation where $|\psi(n)\rangle = U(T)^n|\psi(0)\rangle$ gives a steady state when $n \to \infty$.
- Generic systems, when continuosly driven periodically, heat up to infinite temperature at large n.
- Not true for many body localized systems (Nandkishore, Huse (2014)) and for *certain* integrable models (extensive number of conservations remain present stroboscopically).

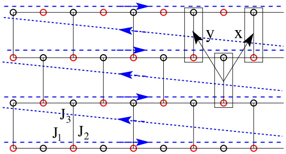
Class of integrable models

- In 1D, transverse field Ising model (TFIM)

$$H = -\sum_j \left(g(t)\sigma_j^x + \sigma_j^z \sigma_{j+1}^z\right)$$

- In 2D, Kitaev model (see Chen+Nussinov, 2008)—

$$H_{2\mathrm{D}} = \sum_{j+l=\mathrm{even}} \left(J_1 \sigma_{j,l}^x \sigma_{j+1,l}^x + J_2 \sigma_{j-1,l}^y \sigma_{j,l}^y + J_3 \sigma_{j,l}^z \sigma_{j,l+1}^z\right)$$

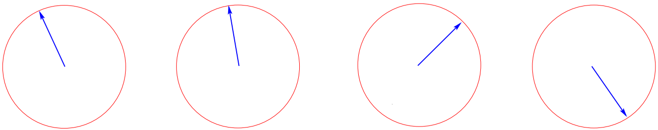


- Jordan-Wigner transformation:

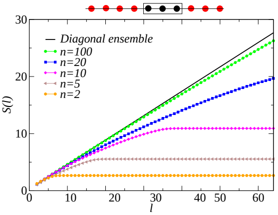
$$\sigma_n^x = 1 - 2c_n^\dagger c_n \quad \sigma_n^z = -(c_n + c_n^\dagger) \prod_{m<n} (1 - 2c_m^\dagger c_m),$$

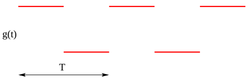
Psuedospin representation

- $H = -\sum_{n=1}^{L}\left(g(t) - 2g(t)c_n^\dagger c_n + c_n^\dagger c_{n+1} + \text{h.c.} + c_n^\dagger c_{n+1}^\dagger + \text{h.c.}\right)$
- Hamiltonian connects $|\uparrow\rangle_{\vec{k}} = c_{\vec{k}}^\dagger c_{-\vec{k}}^\dagger |0\rangle$ with $|\downarrow\rangle_{\vec{k}} = |0\rangle$ where $|0\rangle$ denotes vacuum of c fermions, i.e. $|\cdots \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \cdots\rangle$ (ground state as $g \to \infty$)
- The wavefunction can be expressed as $|\psi(t)\rangle = \otimes_{k>0}|\psi_k(t)\rangle$ where $|\psi_k(t)\rangle = u_k(t)|\uparrow\rangle_k + v_k(t)|\downarrow\rangle_k$ with $k = 2\pi m/L$, $m = 1/2, 3/2, \cdots, (L-1)/2$.
- Dynamics through $H_k = (g(t) - \cos(k))\tau_3 + \sin(k)\tau_1$, which acts as a **time-dependent magnetic field.**



Entanglement generation after n drive cycles





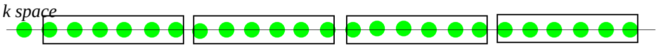
- Need the knowledge of two $l \times l$ matrices to fix the reduced density matrix $\rho_l = \mathrm{Tr}_{L-l}(\rho)$ (Peschel (2003))–

$$C_{ij} = \langle c_i^\dagger c_j \rangle_n = \frac{2}{L} \sum_{k>0} |u_k(t)|^2 \cos(k(i-j))$$

$$F_{ij} = \langle c_i^\dagger c_j^\dagger \rangle_n = \frac{2}{L} \sum_{k>0} u_k^*(t) v_k(t) \sin(k(i-j))$$

$$\mathcal{C}_n(l) = \begin{pmatrix} \mathbf{I} - \mathbf{C} & \mathbf{F} \\ \mathbf{F}^* & \mathbf{C} \end{pmatrix}$$

Coarse-graining in momentum space



- RDM for $l \ll L$ depends *only* on *suitably coarse-grained* variables in k space (Lai+Yang, 2015 used idea for eigenstates)—

$$\left(|u_k(n)|^2\right)_c = \frac{1}{N_c}\sum_{k\in k_{\rm cell}} |u_k(n)|^2$$

$$(u_k^*(n)v_k(n))_c = \frac{1}{N_c}\sum_{k\in k_{\rm cell}} u_k^*(n)v_k(n)$$

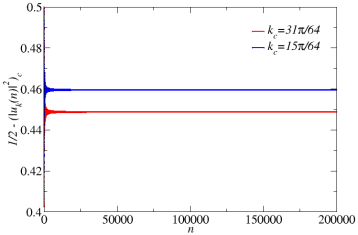
These defined using $N_c(\gg 1)$ consecutive k modes that lie within a cell ($k_{\rm cell}$) which has average momentum k_c and size δk where $1/L \ll \delta k \ll 1/l$.

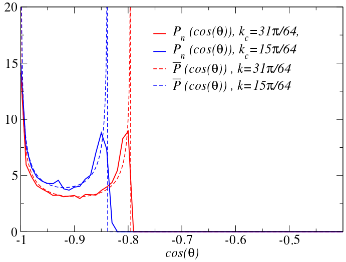
- Since $0 \le |i-j| \le l$, we have $\cos[k(i-j)] \simeq \cos[k_c(i-j)]$ (sim. for sin)

$$C_{ij} \simeq \frac{1}{\mathcal{N}_{\rm cell}}\sum_{k_c}(|u_k(n)|^2)_c \cos(k_c(i-j))$$

$$F_{ij} \simeq \frac{1}{\mathcal{N}_{\rm cell}}\sum_{k_c}(u_k^*(n)v_k(n))_c \sin(k_c(i-j))$$

Behaviour for periodically driven systems



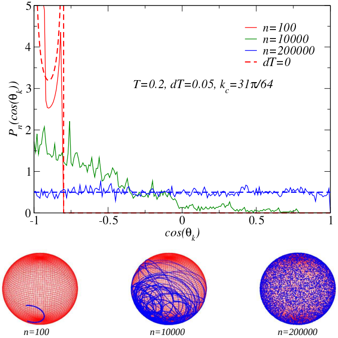


- Due to the 2×2 structure, $U_k(T) = \exp(-i\gamma_k \hat{e}_k \cdot \vec{\sigma}) \Rightarrow$ $\mathcal{I}_k = \langle \psi_k(n) | \hat{e}_k \cdot \vec{\sigma} | \psi_k(n) \rangle$ independent of n
- A single k mode never *thermalizes* since $|u_k|^2 = \sin^2(n\gamma_k)\,(1 - e_{k3}^2)$ but $(|u_k|^2)_c \to (1/2)(1 - e_{3k}^2)$.
- $P_n(\cos(\theta_k)) = \frac{1}{\pi\sqrt{(1+\cos(\theta_k))(1-2e_{3k}^2-\cos(\theta_k))}}$ for $-1 \leq \cos(\theta_k) \leq 1 - 2e_{3k}^2$ and 0 otherwise $(= \overline{P}(\cos\theta_k))$.

Perturbed Floquet dynamics

- Build using either $U(T + dT)$ or $U(T - dT)$ at each n s.t. $g(t)$ is g_i for $(T \pm dT)/2$ and g_f for $(T \pm dT)/2$
- $|\psi_k(n)\rangle = U_k(T + \tau_n dT) U_k(T + \tau_{n-1} dT) \cdots U_k(T + \tau_1 dT) |\psi_k(0)\rangle$ where the sequence $\tau_i = \tau_1, \tau_2, \tau_3, \cdots$ is the same for all the k modes.
- If $dT = 0$, back to periodically driven case.
- For Floquet systems perturbed with random noise, the sequence τ_i is any typical realization of a random process where each τ is chosen randomly to be either $+1$ or -1.
- For Floquet systems perturbed with scale-invariant noise, we take the Thue-Morse (TM) sequence. (Thue 1906, Morse 1921)

Perturbation with random noise-I



$$U_k(T - dT) = \exp(-i\alpha_k \hat{a}_k \cdot \vec{\sigma})$$
$$U_k(T + dT) = \exp(-i\beta_k \hat{b}_k \cdot \vec{\sigma})$$

- $\Delta\phi_{1k} = \arccos(\hat{e}_k \cdot \hat{a}_k)$, $\Delta\phi_{2k} = \arccos(\hat{e}_k \cdot \hat{b}_k)$, $\Delta\phi_k \propto dT \sin(k) \rightarrow \sqrt{n} dT \sin(k)$ control parameter!
- Compact space–Surface of the unit sphere

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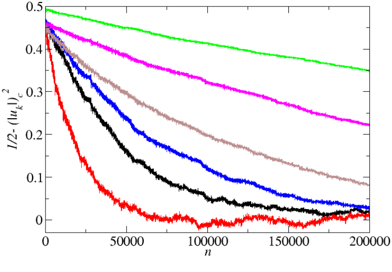
Perturbation with random noise-II



- Behavior of $(|u_k|^2)_c$ shows the *irreversible approach* to an infinite temperature ensemble.
- Relaxation controlled by $\tau_{k,dT} = 1/\left((dT)^2(\sin k_c)^2\right)$

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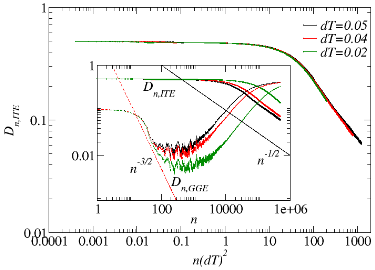
Perturbation with random noise-III



- $\mathcal{D} = \mathrm{Tr}[(\mathcal{C}_{\rm ref}(l) - \mathcal{C}_n(l))^\dagger(\mathcal{C}_{\rm ref}(l) - \mathcal{C}_n(l))]^{1/2}/(2l)$
- $\mathcal{D}_{n,ITE}(l) \sim \int_0^\pi dk \exp(-n(dT)^2 \sin^2(k))$
- At large n, integral dominated by $k = 0$ and $k = \pi$
- $\mathcal{D}_{n,ITE}(l) \sim \mathcal{F}_l(n(dT)^2)$ where $\mathcal{F}_l(x) \sim 1/\sqrt{x}$ for $x \gg 1$ and $\mathcal{O}(1)$ for $x \ll 1$.

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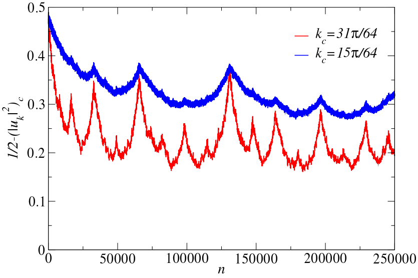
Perturbation with scale invariant noise-I

$$
\begin{aligned}
m &= 0, \tau_1, -1 \\
m &= 1, \tau_1, \tau_2, \boxed{-1}, +1 \\
m &= 2, \tau_1, \cdots, \tau_4, \boxed{-1, +1}, +1, -1 \\
m &= 3, \tau_1, \cdots, \tau_8, \boxed{-1, +1, +1, -1}, +1, -1, -1, +1 \\
m &= 4, \tau_1, \cdots, \tau_{16}, \boxed{-1, +1, +1, -1, +1, -1, -1, +1}, +1, -1, -1, +1, -1, \cdot \\
&\vdots
\end{aligned}
$$

- At each recursion level m, we obtain the first 2^m elements of the infinite sequence.
- Self-similar structure in time (verify by removing every second term)
- Neither periodic nor random in time $\Rightarrow$ *quasiperiodic* sequence

Perturbation with scale invariant noise-II



- Behaviour is *different* from ITE or the periodic-GGE ($dT = 0$).
- Not obvious that the coarse-grained quantities approach time-independent values—so, is there a NESS?

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Perturbation with scale invariant noise-III

- Denote $U_k(T - dT)$ by A_0 and $U_k(T + dT)$ by B_0
- $A_{m+1} = B_m A_m$ and $B_{m+1} = A_m B_m$ for $m \geq 0$

$A_0 B_0$	$B_0 A_0$	$B_0 A_0$	$A_0 B_0$	$B_0 A_0$	$A_0 B_0$	$A_0 B_0$	$B_0 A_0$	$\vert$	$\psi_k(n=0)\rangle$
				$B_1 A_1$	$A_1 B_1$	$A_1 B_1$	$B_1 A_1$	$\vert$	$\psi_k(n=0)\rangle$
						$A_2 B_2$	$B_2 A_2$	$\vert$	$\psi_k(n=0)\rangle$
							$B_3 A_3$	$\vert$	$\psi_k(n=0)\rangle$
							A_4	$\vert$	$\psi_k(n=0)\rangle$

- Evolution operator after exactly 2^m drives is given by A_m.
- Remarkably, $A_m \to B_m$ at large m and we have *emergent* time periodicity!

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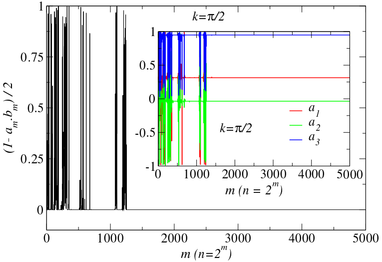
Perturbation with scale invariant noise-IV



- $A_m = \exp(-i\alpha_m \hat{a}_m \cdot \vec{\sigma})$ and $B_m = \exp(-i\beta_m \hat{b}_m \cdot \vec{\sigma})$
- Define $\phi_m = \arccos(\hat{a}_m \cdot \hat{b}_m)$ $(0 \leq \phi_m \leq \pi)$
- Easy to show that $\alpha_m = \beta_m$ for all $m \geq 1$.
- Furthermore, α_m covers $[0, \pi]$ uniformly AND $\phi_m \to 0$ as $m \to \infty$.

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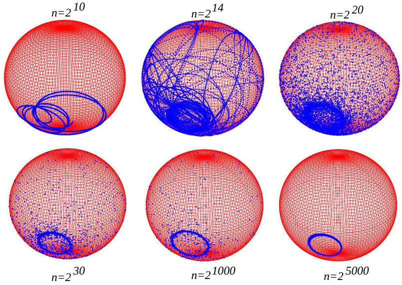
Perturbation with scale invariant noise-V



Emergent conservation law at each k–

$$\begin{aligned}
\mathcal{J}_k(2^m) &= \langle\psi_k(2^m)|\hat{a}_\infty\cdot\vec{\sigma}|\psi_k(2^m)\rangle \\
&= \langle\psi_k(0)|A_m^\dagger(\hat{a}_\infty\cdot\vec{\sigma})A_m|\psi_k(0)\rangle \\
&= \langle\psi_k(0)|e^{i\alpha_m\hat{a}_m\cdot\vec{\sigma}}(\hat{a}_\infty\cdot\vec{\sigma})e^{-i\alpha_m\hat{a}_m\cdot\vec{\sigma}}|\psi_k(0)\rangle
\end{aligned}$$

equals $\mathcal{J}_k(n=0) = \langle\psi_k(0)|\hat{a}_\infty\cdot\vec{\sigma}|\psi_k(0)\rangle$ and becomes independent of m for sufficiently large m at each momentum k.

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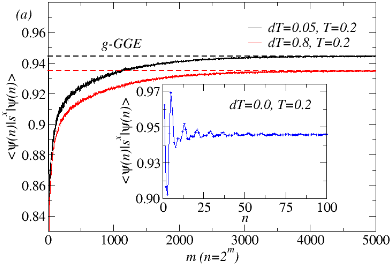
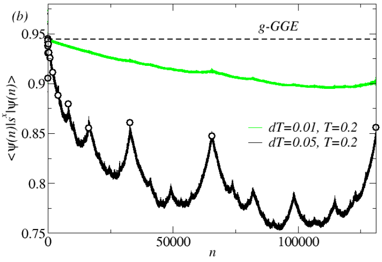
Perturbation with scale invariant noise-VI





- The construction of steady state ⇒ the relevant integrals of motion are now $\mathcal{J}_k$.
- The density matrix of the g-GGE then equals

$$\rho_{\mathrm{g-GGE}} = \frac{1}{Z} \exp(-\sum_k \lambda_k \mathcal{J}_k),$$

where the Lagrange multipliers λ_k are fixed by the condition
$\mathrm{Tr}[\rho_{\mathrm{g-GGE}} \mathcal{J}_k] = \langle \psi_k(n=0) | \mathcal{J}_k | \psi_k(n=0) \rangle$

Conclusions

- NESSs of continually driven systems not understood in general
- Here we construct two examples of driving protocols where there is no periodic structure in time
- **Infinite temperature ensemble** for perturbed Floquet with random noise even though model remains integrable
- Emergent conservation laws at *extremely* late times for perturbed Floquet with scale invariant noise leading to a **geometric generalized Gibbs ensemble**
- More general way of classifying possible NESSs in integrable models (work in progress)

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