# Signature of Ground-State Quantum Phase Transition far from equilibrium

## OQS'17, ICTS

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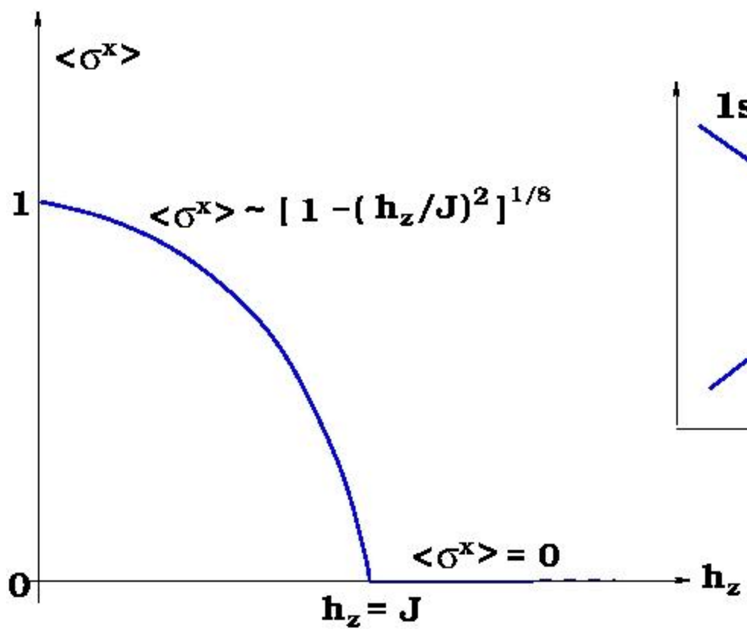
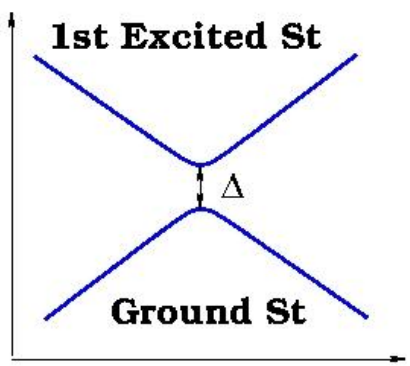
# Continuous Quantum Phase Transition (QPT): A Cursory Recap (T=0)

$$H_{Ising} = -J\sum_{i}^{L}\sigma_i^x\sigma_{i+1}^x - h_z\sum_{i=1}^{L}\sigma_i^z$$

Phase Transition at T = 0 ⟷ Transition in the Ground State Properties

$\lambda \rightarrow$ *Coupling tuned to drives the transition* ($h_z$ in this case)




Usual Non-analyticities

$$\tau \sim 1/\Delta = \frac{1}{|J-h_z|} \Rightarrow \nu z = 1$$

$$m^z \sim |h_z - J|^{1/8} \;\text{ as }\; |h_z| \rightarrow J$$

$$\xi \sim 1/|J-h_z|^{\nu} \sim \frac{1}{|J-h_z|} \Rightarrow \nu = z = 1$$

# These Non-analyticities (as the function of λ) Disappear at Finite Energy Densities, e.g., at $T \neq 0$

For the Ising Chain at Low T in the Ordered Phase ($T \ll |J-h_z|$, $J-h_z > 0$):

$$\xi^{-1} \sim \sqrt{T|J-h_z|}e^{-|J-h_z|/T}$$
$$\tau^{-1} \sim Te^{-|J-h_z|/T}$$
$$m^z = 0$$

These Non-analyticities are Removed at $T \neq 0$

And …

$$F(\lambda = h^z/J) = -\frac{L}{\beta}\left[\ln 2 + \int_0^\pi dk \ln\cosh\left(\frac{1}{2}\beta\omega_k(\lambda)\right)\right]\;;\;\omega_k(\lambda) = J\sqrt{(\cos k + \lambda)^2 + \sin^2 k}$$

⇓

No Singularity in $F$ (hence in *any local observable*) w.r.t. λ

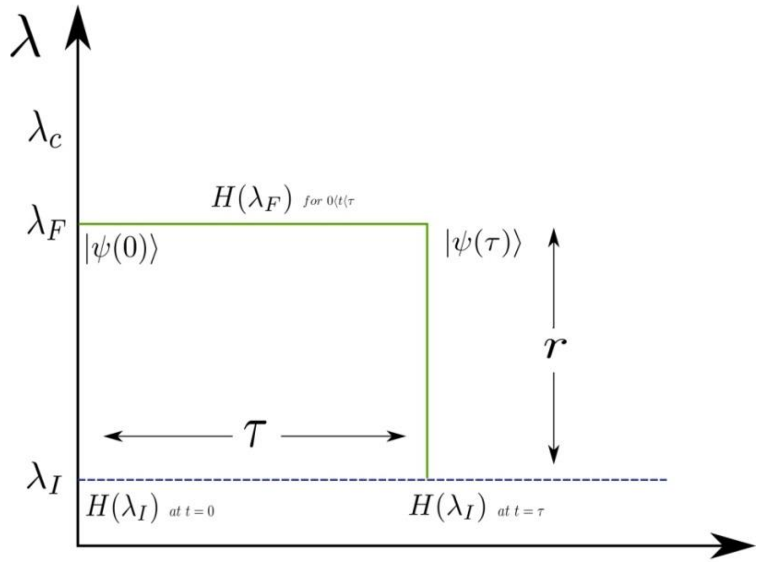
# Two Fundamental (Related) Issues:

- Signatures of a QPT in other Excited States, e.g., in Out-of-Equilibrium Behaviour of a System Driven Across Criticality. Particularly, is there one which can help *locating* a critical point?

- Signature of QPT on Excited States.

# Generating Finite Energy Density States via Finite Quench: Pumping in Finite Energy Density

S. Bhttacharyya, S. Dasgupta, AD, *Sci. Rep.* (2015).


- Initial State : $|\psi(0)\rangle$ = ground state of $H(\lambda_I)$

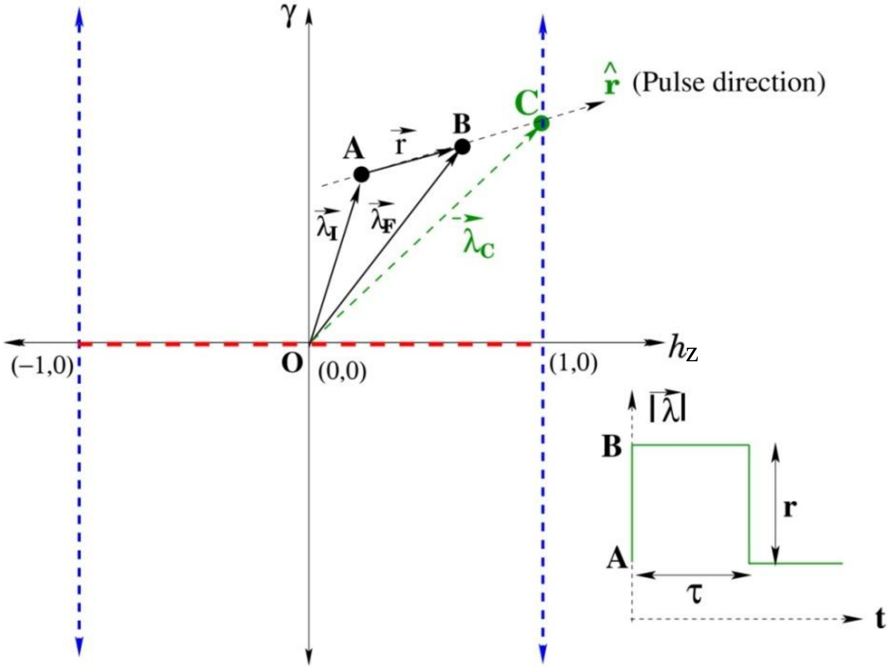
- Evolved with $H(\lambda_F)$ from t = 0 to t = τ (in principle, take $\tau \to \infty$ )

- Measured: Any local correlator that does not commute with $H(\lambda_F)$ at t = τ, for example, we could measure $H(\lambda_I)$

# The XY Chain in Transverse Field as a Test Case:

S. Bhttacharyya, S. Dasgupta, AD, *Sci. Rep.* (2015).

$$H = -\frac{1}{2}\left[(1+\gamma)\sum_{i=1}^{L}\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sum_{i=1}^{L}\sigma_i^y\sigma_{i+1}^y\right] - h_z\sum_{i=1}^{L}\sigma_i^z$$



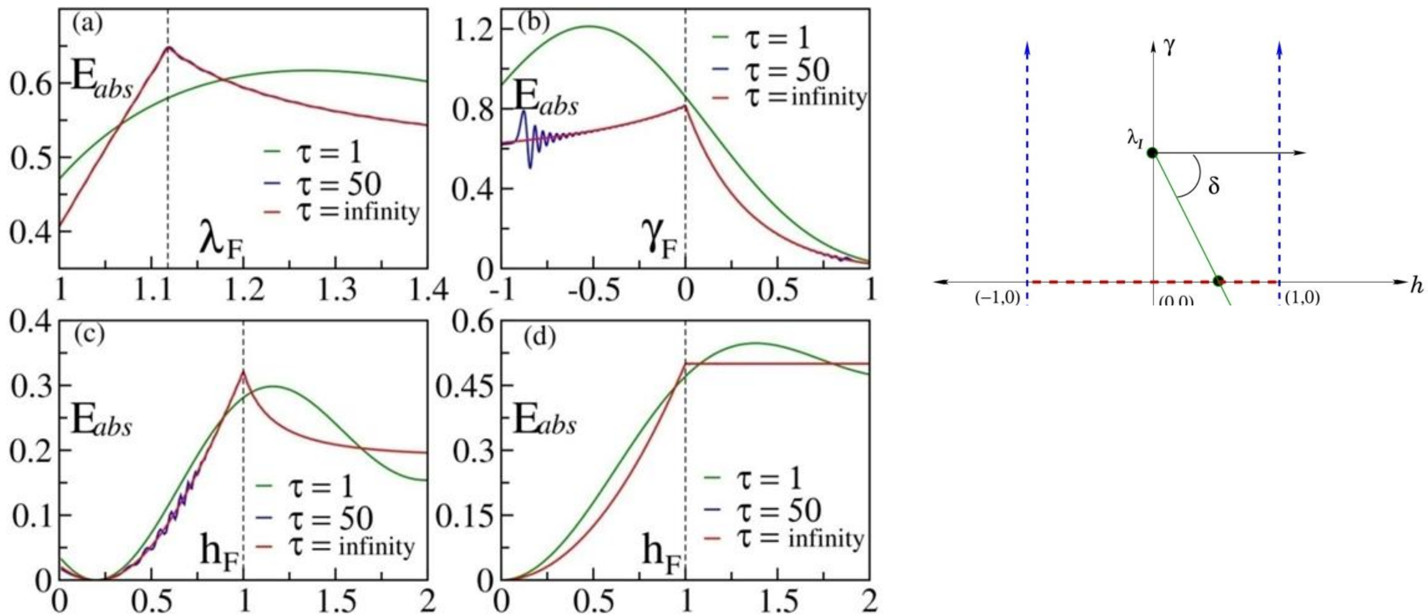

**Some General Properties of $\lim_{\tau\to\infty}|\psi(\tau,\lambda_F)\rangle$**

- It has *finite energy density* w.r.t. the Ground State of $H_F$.
- It has *extensive Entanglement Entropy*.
- *Completely Disordered (Paramagnetic)*: $\langle\sigma^x\rangle = \langle\sigma^y\rangle = 0$ for all $\lambda_F$

# For XY-Chain in Transverse Field

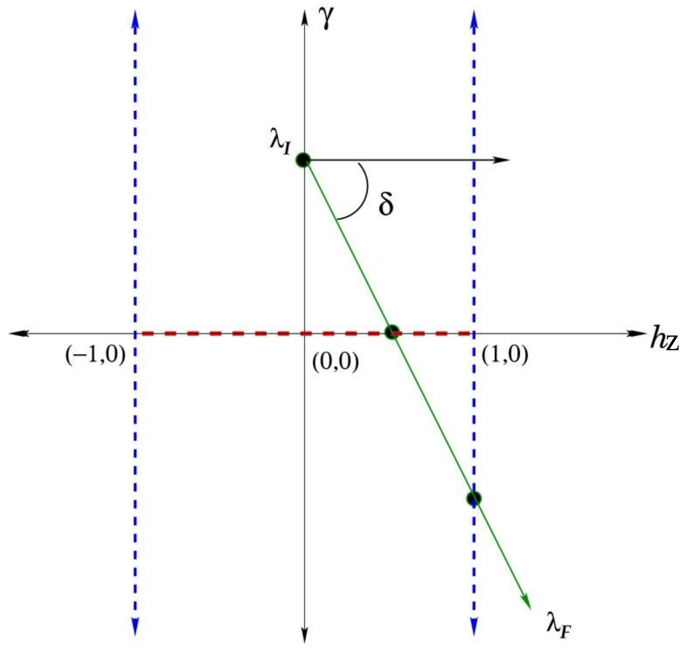
We Measure Energy Absorption after returning to $\lambda = \lambda_I$:

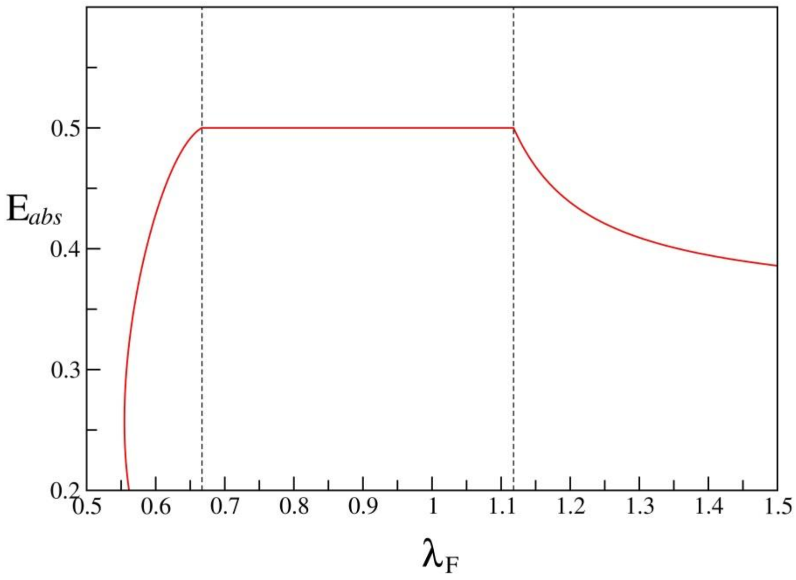
$$E_{abs}(\lambda_F) = \lim_{\tau\to\infty} \langle\psi(\tau,\lambda_F)|H_I|\psi(\tau,\lambda_F)\rangle - \langle\psi(0)|H_I|\psi(0)\rangle$$




(a) Starting from $h_I^z = 0, \gamma = 1,$ in the direction $\delta = \tan^{-1}(-1/2)$.
(b) Starting from $h_I^z = 0.2, \gamma = 1.5,$ along the $\gamma -$ axis.
(c) Starting from $h_I^z = 0.2, \gamma = 0.5,$ along the $\mathrm{h^z} -$ axis.
(d) Starting from $h_I^z = 0.2, \gamma = 1,$ along the $\mathrm{h^z} -$ axis (the Ising case)

# Crossing two Critical Lines:




$$\delta = \tan^{-1}(-3/2)$$

# In Momentum Space …

$$H = -\frac{1}{2}\left[(1+\gamma)\sum_{i=1}^{L}\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sum_{i=1}^{L}\sigma_i^y\sigma_{i+1}^y\right] - h_z\sum_{i=1}^{L}\sigma_i^z$$

(JWT + FT)

$$|\psi(t)\rangle = \bigotimes_{k>0}|\psi_k(t)\rangle; \quad |\psi_k(t)\rangle = u_k|0_k, 0_{-k}\rangle + v_k|1_k, 1_{-k}\rangle.$$

$$i\frac{d}{dt}\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} E_k & i\Delta_k \\ -i\Delta_k & -E_k \end{pmatrix}\begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

$$E_k = h_z + \cos k; \quad \Delta_k = \gamma \sin k$$

$$E_{abs} = \frac{1}{\pi}\int_0^{\pi}\frac{\tilde{W}(h_I, \gamma_I, h_F, \gamma_F)}{\sqrt{\tilde{Q}(h_I, \gamma_I)\tilde{Q}(h_F, \gamma_F)}}dk$$

# In Complex Plane …

$$z = e^{ik}$$

$$E_{\text{abs}} = \frac{i}{4\pi}\mathcal{A}\oint_{C}\frac{W(z,h_I,\gamma_I,h_F,\gamma_F)}{\sqrt{Q(z,h_I,\gamma_I)}\;\; z^2 Q(z,h_F,\gamma_F)}dz$$

$$\mathcal{A} = \frac{1}{(1-\gamma_F^2)\sqrt{1-\gamma_I^2}},$$

$$W = (z^2-1)^2\left[\gamma_I(z^2+2h_F z+1)-\gamma_F(z^2+2h_I z+1)\right]^2,$$

$$Q = (z-z_1)(z-z_2)(z-\frac{1}{z_1})(z-\frac{1}{z_2}) \quad \text{with}$$

$$z_{1,2} = \frac{1}{1-\gamma}[-h \pm \sqrt{h^2+\gamma^2-1}].$$

Crossing the *h* = 1 critical line:

$h_F < 1$ : A, B = $1/z_1^F$, $z_1^F$

$h_F > 1$ : A, B = $z_1^F$, $1/z_1^F$

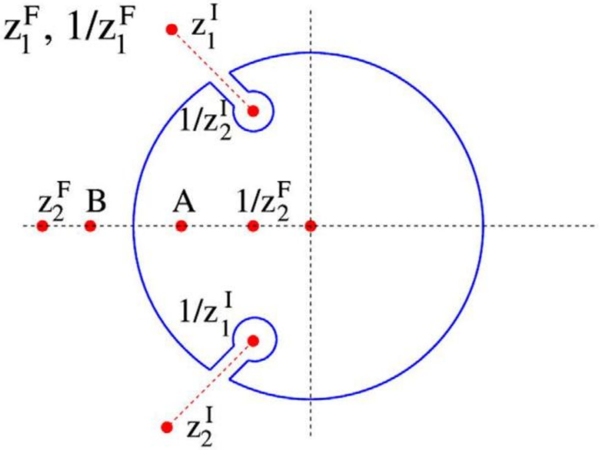
**Movement of the 4 Poles as $h_F$ is changed:**

(1) $1/z_2^F$ is always inside the contour.

(2) $z_2^F$ is always outside the contour.

(3) $1/z_1^F$ is inside for $h_F < 1$ and outside for $h_F > 1$

(4) Opposite happens for the pole $z_1^F$




$0 < h_I < 1$

$0 < \gamma_I < 1$

# The Discontinuity Δ …

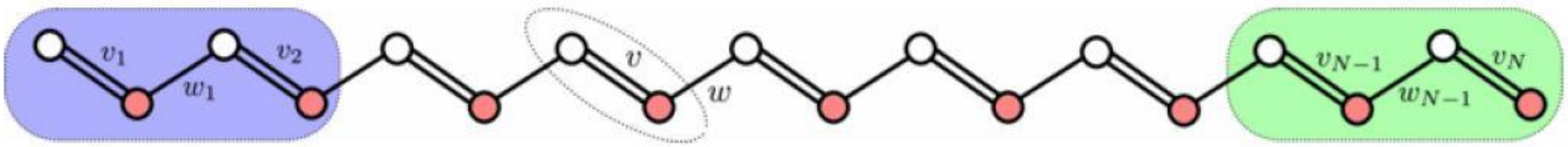
$$\Delta = \lim_{\epsilon \to 0} \left[ \left( \frac{\partial E_{\text{abs}}}{\partial \lambda_F} \right)_{\lambda_c - \epsilon} - \left( \frac{\partial E_{\text{abs}}}{\partial \lambda_F} \right)_{\lambda_c + \epsilon} \right]$$

This is easy to calculate – requires only the residues for the (discontinuously) changing pole configurations within the contour at the critical point approaching it from either sides.

$$\Delta = \frac{(1 - h_I)}{|\gamma_I + m(1 - h_I)|\sqrt{1 + m^2}}$$

$$m = \tan\left( \frac{\gamma_F - \gamma_I}{h_F - h_I} \right)$$

# Locating Topological Transitions: The Su-Schrieffer-Heeger Model




$$H_{SSH} = -\sum_{l=1}[c^{\dagger}_{l,A}c_{l,B} + \lambda c^{\dagger}_{l,B}c_{l+1,A} + h.c.] \quad (v = 1, w = \lambda)$$

$$H_{SSH}(\lambda) = \bigoplus_k H_k(\lambda), \text{ where } \mathrm{H_k}(\lambda) = \vec{d}_{\mathrm{k}}.\vec{\sigma}, \text{ with}$$

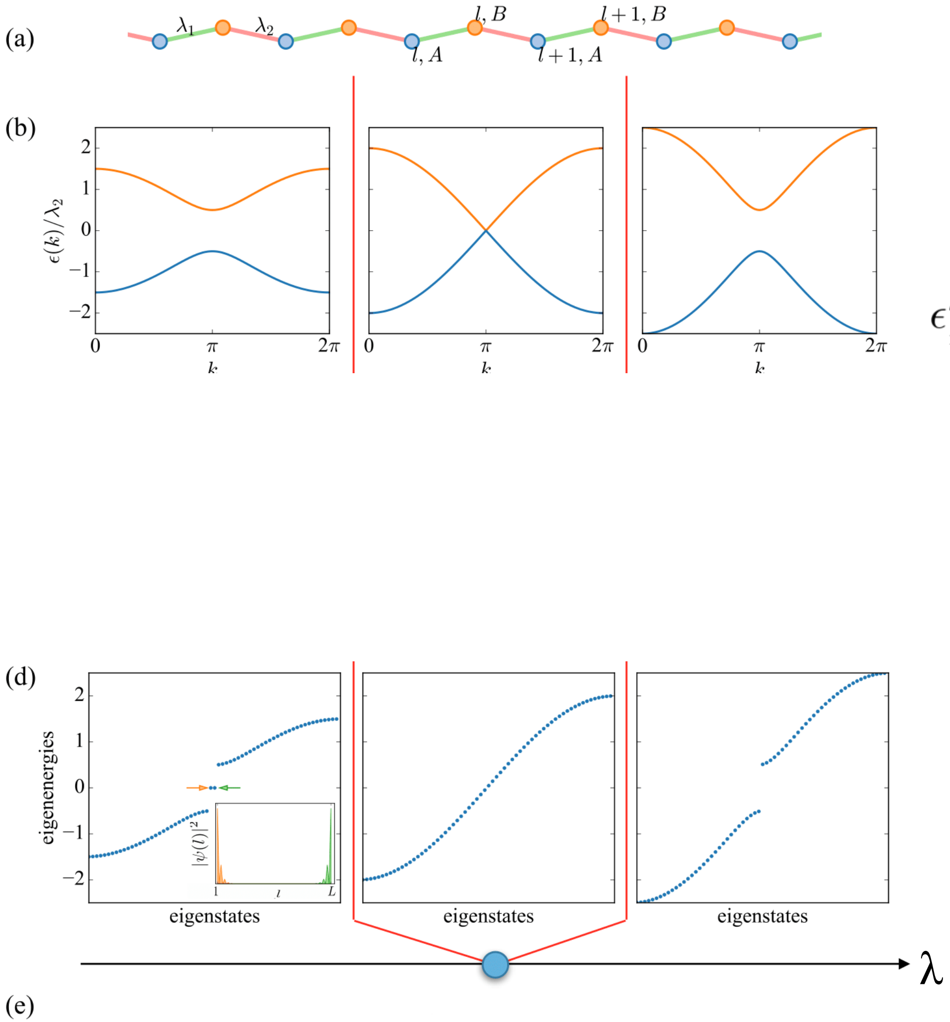
$$d_k^x = 1 + \lambda \cos k$$

$$d_k^y = \lambda \sin k$$

$$d_k^z = 0$$

Critical Points: $\lambda = \pm 1, \;\; k_c = \pi, 0$

# Topological Transition in the SSH model




λ = 1

## Bands

$$\epsilon_{\pm}^{k} = \pm|\overrightarrow{d_k}| = \pm\sqrt{1+\lambda^2+2\lambda\cos k}$$

## Bulk Winding Number

Following the Complex Number

$$\begin{aligned} h(k) &= d_k^x + i d_k^y \\ d_k^x &= 1+\lambda\cos k;\ d_k^y = \lambda\sin k \end{aligned}$$

as k change from –π to +π: depicts a circle with radius λ and center at $d_k^x = 1, d_k^y = 0$.

Winding Number $\mathcal{W}$ = How many times it goes around the origin.

$$\mathcal{W} = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk}\log h(k)$$

:)

# Quenching the SSH …

**Initial States:**

S. Roy, R. Moessner, AD, *Phys Rev. B* (**R**) 2017.

(a) Ground State of the Initial Hamiltonian

(b) Thermal State:

$$\rho(t=0) = \otimes \prod_k \rho_{I,k},$$

$$\text{with } \rho_{\mathrm{I,k}} = W_{-,k}|g_{I,k}\rangle\langle g_{I,k}| + W_{+,k}|e_{I,k}\rangle\langle e_{I,k}|$$

$$W_{\pm,k} = e^{-\beta\epsilon_{\pm}^{k}(I)}/(e^{-\beta\epsilon_{-}^{k}(I)} + e^{-\beta\epsilon_{+}^{k}(I)})$$

**Final State (Diagonal density matrix):** $\rho(t\to\infty) = \otimes \prod_k \rho_{F,k}$

$$\rho_{\infty,k} = \frac{1}{2}\left[I_2 + (W_{+,k} - W_{-,k})\frac{\mathbf{d}_{I,k}\cdot\mathbf{d}_{F,k}}{d_{I,k}d_{F,k}^2}\mathbf{d}_{F,k}\cdot\sigma\right]$$

**Observables:** ***Local Observables in the Bulk*** (also local in quasi-particle):

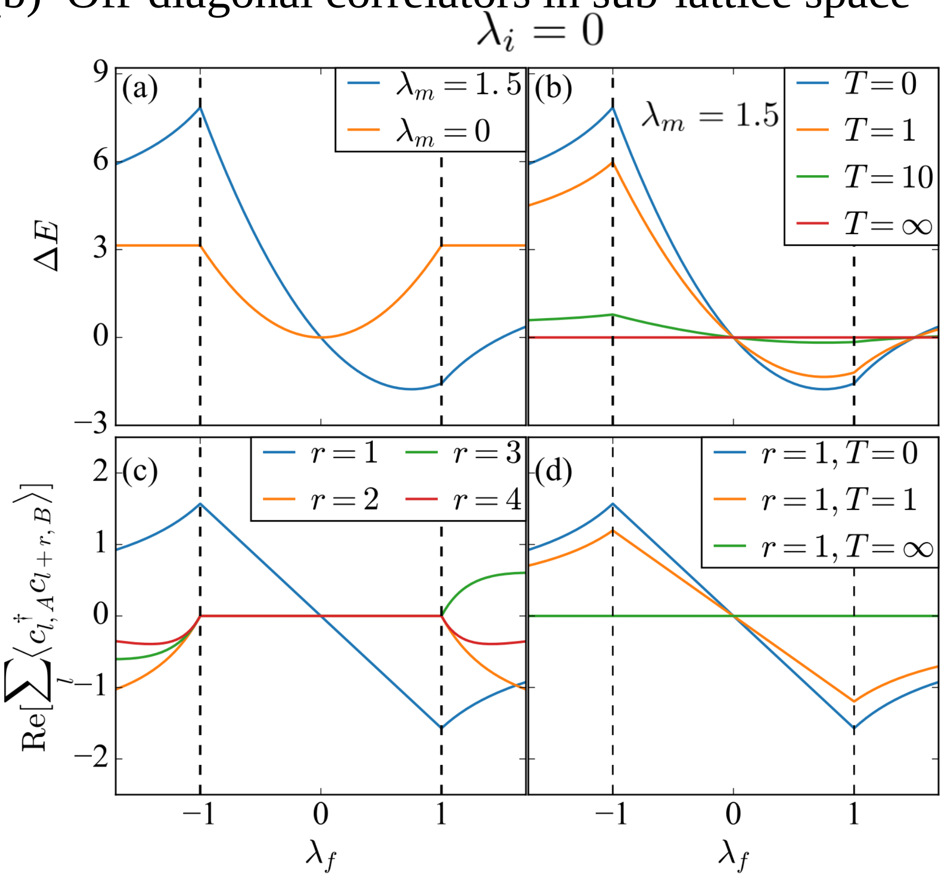
$$\langle\mathcal{O}\rangle = \frac{1}{2\pi}\int dk\ \mathrm{Tr}[\rho_{\infty,k}\hat{\mathcal{O}}_k]$$

# SSH Quench …

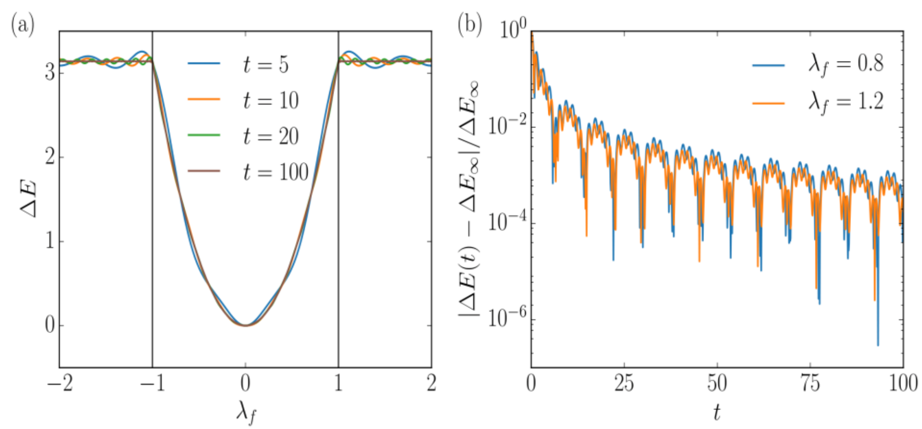
For example, we measure

(a) the energy difference between the initial and final states, with respect to a Hamiltonian corresponding to any point in parameter space: $\Delta E = [H(\lambda_m)\rho_\infty] - [H(\lambda_m)\rho_0]$

(b) Off-diagonal correlators in sub-lattice space




Finite time evolution results



# Signature in the GGE

Generalized Thermalization –
After quench, Local Observables can be described by Generalized Gibbs' Ensemble:

$$\lim_{t\to\infty}\langle\psi(t)|\mathcal{O}|\psi(t)\rangle = [\rho_{GGE},\mathcal{O}]$$

where,

$$\rho_{GGE} = \frac{1}{\mathcal{Z}} e^{-\sum_n \beta_n \mathcal{I}_n}$$

$\mathcal{I}_n$ → Local Conserved Quantities

$\beta_n$ → Lagrange Multipliers

Where does the non-analyticity enter in $\rho_{GGE}(\lambda_F)$ at $\lambda_c$ ?

e.g., for Ising model:

$$\beta_n \sim \frac{2}{n}\left[\pm\frac{1}{\varepsilon_{k=0}} + \frac{(-1)^n}{\varepsilon_{k=\pi}}\right]; \quad n \gg 1$$

where, "+" in the first term holds for quenches within the same phase, while " - " sign applies for quenches between phases.

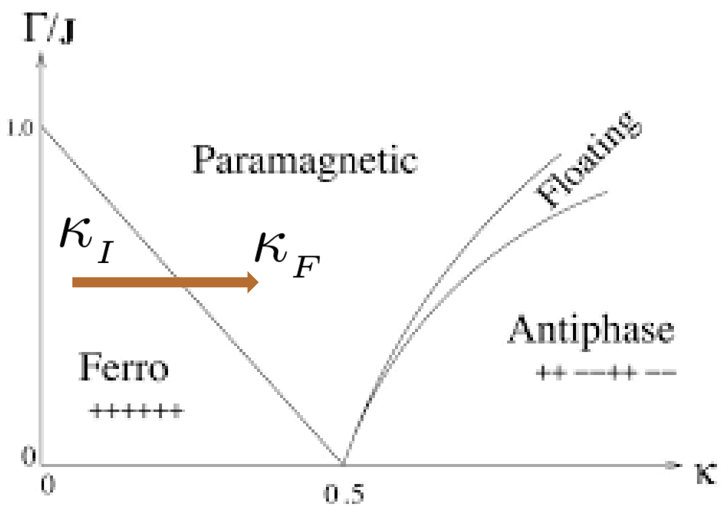
# Signature in Non-Integrable Systems (ANNNI Chain)

$$H = -J\left(\sum_i \sigma_i^x\sigma_{i+1}^x + \kappa\sum_i \sigma_i^x\sigma_{i+2}^x\right) - \Gamma\sum_i \sigma_i^z$$

A. Haldar, F. Pollmann, AD
(ongoing)

$$H = 2\Gamma\sum_i^L c_i^\dagger c_i - \sum_i (c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}) - \kappa\sum_i (c_i^\dagger - c_i)(1 - 2c_{i+1}^\dagger c_{i+1})(c_{i+1}^\dagger + c_{i+2}) - L\Gamma$$

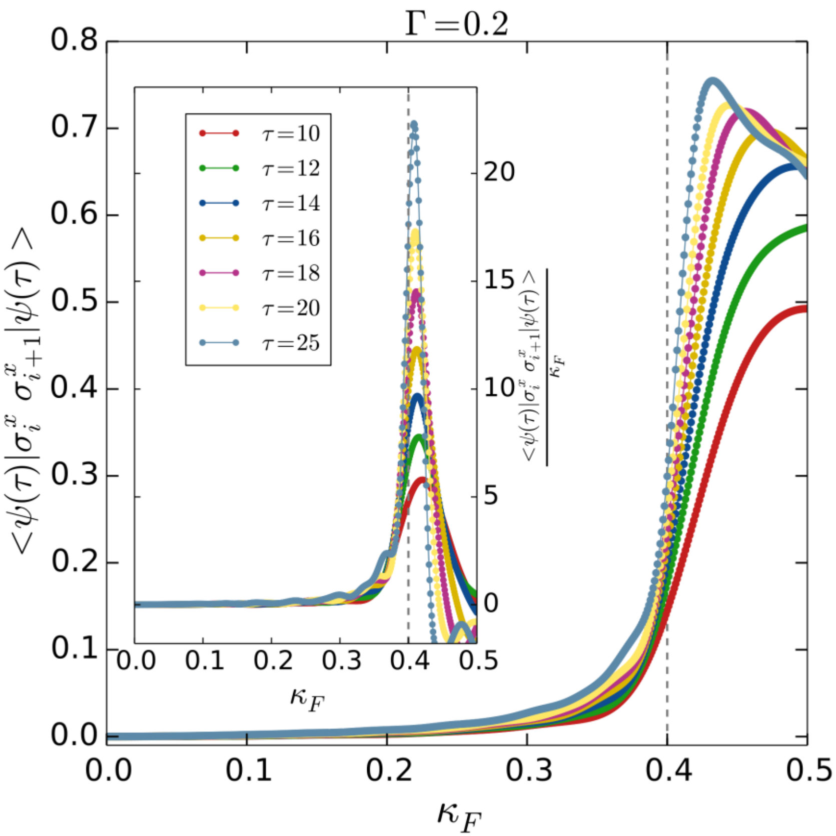
4-Fermion term (non-integrable)



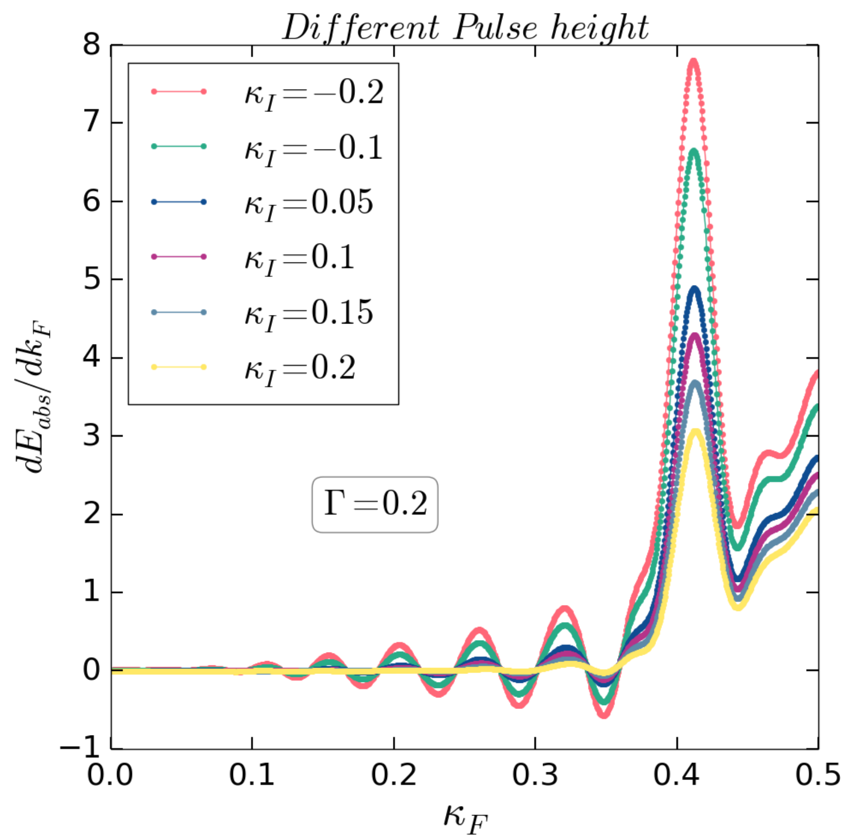

❖ Ferro-Para transition line (the straight line) is determined analytically from perturbation theory for small $\Gamma$ (2007).

❖ The entire line is accurately determined by Quantum Monte Carlo (1991).

# Signature (ANNNI Chain) ...



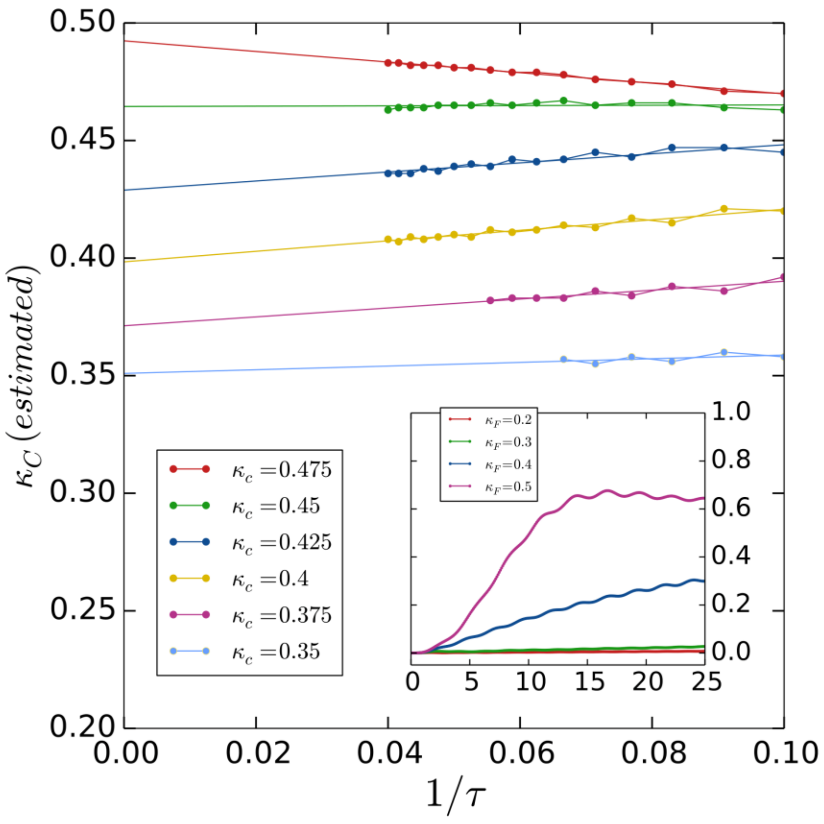

Convergence with increasing $\tau$

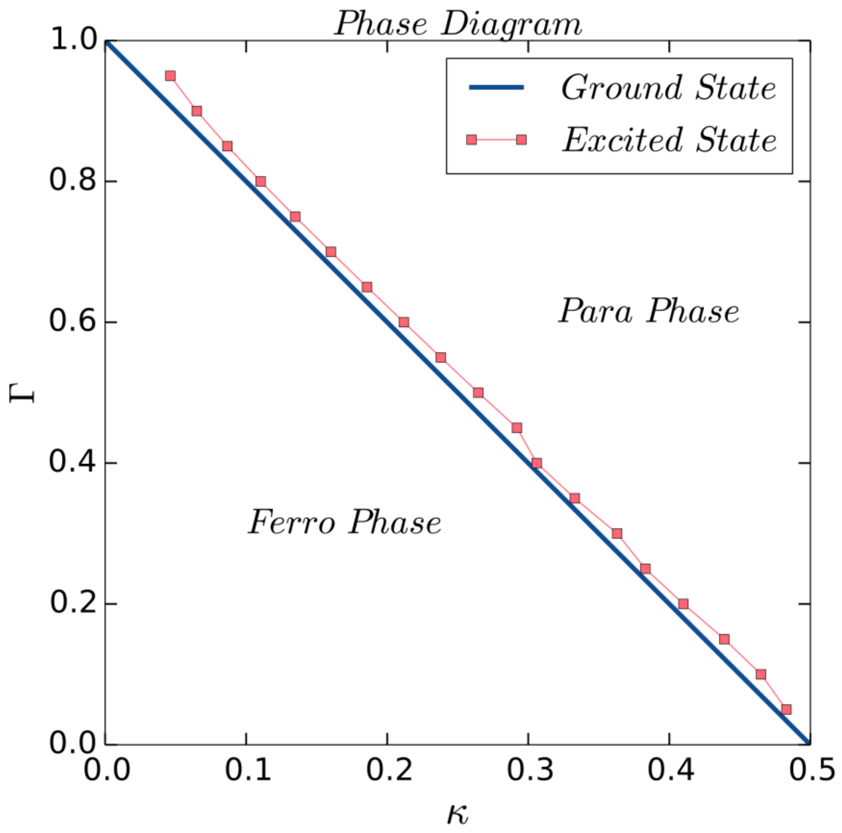



Stronger Signature for Stronger Pulses!

Using **i-TEBD** (for Infinite size)

# Signature (ANNNI Chain) ...



# Conclusion and Outlook

- We have shown strong non-analytic signature of ground state quantum phase transitions are imprinted on a family of highly excited (paramagnetic) non-equilibrium states. The signatures appear in local observables measured over the family.

- Deeper Question: What ensures existence of such signatures in highly excited states? Locality of the Hamiltonian and ETH implies every eigenstate has information about the entire Hamiltonian

- Do these signatures also contain information about *the universality class* of the transition?

- This can be used to detect any gap-closing transition, using local observables in the bulk, interestingly, even when the to transition is topological.

- Do similar signatures occur for first (or other than second) order transitions?

- Quench isn't the only non-equilibrium protocol. There can be other, more convenient ones - might facilitate experimental detection of QPT

# Thanks!

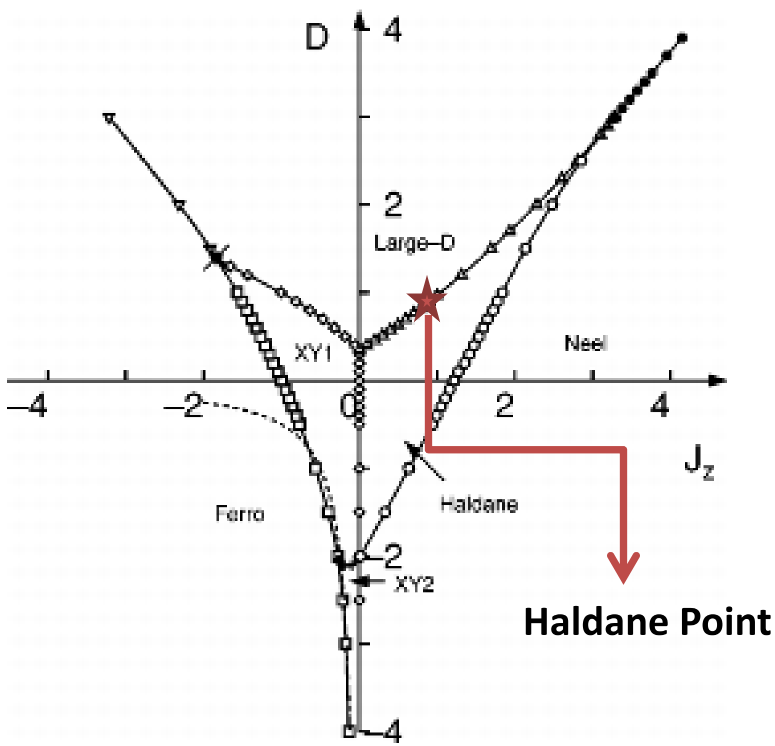
(1876)

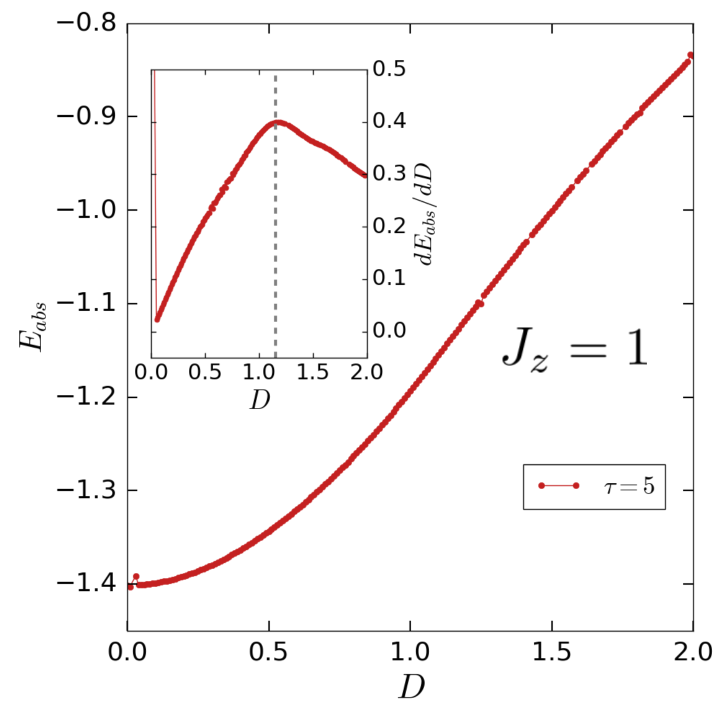
(1907-1930)

$$\mathcal{I}_n^+ = \frac{iJ}{2} \sum_j \left\{ a_{2j} \left[ a_{2j+2n+1} + a_{2j-2n+1} \right] - h_z a_{2j} \left[ a_{2j+2n-1} + a_{2j-2n-1} \right] \right\}$$

# XXZ-Chain

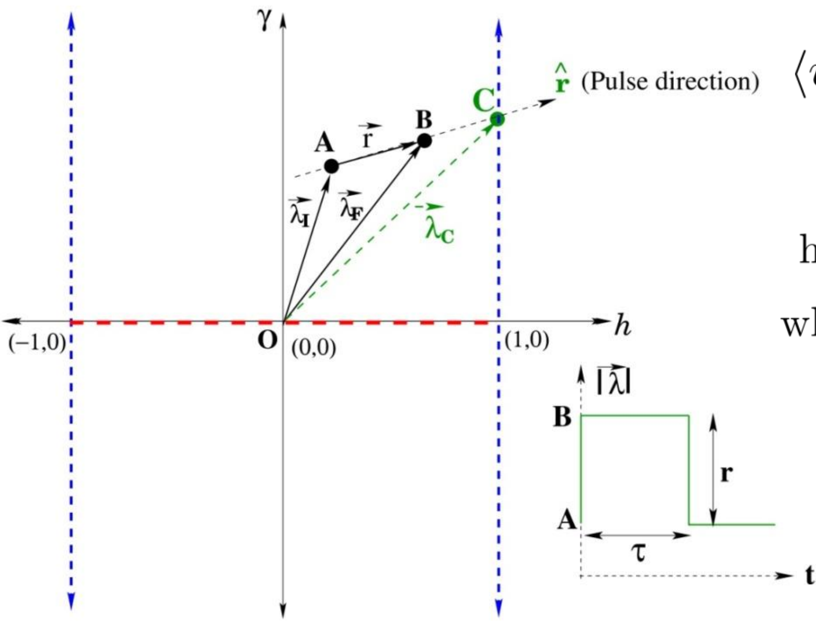
$$H = \sum_{i}^{L} \left[ J(S_i^x S_{i+1}^x S_{i+1}^y S_{i+1}^y) + J_z S_i^z S_{i+1}^z \right] + D \sum_{i}^{L} (S_i^z)^2$$



# Energy Absorption in Pulsed in XY Chain

$$H = -\frac{1}{2}\left[(1+\gamma)\sum_{i}^{L}\sigma_i^x\sigma_{i+1}^x + (1+\gamma)\sum_{i}^{L}\sigma_i^y\sigma_{i+1}^y\right] - h_z\sum_{i}^{L}\sigma_i^z$$




$$\langle\psi(\tau)|H_I|\psi(\tau)\rangle = \sum_{n,k,m} \epsilon_k^I[\chi_{n0}\chi_{m0}^*\chi_{mk}\chi_{nk}^*]e^{i(\epsilon_n^F-\epsilon_m^F)\tau}$$

here, $\chi_{ij} = \langle\epsilon_i^F|\epsilon_I^j\rangle$

where $|\epsilon_i^{I/F}\rangle, \epsilon_i^{I/F} \Rightarrow$ Eigenvectors and of $H(\lambda_{I/F})$

As L → ∞ , the sums will be replaced by integrals, and as τ → ∞ (Dephasing Limit) we keep only the $n = m$ terms (Riemann-Lebesgue Lemma) giving:

$$\mathrm{E_{abs}} = \left(\sum_k \epsilon_k^I \sum_l |\chi_{l0}|^2|\chi_{lk}|^2 - \epsilon_0^I\right)$$

S. Bhttacharyya, S. Dasgupta, AD, **arXiv:1409.0545** (2014).

# Some Standard Signatures of QPT: Non-Analyticity in Ground-State Properties Due to the Gap Closure:

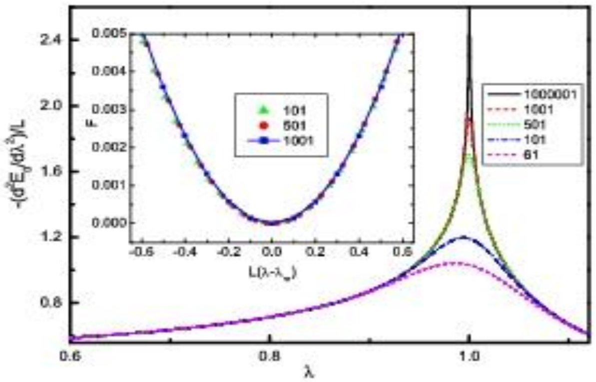
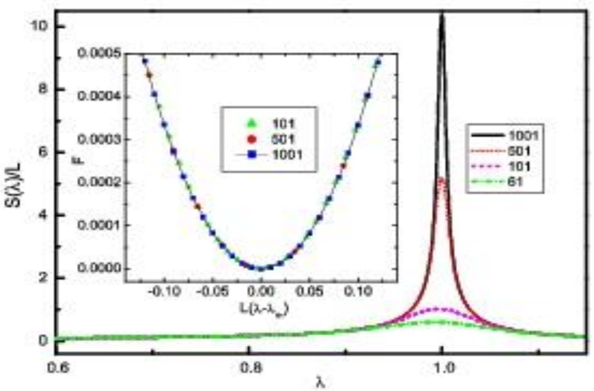



FIG. 1: (color online) The derivatives $\partial^2 E_0(\lambda)/\partial\lambda^2$ as a function of $\lambda$.

$\mathrm{H}(\lambda) = H_0 + \lambda H_1\,;\quad \lambda = h_z/J$ (for Ising Chain)

$$\frac{\partial^2 E_{Gr}(\lambda)}{\partial\lambda^2} = \sum_{n\neq 0} \frac{2|\langle\psi_n(\lambda)|H_1|\psi_{Gr}(\lambda)\rangle|^2}{[E_{Gr}(\lambda)-E_n(\lambda)]}$$

$$\{E_n, |\psi_n\rangle\} \leftrightarrow \text{ Eigenvalues/vectors of } H(\lambda)$$




$$F(\lambda, \lambda+\delta\lambda) = |\langle\psi_{Gr}(\lambda)|\psi_{Gr}(\lambda+\delta\lambda)\rangle|$$

$$\chi_F(\lambda) = -\lim_{\delta\lambda\to 0} \frac{2\ln F}{\delta\lambda^2} = -\frac{\partial^2 F}{\partial(\delta\lambda)^2}$$

$$\chi_F(\lambda) = \sum_{n\neq 0} \frac{|\langle\psi_n(\lambda)|H_1|\psi_{Gr}(\lambda)\rangle|^2}{[E_{Gr}(\lambda)-E_n(\lambda)]^2}$$


S. Chen *et. al*, PRA **77** 032111 (2008)