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Nudging methods in geophysical data assimilation

Hands-on lab

Mathematical and Statistical Foundations of Data Assimilation,

TIFR-CAM, Bangalore, India.

July 04-23, 2011

1. Nudging - Linear ODE

Consider the following ODE in $\mathbb{R}^2$:

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x(t)$$

We assume that the first component of x is observed : $H = (1 \quad 0)$. Then we are looking for a gain matrix

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

such that $F - KH$ has negative eigenvalues :

$$det(\lambda I - (F - KH)) = (\lambda - 1 + k_1)(\lambda - 1) - 1 + k_2 = \lambda^2 + (k_1 - 2)\lambda + (k_2 - k_1).$$

1. Nudging - Linear ODE

Choose for instance $k_1 = 4$ and $k_2 = 5$, which leads to the polynomial $\lambda^2 + 2\lambda + 1$, and the eigenvalues of $F - KH$ are now -1 and -1 : they are both strictly negative (or of strictly negative real part), and the nudging system is now stable. In this case, the error asymptotically decreases in time : $E(t) = e^{-t} E_0$.

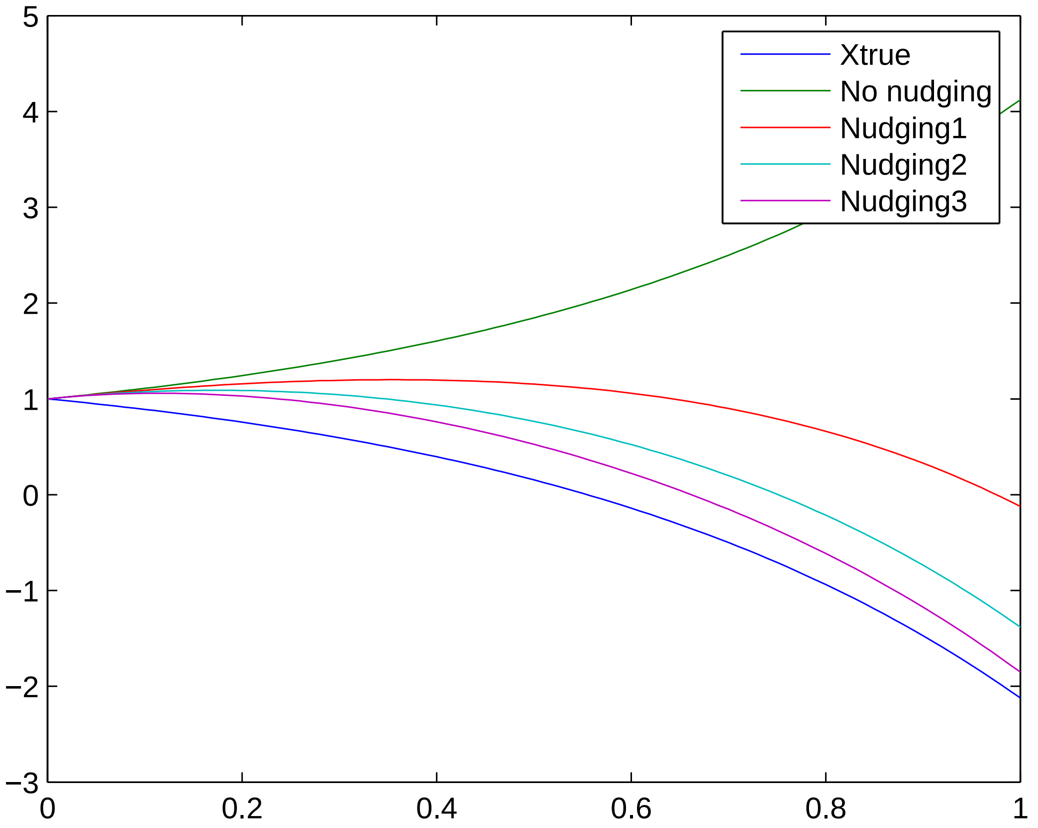
$\Rightarrow$ solve the linear system (explicit Euler) from $t = 0$ to $t = 1$ (or 2), and store the trajectory (x_1 is observed) ;
$\Rightarrow$ solve the linear system from another initial condition, without and with nudging, and compare the solutions for various values of k_1 and k_2.

Explicit Euler :

$$X^{n+1} = X^n + \Delta t \, F X^n$$

1. Nudging - Linear ODE

Numerical tests on this example : $X_{true,0} = [1; -2]$, $X_0 = [1; 0]$.



Case 1 : $k_1 = 2$ and $k_2 = 2 \Rightarrow$ eigenvalues 0 and 0 ;

Case 2 : $k_1 = 4$ and $k_2 = 5 \Rightarrow$ eigenvalues -1 and -1 ;

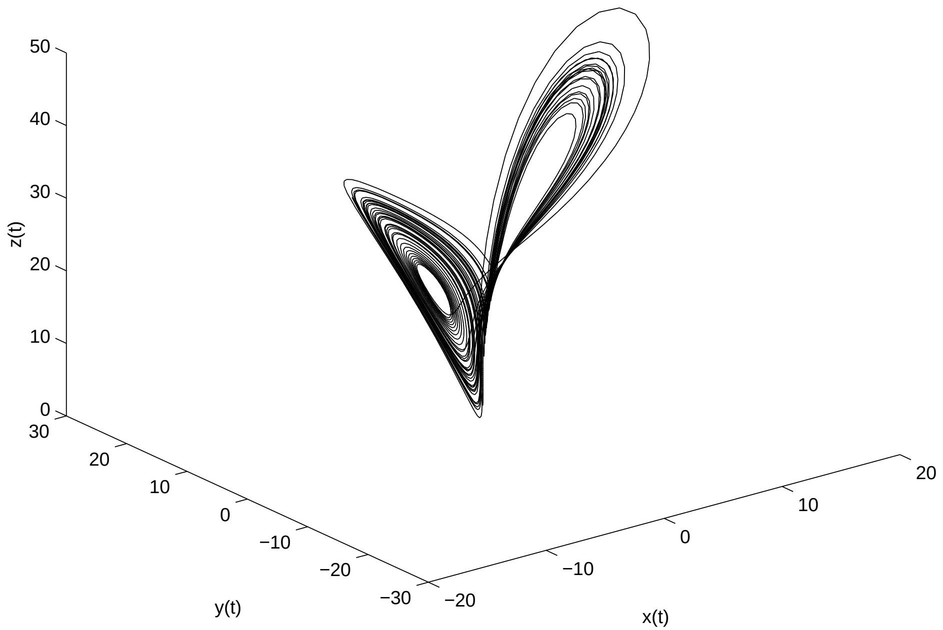
Case 3 : $k_1 = 6$ and $k_2 = 10 \Rightarrow$ eigenvalues -2 and -2.

2. Nudging - Lorenz model

Example : Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = xy - \beta z, \end{cases}$$

with standard values of parameters $\sigma = 10$, $\rho = 28$ and $\beta = \frac{8}{3}$ for a chaotic behavior.



2. Nudging - Lorenz model

Possible observer when only x is observed :

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x_{true} - y - x_{true} z, \\ \dot{z} = x_{true} y - \beta z, \end{cases}$$

which can be rewritten as

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz + \rho(x_{true} - x) - z(x_{true} - x), \\ \dot{z} = xy - \beta z + y(x_{true} - x), \end{cases}$$

The nudging term is the same as in the previous slide (only x is observed, and $K = (0;\rho;0)^T$). There is an additional term : $(0\ ;\ -z\ ;\ y)^T H(X_{true} - X)$

2. Nudging - Lorenz model

$\Rightarrow$ Solve the standard Lorenz model (explicit Euler) and store the reference trajectory ;
$\Rightarrow$ Solve the Lorenz model from another initial condition, with or without nudging ;
$\Rightarrow$ Find the nudging term when only y is observed, and check numerically that it works.

3. BFN - Linear transport equation

Linear transport equation :

$$\begin{cases} \partial_t u + a(x)\partial_x u = -K(u - u_{obs}), & u(x, t=0) = u_0(x) \\ \partial_t \tilde{u} + a(x)\partial_x \tilde{u} = K(\tilde{u} - u_{obs}), & \tilde{u}(x, t=T) = u_T(x) \end{cases}$$

$\Rightarrow$ Solve the linear transport equation with $a(x) = 1$ (Lax-Friedrichs scheme), in forward and backward modes, and store the solution ;
$\Rightarrow$ Solve the same equation from another starting point, with and without nudging ($K = k\mathbb{1}_{[0;0.5]}(x)$), and compare the solutions for various final times T.

Lax-Friedrichs :

$$u_i^{n+1} = \frac{u_{i-1}^n + u_{i+1}^n}{2} - a\frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) - \Delta t\, K u_i^n.$$

$$\tilde{u}_i^n = \frac{\tilde{u}_{i-1}^{n+1} + \tilde{u}_{i+1}^{n+1}}{2} + a\frac{\Delta t}{2\Delta x}(\tilde{u}_{i+1}^{n+1} - \tilde{u}_{i-1}^{n+1}) - \Delta t\, K \tilde{u}_i^{n+1}.$$

4. BFN - Burgers equation

Burgers equation :

$$\begin{cases} \partial_t u + \frac{1}{2}\partial_x(u^2) = -K(u - u_{obs}), & u(x, t = 0) = u_0(x) \\ \partial_t \tilde{u} + \frac{1}{2}\partial_x(\tilde{u}^2) = K(\tilde{u} - u_{obs}), & \tilde{u}(x, t = T) = u_T(x) \end{cases}$$

$\Rightarrow$ Solve Burgers equation (Lax-Friedrichs scheme), in forward and backward modes, and store the solution ;

$\Rightarrow$ Solve the same equation from another starting point, with and without nudging ($K = k\mathbb{1}_{[0;0.5]}(x)$), and compare the solutions for various final times T.

Lax-Friedrichs :

$$u_i^{n+1} = \frac{u_{i-1}^n + u_{i+1}^n}{2} - \frac{1}{2}\frac{\Delta t}{2\Delta x}\left((u_{i+1}^n)^2 - (u_{i-1}^n)^2\right) - \Delta t\, K \tilde{u}_i^n.$$

$$\tilde{u}_i^n = \frac{\tilde{u}_{i-1}^{n+1} + \tilde{u}_{i+1}^{n+1}}{2} + \frac{1}{2}\frac{\Delta t}{2\Delta x}\left((\tilde{u}_{i+1}^{n+1})^2 - (\tilde{u}_{i-1}^{n+1})^2\right) - \Delta t\, K \tilde{u}_i^{n+1}.$$