On the entanglement spectrum: From the Levy distribution to the Tracy-Widom

Arul Lakshminarayan

Department of Physics
Indian Institute of Technology Madras
Chennai, India.

July 28 2017, OQS at ICTS

Outline

- **Introduction**
- **Typical entanglement and quantum chaos**
- **Entanglement transitions in coupled chaotic systems**
- **Summary**

with
Shashi C. L. Srivastava (VECC Kolkata),
Steven Tomsovic (WSU Pullman),
Arnd Bäcker (TU Dresden) and
Roland Ketzmerick (TU Dresden).

Integrable vs Chaotic: Billiards as examples

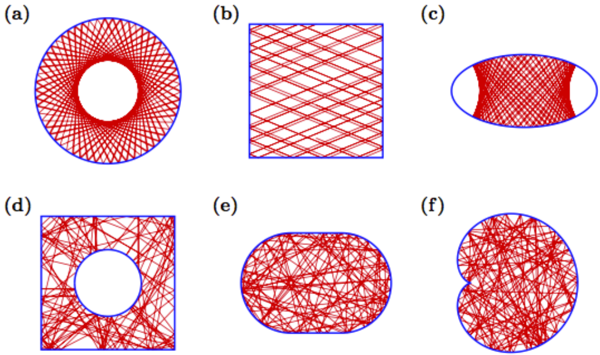


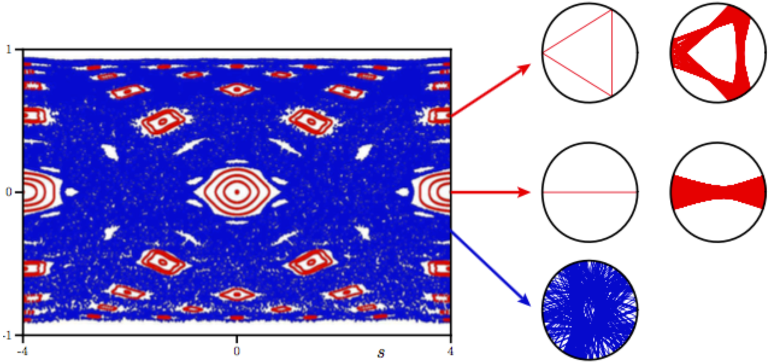
Figure 1: For different billiards 100 successive reflections of one orbit are shown. The regular dynamics for the billiard in the (a) circle, (b) square and (c) ellipse is in contrast to chaotic dynamics for the (d) Sinai billiard, (e) stadium billiard and (f) cardioid billiard.

$$\rho(\phi) = 1 + \cos(\phi)$$

From Arnd Bäcker, Habilitationsschrift, 2007.

Coexistence of regular and chaotic

Limacon billiards



$$\rho(\phi) = 1 + 0.3\cos(\phi)$$

From Arnd Bäcker, Habilitationsschrift, 2007.

Eigenfunctions of Chaos: Quantum ergodicity

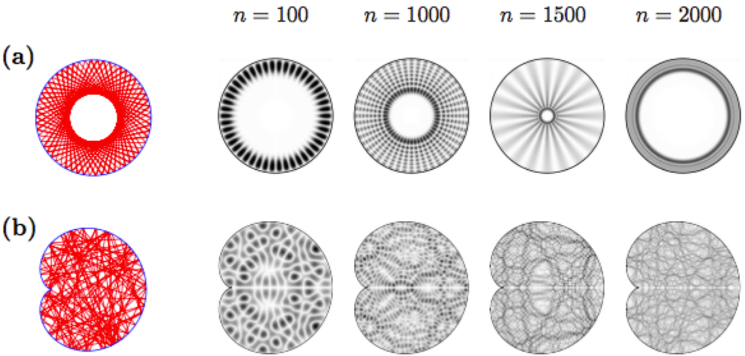


Figure 2: The eigenstates of (a) the integrable circular billiard and (b) the chaotic cardioid billiard reflect the structure of the corresponding classical dynamics. Shown is a density plot of $|\psi_n(\boldsymbol{q})|^2$ where black corresponds to high probability.

From Arnd Bäcker, Habilitationsschrift, 2007.

Ergodic dynamics and thermalization in an isolated quantum system

C. Neill[1],* P. Roushan[2],* M. Fang[1],* Y. Chen[2],* M. Kolodrubetz[3], Z. Chen[1], A. Megrant[2], R. Barends[2], B. Campbell[1], B. Chiaro[1], A. Dunsworth[1], E. Jeffrey[2], J. Kelly[2], J. Mutus[2], P. J. J. O'Malley[1], C. Quintana[1], D. Sank[2], A. Vainsencher[1], J. Wenner[1], T. C. White[2], A. Polkovnikov[3], and J. M. Martinis[1,2†]

[1] *Department of Physics, University of California, Santa Barbara, CA 93106-9530, USA*
[2] *Google Inc., Santa Barbara, CA 93117, USA and*
[3] *Department of Physics, Boston University, Boston, MA 02215, USA*

Statistical mechanics is founded on the assumption that all accessible configurations of a system are equally likely. This requires dynamics that explore all states over time, known as ergodic dynamics. In isolated quantum systems, however, the occurrence of ergodic behavior has remained an outstanding question [1–4]. Here, we demonstrate ergodic dynamics in a small quantum system consisting of only three superconducting qubits. The qubits undergo a sequence of rotations and interactions and we measure the evolution of the density matrix. Maps of the entanglement entropy show that the full system can act like a reservoir for individual qubits, increasing their entropy through entanglement. Surprisingly, these maps bear a strong resemblance to the phase space dynamics in the classical limit; classically chaotic motion coincides with higher entanglement entropy. We further show that in regions of high entropy the full multi-qubit system undergoes ergodic dynamics. Our work illustrates how controllable quantum systems can investigate fundamental questions in

In classical systems, it is chaotic motion which drives the system to ergodically explore the state space [5]. Quantum systems, however, are governed by Schrodinger's equation which is linear and consequently forbids chaotic motion [6]. This poses fundamental questions regarding the applicability of statistical mechanics in isolated quantum systems [1–4]. Do quantum systems exhibit ergodic behavior in the sense of Eq. 1? Do quantum systems act as their own bath in order to approach thermal equilibrium? Extensive experimental efforts have been made to address these fundamental questions [12–18].

Here we investigate ergodic dynamics by considering a simple quantum model whose classical limit is chaotic [19–23]. This model describes a collection of spin-1/2 particles whose collective motion is equivalent to that of a single larger spin with total angular momentum j governed by the Hamiltonian

$$\mathcal{H}(t) = \frac{\pi}{2\tau} J_y + \frac{\kappa}{2j} J_z^2 \sum_{n=1}^{N} \delta(t - n\tau) \qquad (2)$$

where J_y and J_z are angular momentum operators. The

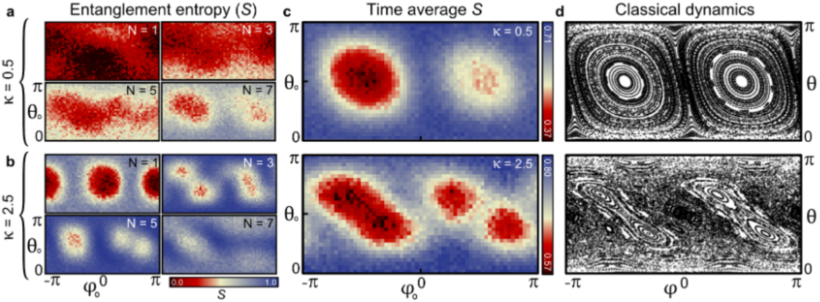


Figure 2. **Entanglement entropy and classical chaos. a,b,** The entanglement entropy (color) of a single qubit (see Eq. 5) averaged over qubits and mapped over a 31 x 61 grid of initial state, for various time steps N and two values of interaction strength κ. The entanglement entropy of a single qubit can range from 0 to 1. **c,** The entanglement entropy averaged over 20 steps for $\kappa = 0.5$ and over 10 steps for $\kappa = 2.5$; for both experiments the maximum pulse sequence is $\approx$ 500 ns. The left/right asymmetry is the result of experimental imperfections and is not present in numerical simulations (see supplement). **d,** A stroboscopic map of the classical dynamics is computed numerically and shown for comparison. The map is generated by randomly choosing initial states, propagating each state forward using the classical equations of motion, and plotting the orientation of the state after each step as a point. We observe a clear connection between regions of chaotic behavior (classical) and high entanglement entropy (quantum).

Neill et. al., Nature Physics 2016

Bipartite pure states: Reduced Density Matrices

$$\mathcal{H} = \mathcal{H}_N^A \otimes \mathcal{H}_M^B, \quad N \leq M \quad |\psi_{AB}\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} a_{i\alpha} |i\rangle \otimes |\alpha\rangle$$

$A = [a_{i\alpha}]_{i=1,\ldots,N;\alpha=1,\ldots,M}$: $N \times M$ matrix.

$\rho_A = \mathrm{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = AA^\dagger$: $N \times N$ matrix

$|\psi_{AB}\rangle$ **unentangled iff** ρ_A (and hence ρ_B) are **pure state** density matrices

Bipartite pure states: Reduced Density Matrices

$$\mathcal{H} = \mathcal{H}_N^A \otimes \mathcal{H}_M^B, \quad N \le M \quad |\psi_{AB}\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} a_{i\alpha} |i\rangle \otimes |\alpha\rangle$$

$$A = [a_{i\alpha}]_{i=1,\ldots,N;\alpha=1,\ldots,M} : N \times M \text{ matrix.}$$

$$\rho_A = \mathrm{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = AA^\dagger : N \times N \text{ matrix}$$

$|\psi_{AB}\rangle$ **unentangled iff** ρ_A (and hence ρ_B) are **pure state** density matrices

Bipartite pure states: Reduced Density Matrices

$$\mathcal{H} = \mathcal{H}_N^A \otimes \mathcal{H}_M^B, \quad N \leq M \quad |\psi_{AB}\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} a_{i\alpha} |i\rangle \otimes |\alpha\rangle$$

$$A = [a_{i\alpha}]_{i=1,\ldots,N;\alpha=1,\ldots,M} : N \times M \text{ matrix.}$$

$$\rho_A = \mathrm{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = AA^\dagger : N \times N \text{ matrix}$$

$|\psi_{AB}\rangle$ **unentangled iff** ρ_A (and hence ρ_B) are **pure state** density matrices

Bipartite pure states: Reduced Density Matrices

$$\mathcal{H} = \mathcal{H}_N^A \otimes \mathcal{H}_M^B, \quad N \leq M \quad |\psi_{AB}\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} a_{i\alpha} |i\rangle \otimes |\alpha\rangle$$

$$A = [a_{i\alpha}]_{i=1,\ldots,N;\alpha=1,\ldots,M} : N \times M \,\text{matrix.}$$

$$\rho_A = \mathrm{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = AA^\dagger : N \times N \;\text{ matrix}$$

$|\psi_{AB}\rangle$ **unentangled iff** ρ_A (and hence ρ_B) are **pure state** density matrices

Bipartite pure states

$$\rho_A = \sum_j \lambda_j \, |\phi_j^A\rangle\langle\phi_j^A|, \quad \rho_B = \sum_j \lambda_j \, |\phi_j^B\rangle\langle\phi_j^B|$$

Schmidt decomposition:

$$|\psi_{AB}\rangle = \sum_{j=1}^{N} \sqrt{\lambda_j} \, |\phi_j^A\rangle|\phi_j^B\rangle. \ \lambda_1 \leq \cdots \leq \lambda_N$$

($\{\lambda_i\}$: eigenvalues of RDM, $\log(\lambda_i)$: entanglement spectrum)

Entanglement in $|\psi_{AB}\rangle =$
$S(\rho_A) = -\mathrm{Tr}(\rho_A \, \log\rho_A) = -\sum_{i=1}^{N} \lambda_i \, \log\lambda_i \in [0, \log N]$

Other entropies: Tsallis [1]

$$S_\alpha = \frac{1 - \mathrm{Tr}(\rho_A^\alpha)}{\alpha - 1} = \frac{1 - \sum_{j=1}^{N} \lambda_j^\alpha}{\alpha - 1}$$

[1] Arnold's law: Everything that is discovered is named after someone else (including Arnold's law).

Bipartite pure states

$$\rho_A = \sum_j \lambda_j \, |\phi_j^A\rangle\langle\phi_j^A|, \quad \rho_B = \sum_j \lambda_j \, |\phi_j^B\rangle\langle\phi_j^B|$$

Schmidt decomposition:

$$|\psi_{AB}\rangle = \sum_{j=1}^{N} \sqrt{\lambda_j} \, |\phi_j^A\rangle|\phi_j^B\rangle. \; \lambda_1 \leq \cdots \leq \lambda_N$$

($\{\lambda_i\}$: eigenvalues of RDM, $\log(\lambda_i)$: entanglement spectrum)

Entanglement in $|\psi_{AB}\rangle =$

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \, \log \rho_A) = -\sum_{i=1}^{N} \lambda_i \, \log \lambda_i \in [0, \log N]$$

Other entropies: Tsallis [1]

$$S_\alpha = \frac{1 - \mathrm{Tr}(\rho_A^\alpha)}{\alpha - 1} = \frac{1 - \sum_{j=1}^{N} \lambda_j^\alpha}{\alpha - 1}$$

[1] Arnold's law: Everything that is discovered is named after someone else (including Arnold's law).

Bipartite pure states

$$\rho_A = \sum_j \lambda_j \, |\phi_j^A\rangle\langle\phi_j^A|, \quad \rho_B = \sum_j \lambda_j \, |\phi_j^B\rangle\langle\phi_j^B|$$

Schmidt decomposition:

$$|\psi_{AB}\rangle = \sum_{j=1}^{N} \sqrt{\lambda_j} \, |\phi_j^A\rangle|\phi_j^B\rangle. \ \lambda_1 \leq \cdots \leq \lambda_N$$

($\{\lambda_i\}$: eigenvalues of RDM, $\log(\lambda_i)$: entanglement spectrum)

Entanglement in $|\psi_{AB}\rangle =$
$S(\rho_A) = -\mathrm{Tr}(\rho_A \, \log \rho_A) = -\sum_{i=1}^{N} \lambda_i \, \log \lambda_i \in [0, \log N]$

Other entropies: Tsallis [1]

$$S_\alpha = \frac{1 - \mathrm{Tr}(\rho_A^\alpha)}{\alpha - 1} = \frac{1 - \sum_{j=1}^{N} \lambda_j^\alpha}{\alpha - 1}$$

[1] Arnold's law: Everything that is discovered is named after someone else (including Arnold's law).

Typical entanglement

Random bipartite pure state $|\psi_{AB}\rangle$ sampled uniformly from the Hilbert space [2] ($\delta(\sum |a_{i\alpha}|^2 - 1)$) $\longrightarrow$ induced reduced density matrix

$$\rho_A = \frac{MM^\dagger}{\mathrm{Tr} MM^\dagger}$$

M has i.i.d. Gaussian random numbers. *Trace constrained Laguerre ensemble.*

Typical Entanglement =average entanglement =

$$-\langle \mathrm{Tr}(\rho_A \log \rho_A) \rangle = \sum_{k=N+1}^{NM} \frac{1}{k} - \frac{M-1}{2N} \approx \log(N) - \frac{N}{2M}$$

Eg.: 2 qubits: $N = M = 2$, $\langle S \rangle = 1/3$

(Seth Lloyd, Heinz Pagels '88; Don Page '93; Sidhartha Sen '94; Elihu Lubkin '78)

[2] Hilbert space is a big place – C. Caves

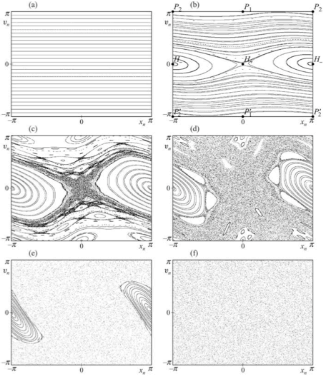
Applications of typical / random states

1. Complex quantum systems
2. Equilibriation of quantum states
3. Foundations of statistical physics
4. Encoding protocols for quantum channels
5. Encryption & Quantum data hiding
6. Information locking
7. Process tomography
8. State distinguishability

P. Hayden, D. Leung, P. W. Shor, and A. Winter, Communications in Mathematical Physics, 250, 371 (2004)

Typical entanglement and quantum chaos[3]

A simple model: Kicked pendulum *a.k.a* standard map



$K = 0,\ .3,\ 1,\ 2,\ 4,\ 7$

$$H = \frac{p^2}{2} - K \cos q \sum_{n=-\infty}^{\infty} \delta(t-n)$$

Classical Map: $(q,p) \to (q+p, p+K\sin(q+p))$

Figure: $K = 0,\ .3,\ 1,\ 2,\ 4,\ 7$

Quantum Map (Floquet operator):

$$U = \exp(-i\hat{p}^2/2\hbar)\ \exp(iK\cos\hat{q}/\hbar)$$

(Berry et. al. 1979, Casati et. al. 1979)

[3]Studying non-linear dynamics is like studying non-elephant biology. – Stanislaw Ulam

Two coupled kicked pendula

$$H = \frac{\mathbf{p}^2}{2} - [K_1 \cos q_1 + K_2 \cos q_2 + b \cos(q_1 + q_2)] \sum_{n=-\infty}^{\infty} \delta(t - n)$$

Classical: 4D symplectic maps (Froeschlé, Astron. Astrophysics, 1970, Richter et. al. Phys. Rev. E., 2014)

Quantize to get unitary Floquet propagator $\mathcal{U}$:

$$\mathcal{U} = [U_1(K_1) \otimes U_2(K_2)] \, U_{12}(b)$$

Compactify phase space as **Torus**:

$U_{1,2}$: $N \times N$ matrices. $\mathcal{U}$: $N^2 \times N^2$

$N = 1/h_{eff}$

Entanglement in Eigenstates: $\mathcal{U}|\phi_k\rangle = \exp(i\phi_k)|\phi_k\rangle$ as a function of K_i and b??

Two coupled kicked pendula

$$H = \frac{\mathbf{p}^2}{2} - [K_1 \cos q_1 + K_2 \cos q_2 + b \cos(q_1 + q_2)] \sum_{n=-\infty}^{\infty} \delta(t-n)$$

Classical: 4D symplectic maps (Froeschlé, Astron. Astrophysics, 1970, Richter et. al. Phys. Rev. E., 2014)

Quantize to get unitary Floquet propagator $\mathcal{U}$:

$$\mathcal{U} = [U_1(K_1) \otimes U_2(K_2)]\, U_{12}(b)$$

Compactify phase space as **Torus**:

$U_{1,2}$: $N \times N$ matrices. $\mathcal{U}$: $N^2 \times N^2$

$N = 1/h_{eff}$

Entanglement in Eigenstates: $\mathcal{U}|\phi_k\rangle = \exp(i\phi_k)|\phi_k\rangle$ as a function of K_i and b??

Two coupled kicked pendula

$$H = \frac{\mathbf{p}^2}{2} - [K_1 \cos q_1 + K_2 \cos q_2 + b\, \cos(q_1 + q_2)] \sum_{n=-\infty}^{\infty} \delta(t-n)$$

Classical: 4D symplectic maps (Froeschlé, Astron. Astrophysics, 1970, Richter et. al. Phys. Rev. E., 2014)

Quantize to get unitary Floquet propagator $\mathcal{U}$:

$$\mathcal{U} = [U_1(K_1) \otimes U_2(K_2)]\, U_{12}(b)$$

Compactify phase space as **Torus**:

$U_{1,2}$: $N \times N$ matrices. $\mathcal{U}$: $N^2 \times N^2$

$N = 1/h_{eff}$

Entanglement in Eigenstates: $\mathcal{U}|\phi_k\rangle = \exp(i\phi_k)|\phi_k\rangle$ as a function of K_i and b??

Two coupled kicked pendula

$$H = \frac{\mathbf{p}^2}{2} - [K_1 \cos q_1 + K_2 \cos q_2 + b\, \cos(q_1+q_2)] \sum_{n=-\infty}^{\infty} \delta(t-n)$$

Classical: 4D symplectic maps (Froeschlé, Astron. Astrophysics, 1970, Richter et. al. Phys. Rev. E., 2014)

Quantize to get unitary Floquet propagator $\mathcal{U}$:

$$\mathcal{U} = [U_1(K_1) \otimes U_2(K_2)]\, U_{12}(b)$$

Compactify phase space as **Torus**:

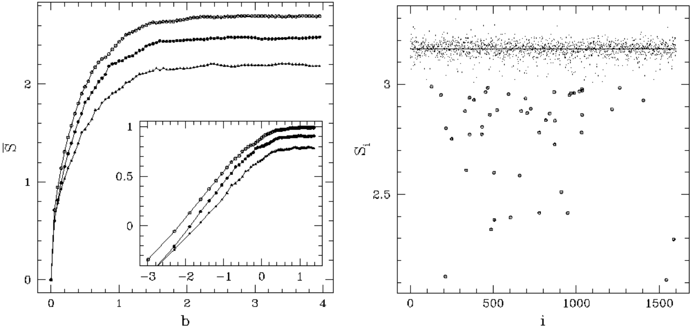
$U_{1,2}$: $N \times N$ matrices. $\mathcal{U}$: $N^2 \times N^2$

$N = 1/h_{eff}$

Entanglement in Eigenstates: $\mathcal{U}|\phi_k\rangle = \exp(i\phi_k)|\phi_k\rangle$ as a function of K_i and b??

Entanglement growth [4]

Use $K_1 = 0.1, K_2 = 0.15$ **Left: Avg. ent. for** $N = 15, 20, 25$**.**
Right: ent. across a spectrum $N = 40, b = 2.0$**.**



Saturating $\overline{S} = \log N - 1/2$

(AL, *Entangling power of quantum chaos*, Phys. Rev. E, vol. 64, 2001)

[4] Entropy is not what it used to be – Anon

Noninteracting to Interacting entanglement transitions, when **all** limits are nonintegrable.

Examples:

- Interacting particles in a quantum dot or billiard: **Circular (no) vs Stadium (yes)?**
- Spin chains/ladders each is in ETH/Random phase
- Coupled kicked rotors: K_1 and K_2 large.

(1) Noninteracting limit still has Poisson statistics and small level spacings are highly likely. (2) Resonances can dominate in the small interaction regime. (3) Eigenvalues are like that of an integrable system, eigenfunctions are randomized.

Noninteracting to Interacting entanglement transitions, when **all** limits are nonintegrable.

Examples:

- Interacting particles in a quantum dot or billiard: **Circular (no) vs Stadium (yes)?**
- Spin chains/ladders each is in ETH/Random phase
- Coupled kicked rotors: K_1 and K_2 large.

(1) Noninteracting limit still has Poisson statistics and small level spacings are highly likely. (2) Resonances can dominate in the small interaction regime. (3) Eigenvalues are like that of an integrable system, eigenfunctions are randomized.

Noninteracting to Interacting entanglement transitions, when **all** limits are nonintegrable.

Examples:

- Interacting particles in a quantum dot or billiard: **Circular (no) vs Stadium (yes)?**
- Spin chains/ladders each is in ETH/Random phase
- Coupled kicked rotors: K_1 and K_2 large.

(1) Noninteracting limit still has Poisson statistics and small level spacings are highly likely. (2) Resonances can dominate in the small interaction regime. (3) Eigenvalues are like that of an integrable system, eigenfunctions are randomized.

Noninteracting to Interacting entanglement transitions, when **all** limits are nonintegrable.

Examples:

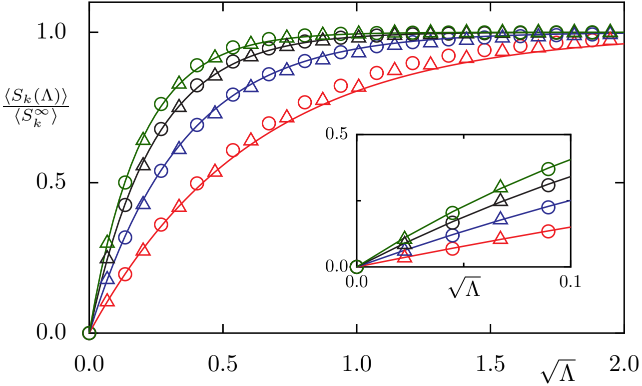
- Interacting particles in a quantum dot or billiard: **Circular (no) vs Stadium (yes)?**
- Spin chains/ladders each is in ETH/Random phase
- Coupled kicked rotors: K_1 and K_2 large.

(1) Noninteracting limit still has Poisson statistics and small level spacings are highly likely. (2) Resonances can dominate in the small interaction regime. (3) Eigenvalues are like that of an integrable system, eigenfunctions are randomized.

Case of coupled kicked rotors vs Theory

$$\Lambda[\mathcal{U}_{SM}(b)] = \frac{N^4 b^2}{32\pi^4}, \quad \langle S_k \rangle \text{ for } k = 1, 2, 3, 4.$$

$K_1 = 9, K_2 = 10, \ N = 50.$



$\mathcal{U}_{\mathrm{RMT}}$: Triangles $\mathcal{U}_{\mathrm{SM}}$: Circles

(AL, Srivastava, Ketzmerick, Bäcker, Tomsovic *Phys. Rev. E. (R) 2016*)

Universal entanglement for weakly interacting strongly chaotic systems

$$\langle S_\alpha(\Lambda)\rangle = \pi\frac{\Gamma(\alpha-1/2)}{\Gamma(\alpha)}\sqrt{\Lambda}+\text{h.o.t.}.$$

Entanglement: von Neumann and Linear entropies

$$\langle S_1\rangle = \pi^{3/2}\sqrt{\Lambda}+\text{h.o.t.},\quad \langle S_2\rangle = \frac{1}{2}\pi^{3/2}\sqrt{\Lambda}+\text{h.o.t.}$$

Nonpertubative regime: a simple law emerges

$$\langle S_\alpha(\Lambda)\rangle = \left(1-\exp\left(-\frac{c(\alpha)}{\langle S_\alpha^\infty\rangle}\sqrt{\Lambda}\right)\right)\langle S_\alpha^\infty\rangle,$$

Universal entanglement for weakly interacting strongly chaotic systems

$$\langle S_\alpha(\Lambda)\rangle = \pi\frac{\Gamma(\alpha-1/2)}{\Gamma(\alpha)}\sqrt{\Lambda} + \text{h.o.t.}.$$

Entanglement: von Neumann and Linear entropies

$$\langle S_1\rangle = \pi^{3/2}\sqrt{\Lambda} + \text{h.o.t.}, \quad \langle S_2\rangle = \frac{1}{2}\pi^{3/2}\sqrt{\Lambda} + \text{h.o.t.}$$

Nonpertubative regime: a simple law emerges

$$\langle S_\alpha(\Lambda)\rangle = \left(1 - \exp\left(-\frac{c(\alpha)}{\langle S_\alpha^\infty\rangle}\sqrt{\Lambda}\right)\right)\langle S_\alpha^\infty\rangle,$$

Universal entanglement for weakly interacting strongly chaotic systems

$$\langle S_\alpha(\Lambda)\rangle = \pi \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha)}\sqrt{\Lambda} + \text{h.o.t.}.$$

Entanglement: von Neumann and Linear entropies

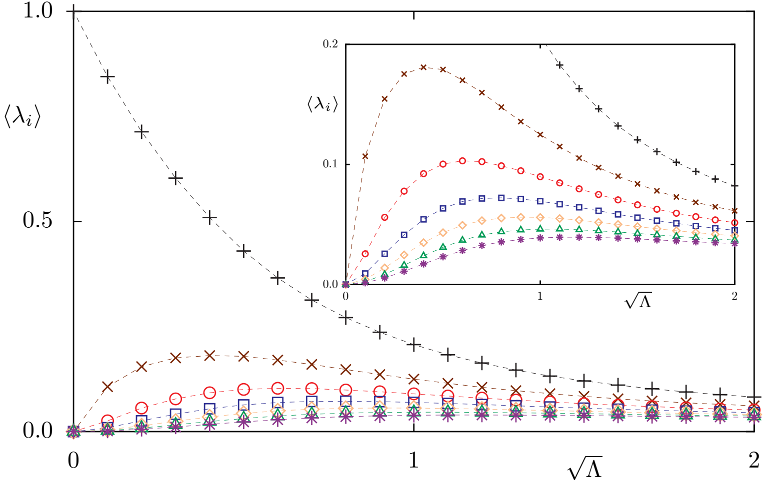
$$\langle S_1\rangle = \pi^{3/2}\sqrt{\Lambda} + \text{h.o.t.}, \quad \langle S_2\rangle = \frac{1}{2}\pi^{3/2}\sqrt{\Lambda} + \text{h.o.t.}$$

Nonpertubative regime: a simple law emerges

$$\langle S_\alpha(\Lambda)\rangle = \left(1 - \exp\left(-\frac{c(\alpha)}{\langle S_\alpha^\infty\rangle}\sqrt{\Lambda}\right)\right)\langle S_\alpha^\infty\rangle,$$

Ent. spectrum across the transition

From perturbation theory to random matrices and "thermalization":



Perturbation theory and entanglement

$$H(\epsilon) = H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B + \epsilon H_{AB}, \ |j_0 k_0\rangle \to |\phi_{j_0 k_0}\rangle$$

Largest eigenvalue of the reduced densty matrix of $|\phi_{j_0 k_0}\rangle$

$$\lambda_1 \approx 1 - \epsilon^2 \sum_{jk \neq j_0 k_0} \frac{|\langle j_0 k_0 | H_{AB} | jk \rangle|^2}{(E^0_{jk} - E^0_{j_0 k_0})^2},$$

Second largest

$$\lambda_2 \approx \epsilon^2 \frac{|\langle j_0 k_0 | H_{AB} | j_1 k_1 \rangle|^2}{(E^0_{j_1 k_1} - E^0_{j_0 k_0})^2}.$$

IF subsystems are chaotic, then $\langle j_0 k_0 | H_{AB} | j_1 k_1 \rangle := \mathrm{N}(0, v^2)$.

$$\frac{\epsilon^2 |\langle j_0 k_0 | H_{AB} | j_1 k_1 \rangle|^2}{(E^0_{j_1 k_1} - E^0_{j_0 k_0})^2} \sim \frac{\epsilon^2 v^2}{D^2} \frac{w}{s^2} = \frac{\Lambda w}{s^2}.$$

Λ: **Dimensionless transition parameter**

... and it blows up in our face

$w := \exp(-w)$.

$$\langle \lambda_2 \rangle = \Lambda \int_0^\infty \frac{w}{s^2} e^{-w} P_t(s)\, dw\, ds = \infty.$$

With the "truly nearest neighbor" $P_t(s) = 2\exp(-2s)$.
Problem: $s = 0$, needs *regularization*.

$$\frac{\Lambda w}{s^2} \mapsto \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \frac{4\Lambda w}{s^2}}}\right).$$

Accounts for binary resonances at all orders. (Also in Symmetry breaking and Nuclear spectra: French, Kota, Pandey, Tomsovic, Ann.of Phys. 1988)

... and it blows up in our face

$w := \exp(-w)$.

$$\langle \lambda_2 \rangle = \Lambda \int_0^\infty \frac{w}{s^2} e^{-w} P_t(s)\, dw\, ds = \infty.$$

With the "truly nearest neighbor" $P_t(s) = 2\exp(-2s)$.
Problem: $s = 0$, **needs** ***regularization*****.**

$$\frac{\Lambda w}{s^2} \mapsto \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \frac{4\Lambda w}{s^2}}}\right).$$

Accounts for binary resonances at all orders. (Also in Symmetry breaking and Nuclear spectra: French, Kota, Pandey, Tomsovic, Ann.of Phys. 1988)

Moments tamed

$$\langle \lambda_1 \rangle = 1 - \sqrt{\pi \Lambda} + \mathcal{O}(\Lambda),$$
$$\langle \lambda_2 \rangle = \sqrt{\pi \Lambda} + \mathcal{O}(\Lambda \ln \Lambda)$$

$$P_2 = \left\langle \sum_{j=1}^{N} \lambda_j^2 \right\rangle = 1 - \frac{\pi^{3/2}}{2} \sqrt{\Lambda} + \text{h.o.t.}$$

$$P_\alpha = \left\langle \sum_{j=1}^{N} \lambda_j^\alpha \right\rangle = 1 - C(\alpha)\sqrt{\Lambda} + \text{h.o.t.}. \quad C(\alpha) = \pi \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha - 1)}.$$

Valid for $\alpha > 1/2$

Moments tamed

$$\langle \lambda_1 \rangle = 1 - \sqrt{\pi \Lambda} + \mathcal{O}(\Lambda),$$
$$\langle \lambda_2 \rangle = \sqrt{\pi \Lambda} + \mathcal{O}(\Lambda \ln \Lambda)$$

$$P_2 = \left\langle \sum_{j=1}^{N} \lambda_j^2 \right\rangle = 1 - \frac{\pi^{3/2}}{2} \sqrt{\Lambda} + \text{h.o.t.}$$

$$P_\alpha = \left\langle \sum_{j=1}^{N} \lambda_j^\alpha \right\rangle = 1 - C(\alpha) \sqrt{\Lambda} + \text{h.o.t.}. \quad C(\alpha) = \pi \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha - 1)}.$$

Valid for $\alpha > 1/2$

Moments tamed

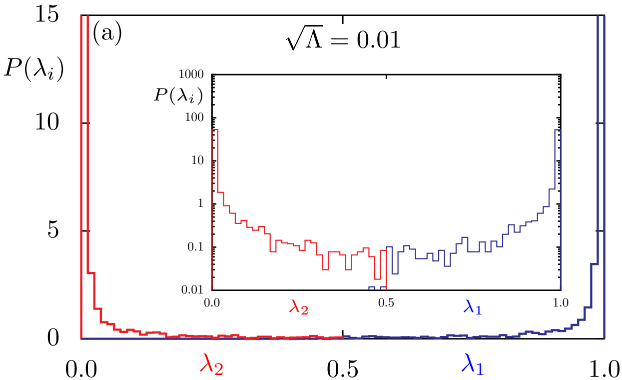
$$\langle \lambda_1 \rangle = 1 - \sqrt{\pi \Lambda} + \mathcal{O}(\Lambda),$$
$$\langle \lambda_2 \rangle = \sqrt{\pi \Lambda} + \mathcal{O}(\Lambda \ln \Lambda)$$

$$P_2 = \left\langle \sum_{j=1}^{N} \lambda_j^2 \right\rangle = 1 - \frac{\pi^{3/2}}{2} \sqrt{\Lambda} + \text{h.o.t.}$$

$$P_\alpha = \left\langle \sum_{j=1}^{N} \lambda_j^\alpha \right\rangle = 1 - C(\alpha)\sqrt{\Lambda} + \text{h.o.t.}. \quad C(\alpha) = \pi \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha - 1)}.$$

Valid for $\alpha > 1/2$

Distributions of $\lambda_{1,2}$

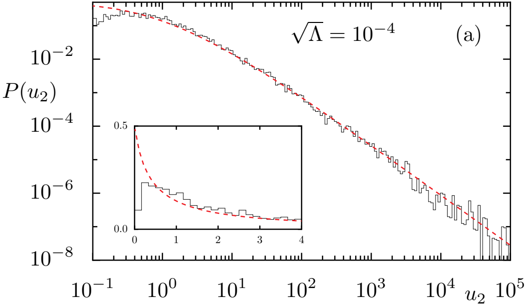


Eigenstates of coupled standard maps; $K_1 = 9, K_2 = 10,$ $N = 100$

Distributions of λ_2: Power laws

$$u_2 = \frac{\lambda_2(1-\lambda_2)}{\Lambda(1-2\lambda_2)^2} \approx \frac{\lambda_2}{\Lambda}$$

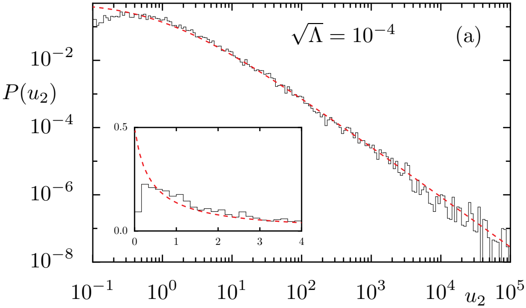
$$P(u_2) = \frac{1}{4}\int_0^\infty t^2 e^{-u_2 t^2/4} e^{-t}\, dt \to 1/u_2^{3/2}$$



Distributions of λ_2: Power laws

$$u_2 = \frac{\lambda_2(1-\lambda_2)}{\Lambda(1-2\lambda_2)^2} \approx \frac{\lambda_2}{\Lambda}$$

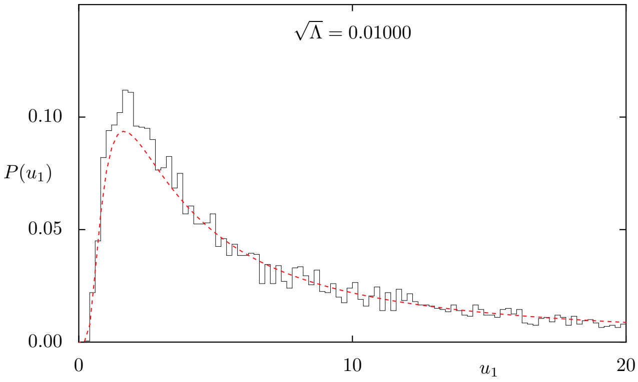
$$P(u_2) = \frac{1}{4}\int_0^\infty t^2 e^{-u_2 t^2/4} e^{-t}\, dt \to 1/u_2^{3/2}$$



Distribution of λ_1:= Lévy distribution

On Regularization: $1-\lambda_1$=**Sum of fat tailed variables** $\sim 1/x^{3/2}$. Generalized versions of the CLT leads to

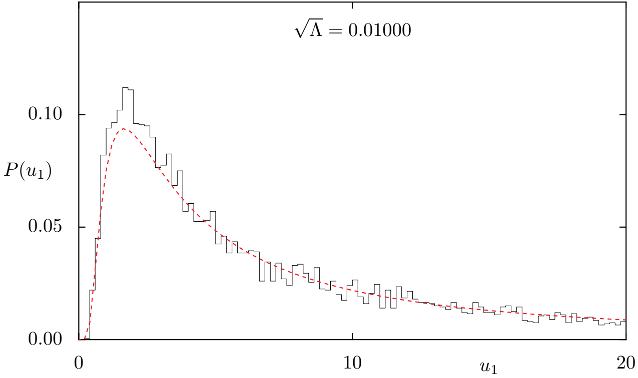
$$u_1 = \frac{\lambda_1(1-\lambda_1)}{\Lambda(1-2\lambda_1)^2} \approx \frac{1-\lambda_1}{\Lambda}. \quad P(u_1 = x) = \frac{\sqrt{\pi}}{2x^{3/2}} \exp\left(-\frac{\pi^2}{4x}\right).$$



Distribution of λ_1:= Lévy distribution

On Regularization: $1-\lambda_1$=**Sum of fat tailed variables** $\sim 1/x^{3/2}$. Generalized versions of the CLT leads to

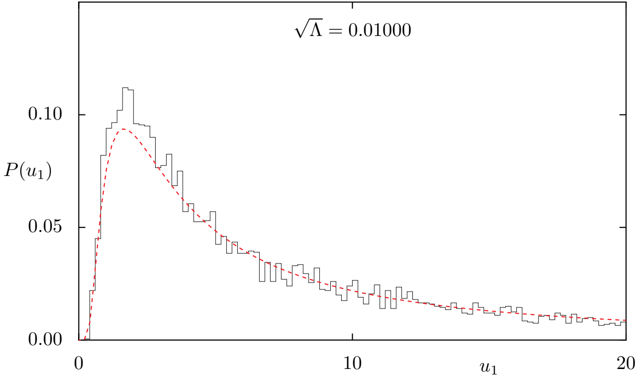
$$u_1 = \frac{\lambda_1(1-\lambda_1)}{\Lambda(1-2\lambda_1)^2} \approx \frac{1-\lambda_1}{\Lambda}.\ P(u_1 = x) = \frac{\sqrt{\pi}}{2x^{3/2}} \exp\left(-\frac{\pi^2}{4x}\right).$$



Distribution of λ_1:= Lévy distribution

On Regularization: $1-\lambda_1$=**Sum of fat tailed variables** $\sim 1/x^{3/2}$. Generalized versions of the CLT leads to

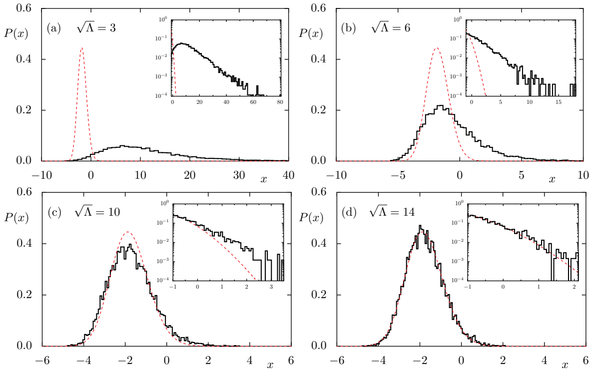
$$u_1 = \frac{\lambda_1(1-\lambda_1)}{\Lambda(1-2\lambda_1)^2} \approx \frac{1-\lambda_1}{\Lambda}. \quad P(u_1 = x) = \frac{\sqrt{\pi}}{2x^{3/2}} \exp\left(-\frac{\pi^2}{4x}\right).$$



Strong interactions: $\lambda_1 :=$ Tracy-Widom

$$\langle \lambda_1 \rangle \longrightarrow 4/N \text{ as } \Lambda \longrightarrow \gg 1, \quad x = \frac{\lambda_1 - 4/N}{2^{4/3} N^{-5/3}}$$

For $\Lambda \gg 1$ $x :=$ **Extreme value statistics of Tracy-Widom universality class**. Very late "thermalization".



Summary

- When **fully chaotic** subsystems start to weakly interact, entanglement production is rapid, analytically calculable, and governed by the same transition parameter as the level fluctuations. (AL et. al., PRE, 2016)
- In the perturbative regimes the distribution of significant Schmidt coefficients, **as well as entropies** are heavy tailed and λ_1 is Levy distributed.
- Transition of λ_1, largest eigenvalue of the RDM, is most sensitive and thermalizes last
- Coupling integrable systems, or intermediate ones?
- Many-body systems?

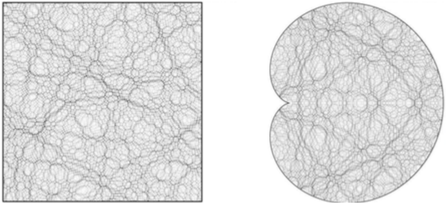
Thank you

Summary

- When **fully chaotic** subsystems start to weakly interact, entanglement production is rapid, analytically calculable, and governed by the same transition parameter as the level fluctuations. (AL et. al., PRE, 2016)
- In the perturbative regimes the distribution of significant Schmidt coefficients, **as well as entropies** are heavy tailed and λ_1 is Levy distributed.
- Transition of λ_1, largest eigenvalue of the RDM, is most sensitive and thermalizes last
- Coupling integrable systems, or intermediate ones?
- Many-body systems?

Thank you

Random waves, Normal distribution of amplitudes



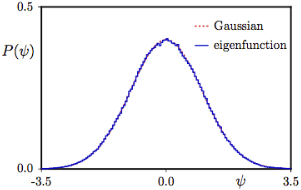


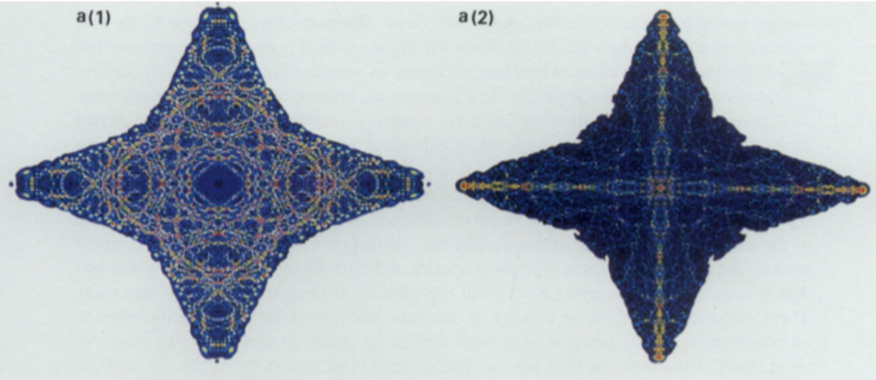
Figure 3: Example of a random wave (4) in comparison with the 6000$^{\text{th}}$ eigenfunction of the cardioid billiard (of odd symmetry). For this state one observes excellent agreement of the amplitude distribution with the expected Gaussian.

From Arnd Bäcker, Habilitationsschrift, 2007.

Other examples: Coupled quartic oscillators

$$H = \mathbf{p}^2 + x^4 + y^4 + \alpha\, x^2 y^2$$

Two highly excited eigenstate intensities in position representation.

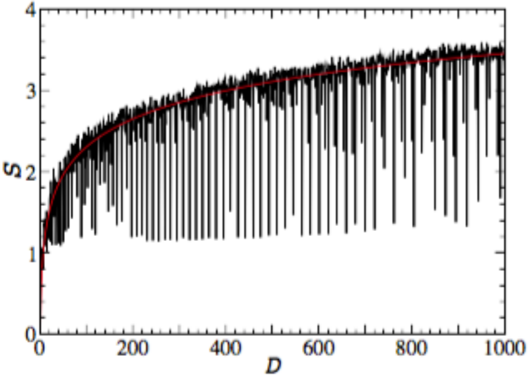


(Santhanam, Sheorey, AL, Pramana '97, Phys. Rev. E. '98)

Continuous variables: Coupled quartic oscillators

$$H = \mathbf{p}^2 + x^4 + y^4 + \alpha\, x^2 y^2. \quad N_{eff} = \sqrt{D},\ \langle S \rangle = \log N_{eff} - 1/2$$

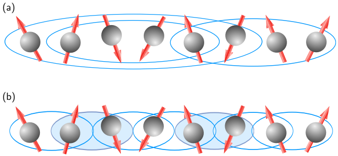
Entanglement in the first 1000 eigenstates

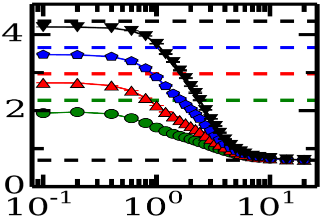


(Santhanam, Sheorey, AL, Phys. Rev. E. 2008)

MBL: Ising spins with NNN interactions

$$H = -\sum_{i=1}^{L-1} J_i \sigma_i^z \sigma_{i+1}^z + J_2 \sum_{i=1}^{L-2} \sigma_i^z \sigma_{i+2}^z + h \sum_{i=1}^{L} \sigma_i^x$$





(Ent. of $L/2$ block vs disorder strength)

Saturating $\langle S \rangle = \log 2^{L/2} - 1/2 = (L/2)\log 2 - 1/2$.

"Volume law" is typical entanglement

(Kjall, Bardarsson, Pollmann, PRL, 2014).