Kinks, Cusps, and Plateaus in the Transition Dynamics of a Bloch State

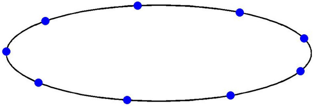
Jiang-min Zhang

Fu-Jian Normal University

July 21, 2017

Outline

- A very simple problem on a very simple model
  - transition dynamics of a Bloch state

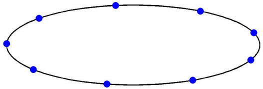


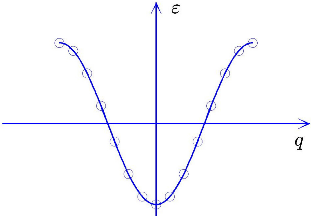
- Two scenarios
  - periodically driving case
    - kinks in the *perturbative* (weakly driving) regime
  - sudden quench case
    - cusps in the *non-perturbative* regime
- Open problems

The 1D tight binding model: an overview

The Hamiltonian

$$\hat{H}_{TBM} \quad = \quad -\sum_{n=0}^{N-1} (|n\rangle\langle n+1| + |n+1\rangle\langle n|).$$





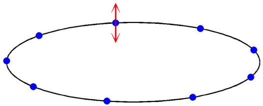
The eigenvalues and eigenstates (Bloch states)

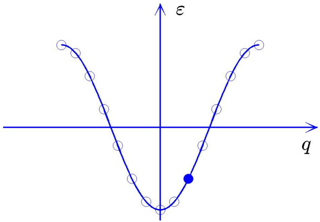
$$\begin{aligned} \varepsilon(q) &= -2\cos q, \quad q = 2\pi k/N, \\ |k\rangle &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{iqn}|n\rangle. \\ k &= 0, 1, \ldots, N-1. \end{aligned}$$

Scenario I: Local periodic driving

Initially, the particle is in some Bloch state $|\Psi(0)\rangle = |k_i\rangle$. Now start driving the system harmonically:

$$\begin{aligned} \hat{H}(t) &= \hat{H}_{TBM} + \hat{V}(t), \\ \hat{V}(t) &= U \sin \omega t |0\rangle\langle 0|. \end{aligned}$$





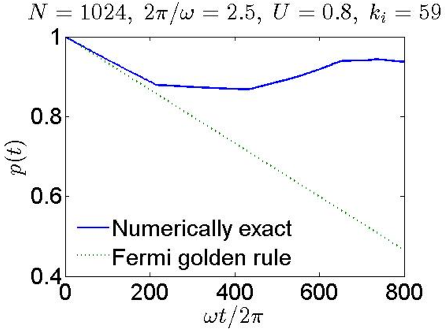
Question: how does the survival probability

$$P(t) = |\langle k_i|\Psi(t)\rangle|^2.$$

evolve?

Fermi golden rule fails

An example



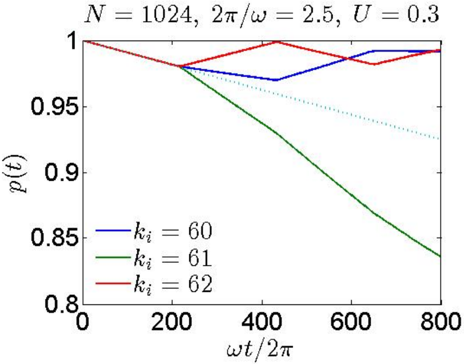
discrepancy is clear

But the solid curve is a little bit weird

Fermi golden rule **breaks down** and the problem has to be solved **non-perturbatively** if the driving amplitude U is large.

Kinks and Sudden bifurcation

A relatively weak driving (U small):

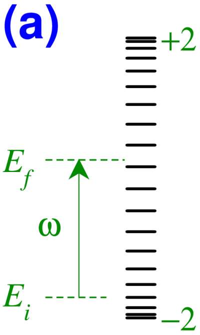


Three successive Bloch state $|k_i\rangle$

Dotted line: Fermi golden rule

- Kinks appear periodically in time.
- Piece-wise linear
- Initial level dependence (beyond some critical time)

Essential features of the model



- Equally spaced spectrum (locally)

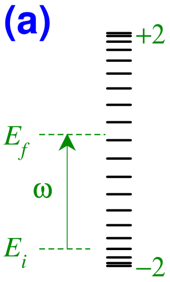
$$E_{n+1} - E_n = \Delta.$$

- Equal coupling

$$|\langle k_2|\hat{V}|k_1\rangle| = g = \frac{U}{N}.$$

- Small U means it is a *perturbative* effect.

A virtual spectrum



The transition probability, in the 1st order approximation, is

$$\begin{aligned} P_{trans} &= 1 - P(t) \\ &\simeq 4g^2 \sum_m \frac{\sin^2[(E_m - E_f)t/2]}{(E_m - E_f)^2}. \end{aligned}$$

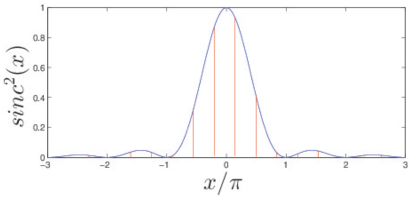
The near resonant levels dominate. Hence,

$$P_{trans} \simeq 4g^2 \sum_{m\in\mathbb{Z}} \frac{\sin^2[(E_m - E_f)t/2]}{(E_m - E_f)^2},$$

with a virtual spectrum $E_m = m\Delta$, $m \in \mathbb{Z}$.

- equally spaced
- extending from $-\infty$ to $+\infty$

Uniform sampling of the $\mathrm{sinc}^2\, x$ function



Introducing the rescaled, dimensionless time $T \equiv \Delta t/2$,

$$P_{trans} = 1 - P(t) \simeq \left(\frac{4g^2}{\Delta^2}\right) W_\alpha(T),$$

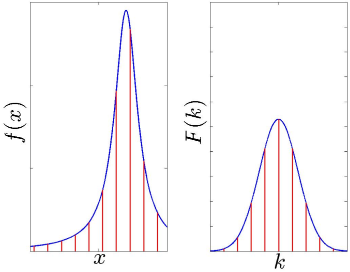
with

$$W_\alpha(T) \equiv T^2 \sum_{m\in\mathbb{Z}} \mathrm{sinc}^2[(m-\alpha)T]$$

with $\alpha = (E_{final} - E_n)/(E_{n+1} - E_n)$, $E_n < E_{final} < E_{n+1}$.

Poisson summation formula I

$$\sum_{m=-\infty}^{+\infty} f(a+mT) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} F\left(\frac{2\pi m}{T}\right) \exp\left(\frac{i2\pi m a}{T}\right).$$



Periodic sampling in **real** space

$\iff$ Periodic sampling in **momentum** space (simpler?)

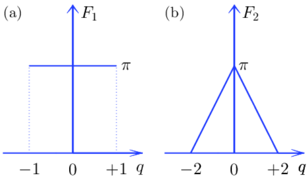
Poisson summation formula II

For our specific problem, we sample the sinc^2 function,

$$\begin{aligned} W_\alpha(T) &\equiv T^2 \sum_{m\in\mathbb{Z}} \mathrm{sinc}^2[(m-\alpha)T] \\ &= T \sum_{n\in\mathbb{Z}} F_2\left(\frac{2\pi n}{T}\right) \exp\left(-i2\pi n\alpha\right), \end{aligned}$$

with

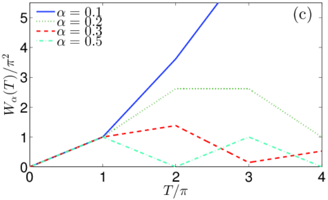
$$\begin{aligned} F_1(q) &\equiv \int_{-\infty}^{+\infty} dx e^{-iqx} \left(\frac{\sin x}{x}\right), \\ F_2(q) &\equiv \int_{-\infty}^{+\infty} dx e^{-iqx} \left(\frac{\sin x}{x}\right)^2, \end{aligned}$$



Both F_1 and F_2 are of **finite** supports (Paley-Wiener theorem)

Only a **finite** number of terms contribute to the summation

A piece-wise linear function of time



- If $0 < T \leq \pi$,

$$W_\alpha(T) = \pi T.$$

  Fermi golden rule. No α dependence.

- If $m\pi < T \leq (m+1)\pi$,

$$W_\alpha(T) \quad = \quad \pi T \sum_{n=-m}^{m} \exp(in\theta) - \sum_{n=1}^{m} 2\pi^2 n \cos(n\theta).$$

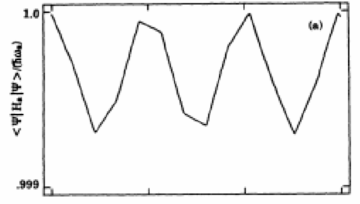
  The slope of this linear function is both m-dependent and α-dependent ($\theta = 2\pi\alpha$)

- The period is $2\pi/\Delta$, the so-called Heisenberg time.

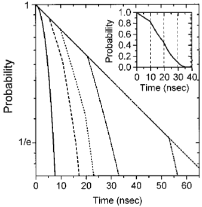
But kinks were discovered long ago in quantum optics

Spontaneous decay of a two-level atom in a multi-mode cavity

$$H = \omega\sigma_z + \sum_{m\in\mathbb{Z}} m\Delta \hat{a}_m^\dagger \hat{a}_m + g\sum_{m\in\mathbb{Z}} (\hat{a}_m^\dagger\sigma^- + \hat{a}_m\sigma^+)$$



J. Parker and C. R. Stroud, Jr., Phys. Rev. A **35**, 4226 (1987).

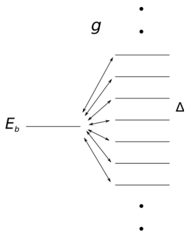


H. Giessen, J. D. Berger, G. Mohs, P. Meystre, Phys. Rev. A **53**, 2816 (1996).

They just attribute the kinks to interference, in a hand-waving way.

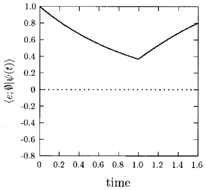
An exactly solvable model and more robust kinks

$$
\begin{aligned}
H &= E_b|b\rangle\langle b| + \sum_{m\in\mathbb{Z}} m\Delta|m\rangle\langle n| + g\sum_{m\in\mathbb{Z}}\left(|b\rangle\langle m| + |m\rangle\langle b|\right) \\
\psi(0) &= |b\rangle, \quad \langle b|\psi(t)\rangle = \textcolor{red}{?}
\end{aligned}
$$



Stey and Gibberd, Physica **60**, 1 (1972).

Realization: a single two-level atom in a multi-mode optical cavity



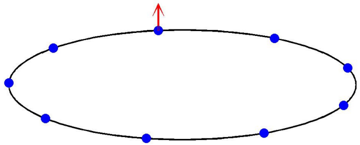
Ligare and Oliveri, Am. J. Phys. **70**, 58 (2002).

JMZ and Y. Liu, *Fermi's golden rule: its derivation and breakdown by an ideal model*, Eur. J. Phys. **37**, 065406 (2016).

Scenario II: Sudden quench

Initially, the particle is in some Bloch state $|\Psi(0)\rangle = |k_i\rangle$. Now turn on the potential at site $n = 0$ suddenly:

$$\hat{H}(t > 0) \;=\; \hat{H}_{TBM} + U|0\rangle\langle 0|.$$



Question: how do the **survival probability**

$$P_i(t) \;=\; |\langle +k_i|\Psi(t)\rangle|^2.$$

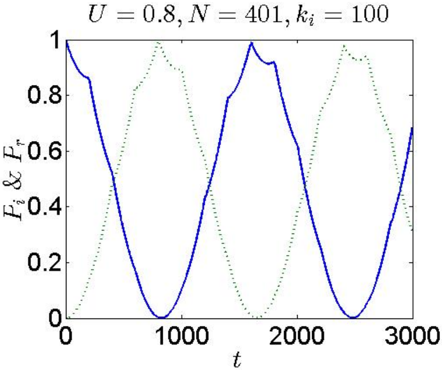
and the **reflection probability**

$$P_r(t) \;=\; |\langle -k_i|\Psi(t)\rangle|^2.$$

evolve?

Cusps!

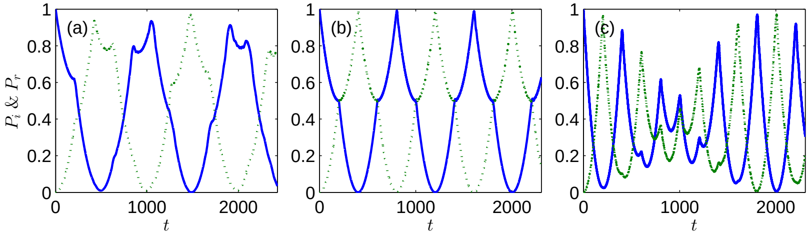
Subtle structures in the overall picture of Rabi oscillation



Apparently a non-perturbative phenomenon

- here, P_i can drop to zero constantly
- while in the previous case, P_i should be close to 1 for the kinks to be sharp

More examples



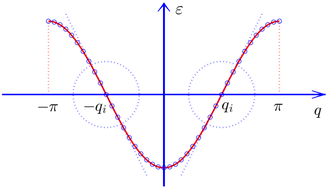
(a) $N = 401$, $k_i = 80$, $\quad U = 1.5$;

(b) $N = 401$, $k_i = 100$, $U = 2$, especially regular;

(c) $N = 401$, $k_i = 100$, $U = 12$, especially sharp.

A linearized model

Key numerical observation: Only those Bloch states with energy $\varepsilon(q) \simeq \varepsilon(q_i)$ participate significantly in the dynamics.



Two groups of Bloch states

- Right going: $|R_n\rangle = |+k_i + n\rangle$;
- Left going: $|L_n\rangle = |-k_i - n\rangle$, $-\infty \le n \le +\infty$.

The dispersion curve at around $q = \pm k_i$ can be linearized.

Similar ideas in the Luttinger liquid theory.

Taking into account the parity symmetry

A new basis (even and odd states)

$$|A_n^{\pm}\rangle \;=\; \frac{1}{\sqrt{2}}(|R_n\rangle \pm |L_n\rangle).$$

In the yet to be diagonalized subspace of $\{|A_n^+\rangle\}$, the matrix elements of $\hat{H}_0 = \hat{H}_{TBM}$ and $\hat{H}_1 = U|0\rangle\langle 0|$ are

$$\langle A_n^+|\hat{H}_0|A_n^+\rangle = n\Delta, \quad \langle A_{n_1}^+|\hat{H}_1|A_{n_2}^+\rangle = 2g, \; \forall\, n_1 \;\&\; n_2.$$

Initially, $|\Psi(0)\rangle = |R_0\rangle = \frac{1}{\sqrt{2}}(|A_0^-\rangle + |A_0^+\rangle)$. Some time later,

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}|A_0^-\rangle + \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\psi_n(t)|A_n^+\rangle,$$

since $|A_0^-\rangle$ is already an eigenstate of $\hat{H}$. The quantity wanted is $\psi_0(t)$, in terms of which the probabilities P_i and P_r are

$$P_i = \frac{1}{4}\left|1+\psi_0\right|^2, \quad P_r = \frac{1}{4}\left|1-\psi_0\right|^2.$$

Solution strategy

The Schrödinger equation for the ψ's is

$$i\frac{\partial}{\partial t}\psi_n = n\Delta\psi_n + 2g\sum_{m=-\infty}^{\infty}\psi_m. \tag{1}$$

Note that the term in the summation is independent of n.
Therefore, we define the collective, auxiliary quantity

$$S(t) = \sum_{m=-\infty}^{\infty}\psi_m(t). \tag{2}$$

The solution of (1) is then

$$\psi_n(t) = e^{-in\Delta t}\delta_{n,0} - i2g\int_0^t d\tau e^{-in\Delta(t-\tau)}S(\tau). \tag{3}$$

Plugging this into (2), we get an integral equation of S,

$$S(t) = 1 - i2g\int_0^t d\tau\left(\sum_{n=-\infty}^{\infty}e^{-in\Delta(t-\tau)}\right)S(\tau). \tag{4}$$

Solution strategy continued

We have (again by the Possion summation formula)

$$\sum_{n=-\infty}^{+\infty} e^{-in\Delta(t-\tau)} = T \sum_{n=-\infty}^{+\infty} \delta(t-\tau-nT),$$

where the period $T \equiv 2\pi/\Delta$ is the Heisenberg time. The integral equation is then easily solve. We have

$$S(t) = \frac{1}{1+igT},$$

which is a constant, for $0 < t < T$. In turn, we get

$$\psi_0(t) = 1 - \frac{i2gt}{1+igT} = \frac{1-i2g(t-T/2)}{1+igT},$$

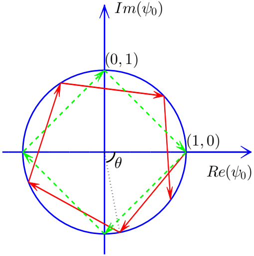
which is linear in t. We note that as $t \to T^-$,

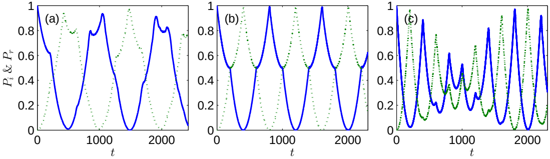
$$\psi_0(t) \to \frac{1-igT}{1+igT} = e^{-i\theta}.$$

After one period, ψ_0 returns to its initial value, except for a phase accumulated. The same evolution then repeats!

Trajectory of ψ_0—A ball in a circular billiard

Irrational rotation





- The overall picture is Rabi oscillation
- Instead of piece-wise linear, now it is piece-wise quadratic
- When the cusps show up, $P_i + P_r = 1$.

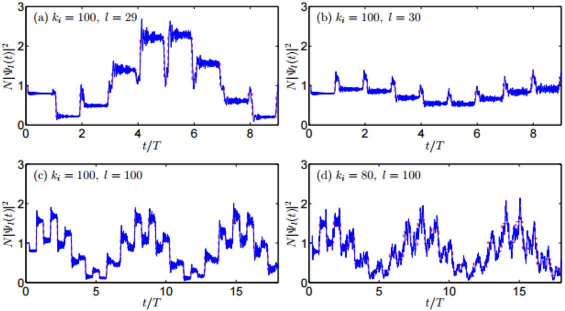
Survival probability
$P_i(t) = \frac{1}{4}\left|1+\psi_0\right|^2$

Reflection probability
$P_r(t) = \frac{1}{4}\left|1-\psi_0\right|^2$

How is it in the real space? Sudden jumps between plateaus

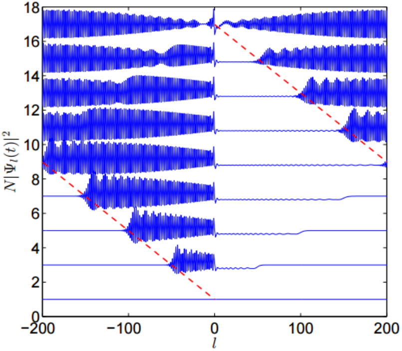
So far, the discussion is confined to the momentum space.

Probability density at an arbitrary site $|\psi_l(t)|^2$:



$N = 401$, $U = 1$.

Scattering waves



- The wave fronts are well shaped.
- $|\psi_l|^2$ jumps when a wave front passes by.

Summary

- Transition dynamics of a Bloch state
  - perturbative kinks in the weak periodical driving scenario
    - a crystal-clear derivation of the Fermi golden rule
  - non-perturbative cusps in the sudden quench scenario
    - reminiscent of the so-called "dynamical phase transition"
- Lesson: simple models are not easy
  - especially when it comes to dynamics
- Many open problems
  - what if it is a Fermi sea?

JMZ and Y. Liu, Eur. J. Phys. 37, 065406 (2016).

JMZ and H. T. Yang, EPL 114, 60001 (2016).

JMZ and H. T. Yang, EPL 116, 10008 (2016).