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Nudging methods in geophysical data assimilation

Part 1 : Standard nudging and asymptotic observers

Mathematical and Statistical Foundations of Data Assimilation,

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Talk overview

1. Nudging
2. Linear case and Luenberger observer
3. Non-linear case : Lorenz model
4. Observer design : shallow water model

Motivations for data assimilation

Environmental and geophysical studies : forecast the natural evolution
$\rightsquigarrow$ retrieve at best the current state (or initial condition) of the environment.

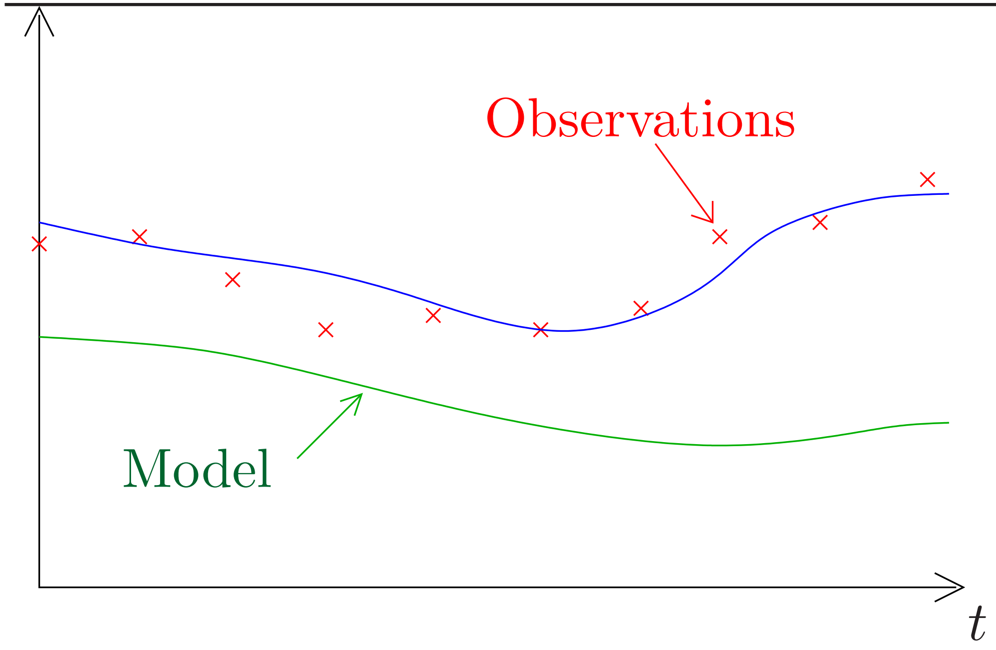
Geophysical fluids (atmosphere, oceans, ...) : turbulent systems $\Longrightarrow$ high sensitivity to the initial condition $\Longrightarrow$ need for a precise identification (much more than observations)

Environmental problems (ground pollution, air pollution, hurricanes, ...) : problems of huge dimension, generally poorly modelized or observed

Data assimilation consists in combining in an optimal way the observations of a system and the knowledge of the physical laws which govern it.

Main goal : identify the initial condition, or estimate some unknown parameters, and obtain reliable forecasts of the system evolution.

Data assimilation



combination

model + observations

$\Downarrow$

identification of the initial condition

of a geophysical system

Fundamental for a chaotic system (atmosphere, ocean, . . .)

Issue : These systems are generally irreversible

Goal : Combine models and data

Typical inverse problem : retrieve the system state from sparse and noisy observations

⇒ 1. Nudging

2. Linear case and Luenberger observer

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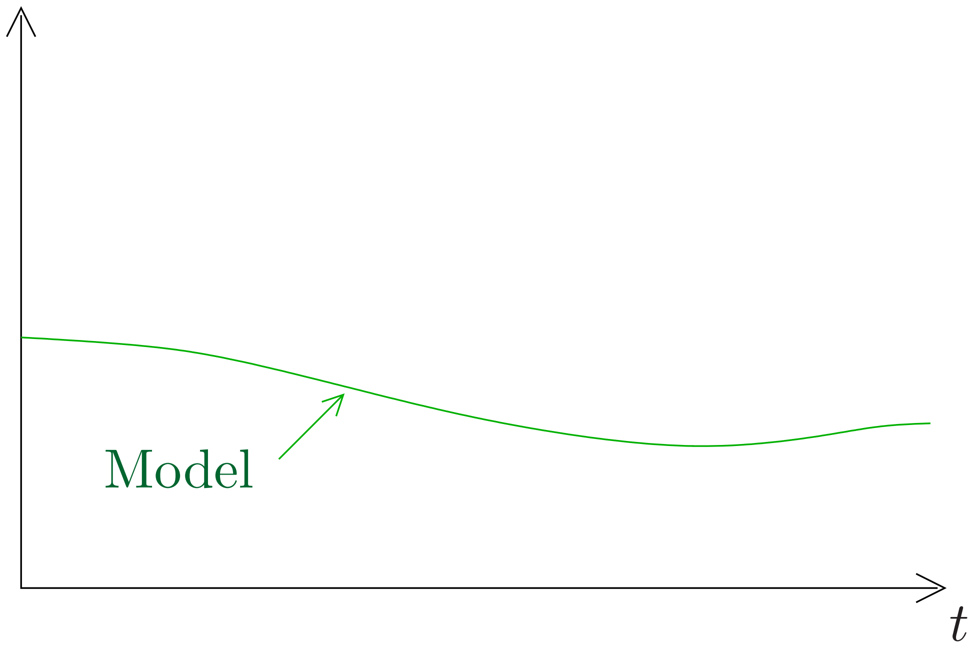
Model

Let us consider a model governed by a system of ordinary differential equations (ODEs) (similar for partial differential equations, PDEs) :

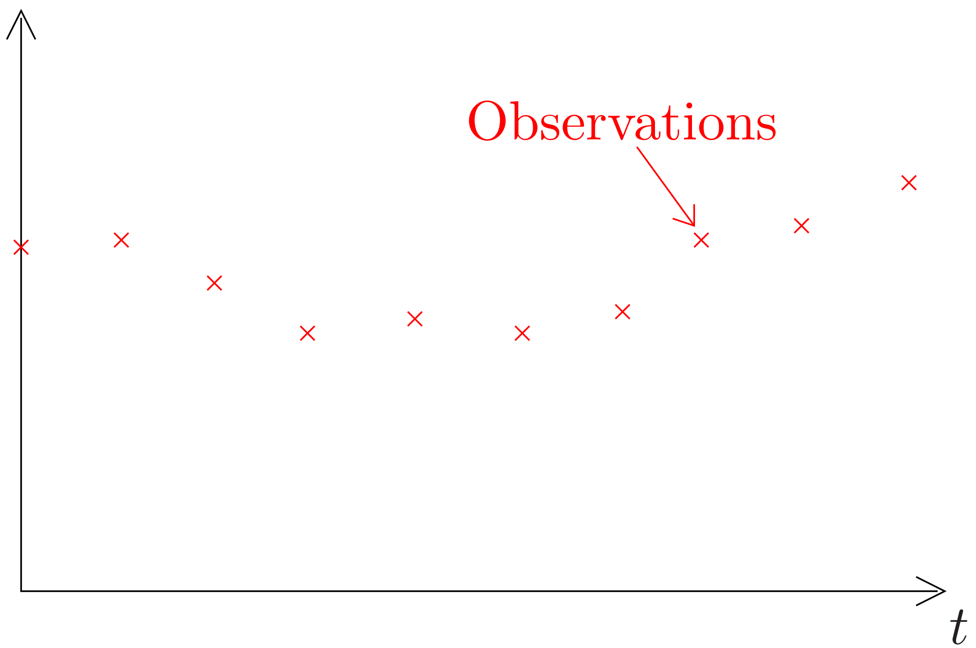
$$\frac{dX}{dt} = F(X), \quad 0 < t < T,$$

with an initial condition $X(0) = x_0$.



Observations

$Y_{obs}(t)$: observations of the system, sparse in time and space, generally noisy, and may not be physically related to the model state.

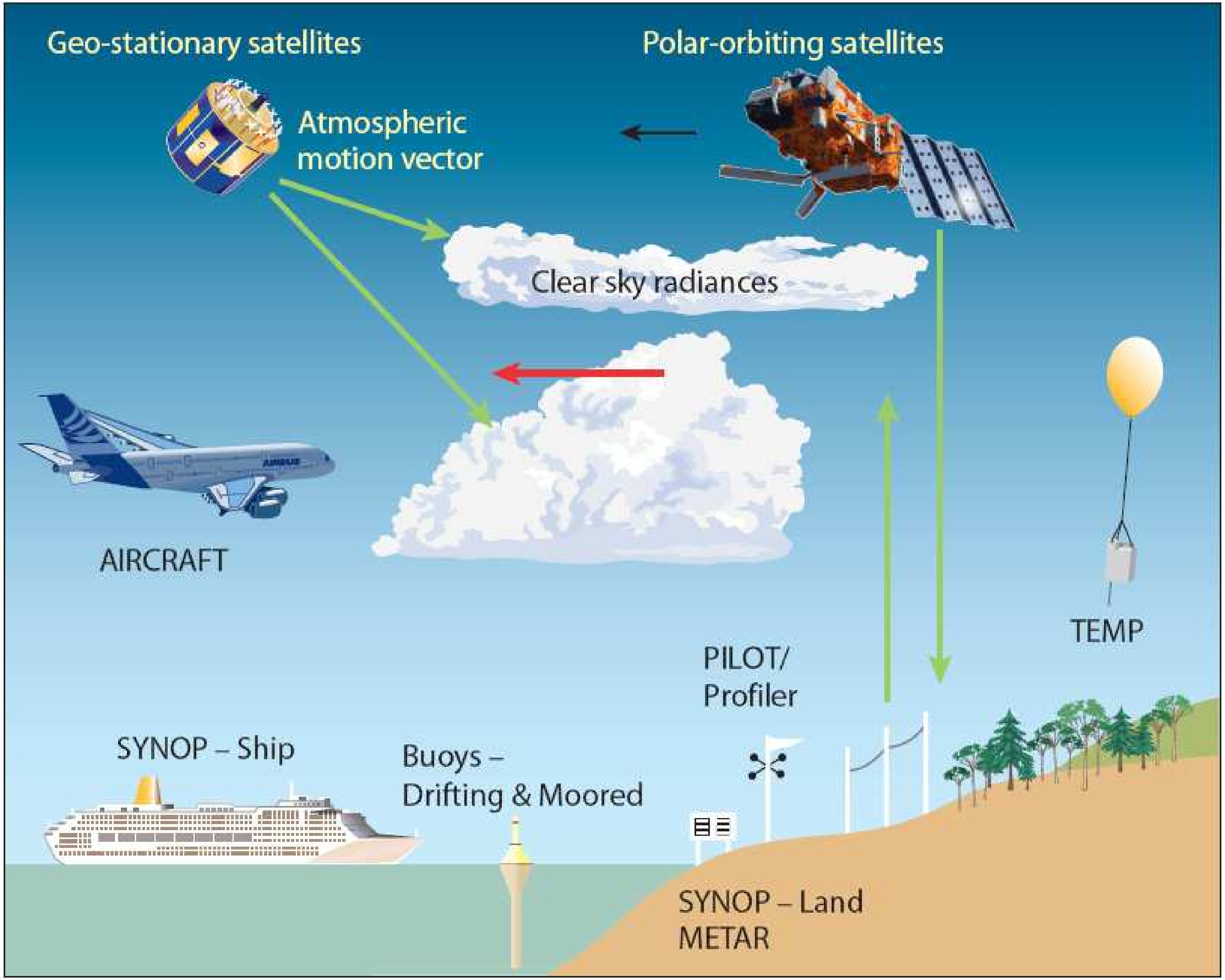


H : observation operator. It allows one to compare an observation $Y_{obs}(t)$ with the corresponding quantity computed by the model $H(X(t))$. It generally contains two parts :

– interpolation/extrapolation

– change of physical variables

Observation types



Nudging

Model :

$$\begin{cases} \dfrac{dX}{dt} = F(X), \quad 0 < t < T, \\ X(0) = X_0, \end{cases}$$

Standard nudging applied to this model :

$$\begin{cases} \dfrac{dX}{dt} = F(X) + K(Y_{obs} - H(X)), \quad 0 < t < T, \\ X(0) = X_0, \end{cases}$$

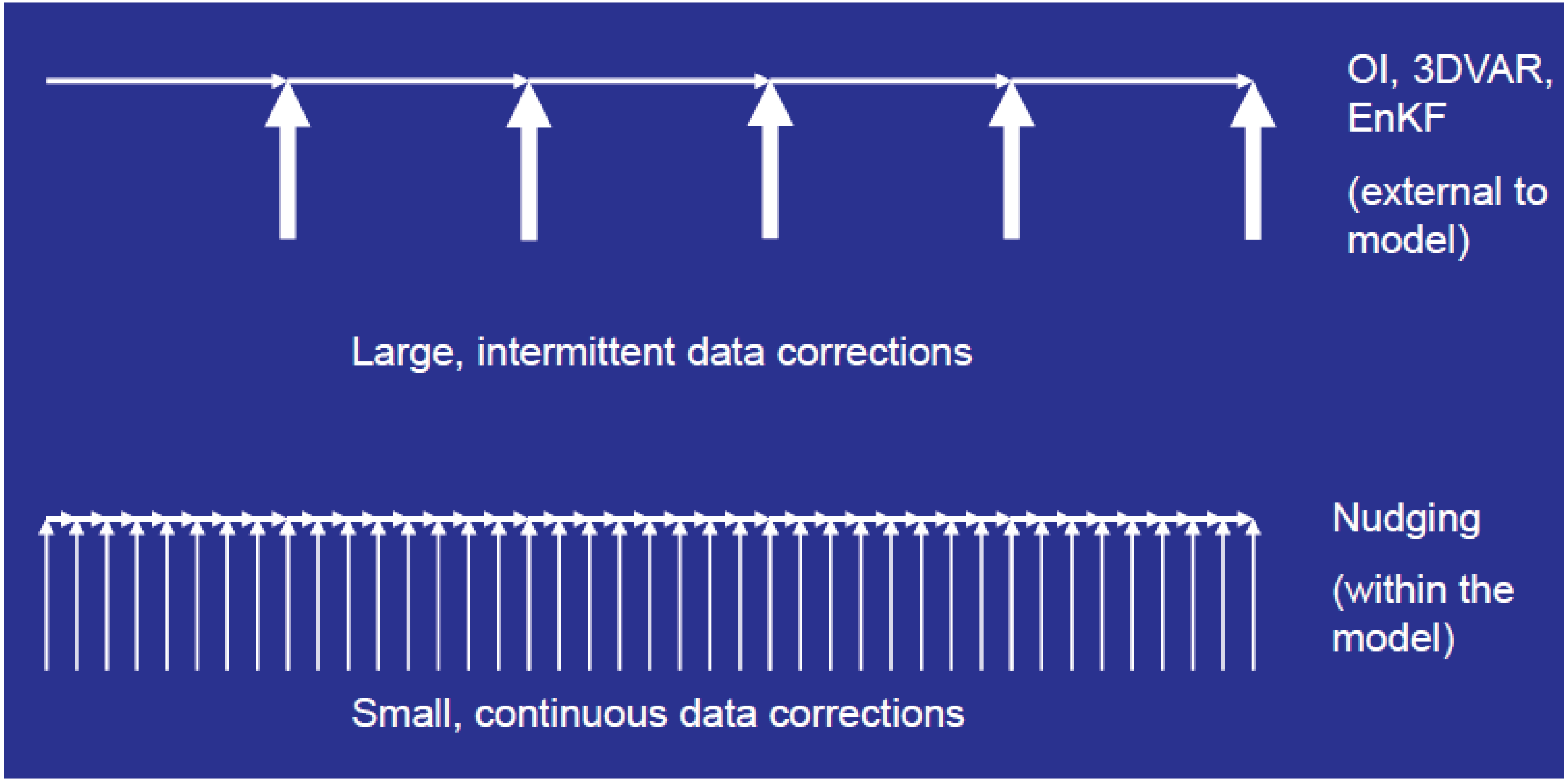
where K is the nudging (or gain) matrix.

Nudging

Nudging is a four-dimensional data assimilation method that uses dynamical relaxation to adjust toward observations (observation nudging) or toward an analysis (analysis nudging).

Nudging is accomplished through the inclusion of a forcing term in the model dynamics, with a tunable coefficient that represents the relaxation time scale. Computationally inexpensive, nudging is based on both heuristic and physical considerations.

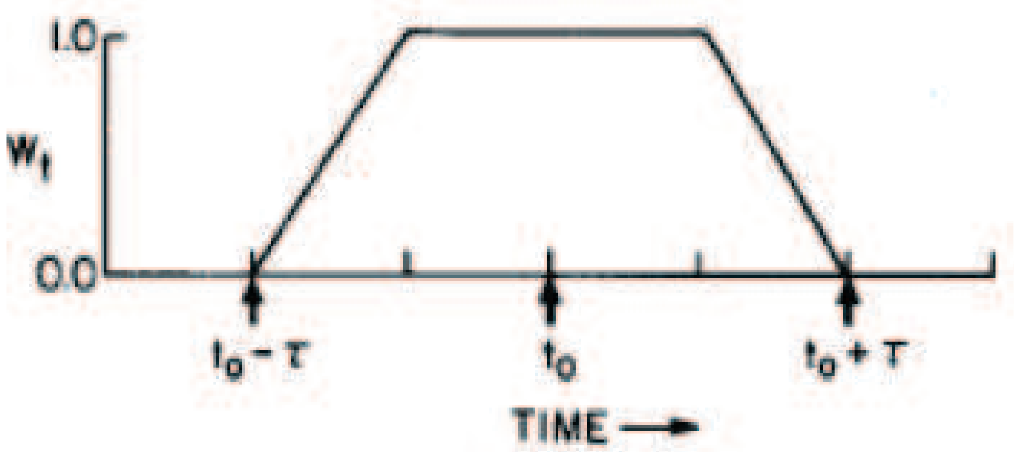
Comparison with other DA techniques



Discrete observations

In general cases, the observations are discrete in time and space. Thus, it is not possible to add a nudging term everywhere and everytime $\Longrightarrow$ nudging is performed at more than the observation time (and location) :

• in time, a same observation is used over several time steps :



• in space, the observation information is spread to a local neighborhood (interpolation, using a radius of influence).

References

– Meteorology (1D shallow water model) :
  Anthes (1974), Hoke (1976), Hoke-Anthes (1976)

– Oceanography (quasi-gesotrophic model) :
  Verron-Holland (1989), Blayo-Verron-Molines (1994)

– Atmosphere (meso-scale FDDA) : Stauffer-Seaman (1990)

– Optimal determination of the nudging coeffcients :
  Zou-Navon-Le Dimet (1992), Stauffer-Bao (1993),
  Vidard-Le Dimet-Piacentini (2003)

– Sensitivity studies (adjoint method) : Bao-Errico (1997)

Variational interpretation

Model equation with nudging :

$$\frac{dX}{dt} = FX + K(Y_{obs} - HX)$$

Implicit discretization, with X^n at time n and X^{n+1} at time $n+1$:

$$\frac{X^{n+1} - X^n}{\Delta t} = FX^{n+1} + K(Y_{obs}^{n+1} - HX^{n+1}).$$

X^{n+1} is solution of the following equation :

$$X - X^n = \Delta t\, FX + \Delta t\, K(Y_{obs}^{n+1} - HX)$$

Assume that

$$K = H^T R^{-1}$$

where R is the covariance matrix of the errors of observations.

Variational interpretation

$$X - X^n = \Delta t\, F X + \Delta t\, H^T R^{-1}(Y_{obs}^{n+1} - HX)$$

Assume the model F is derived from an energy principle.

Variational interpretation : direct nudging is a compromise between the minimization of the energy of the system and the quadratic distance to the observations :

$$\min_X \left[\frac{1}{2}\langle X - X^n, X - X^n\rangle - \frac{\Delta t}{2}\langle FX, X\rangle + \frac{\Delta t}{2}\langle R^{-1}(Y_{obs}^{n+1} - HX), Y_{obs}^{n+1} - HX\rangle \right],$$

[Auroux-Blum, Nonlin. Proc. Geophys. 2008]

Example : heat equation $F = \Delta$ (Laplacian), and the energy is $-\langle \Delta X, X\rangle = \|\nabla X\|^2$.

Sequential interpretation

It is also possible to give a sequential interpretation of the standard nudging algorithm by seeing it as a Kalman filter. Indeed, when no observations are available, the nudging method simply consists of solving the model equations, like Kalman filters.

On the other hand, when some observations are available, in both nudging and Kalman filters, the model solution is corrected with the innovation vector, i.e. the difference between the observations and the corresponding model state.

If at any time, the nudging matrices are set in an optimal way, then the standard nudging method is equivalent to the standard Kalman filter. In the other cases, it can be seen as a suboptimal Kalman filter.

$\Rightarrow$ Talk 2

However, the iterative and alternative resolutions of forward and backward models appreciably improves the efficiency of the standard nudging method.

$\Rightarrow$ Talk 3

Nudging : linear case

We will first study nudging in a fully linear situation, in order to understand how and why it works.

Linear model :

$$\begin{cases} \dfrac{dX}{dt} = FX, \quad 0 < t < T, \\ X(0) = X_0, \end{cases}$$

where F is the linear model operator (matrix).

Standard nudging applied to this model :

$$\begin{cases} \dfrac{dX}{dt} = FX + K(Y_{obs} - HX), \quad 0 < t < T, \\ X(0) = X_0, \end{cases}$$

where K is the nudging (or gain) matrix, and H is the linear observation operator.

Nudging : linear case

Assume that the observations are extracted from a true solution of the model :

$$\begin{cases} \dfrac{dX_{true}}{dt} = FX_{true}, \quad 0 < t < T, \\ \\ X_{true}(0) = X_{true,0}, \end{cases}$$

and $Y_{obs} = HX_{true}$.

Then the nudging equation

$$\begin{cases} \dfrac{dX}{dt} = FX + K(Y_{obs} - HX), \quad 0 < t < T, \\ \\ X(0) = X_0 \end{cases}$$

can be rewritten :

$$\begin{cases} \dfrac{dX}{dt} = FX + KH(X_{true} - X), \quad 0 < t < T, \\ \\ X(0) = X_0 \end{cases}$$

Error equation

Let E be the error between the true state X_{true} and the nudging state X : $E = X - X_{true}$. Then,

$$\begin{cases} \dfrac{dE}{dt} = \dfrac{dX}{dt} - \dfrac{dX_{true}}{dt} = FX + KH(X_{true} - X) - FX_{true}, \quad 0 < t < T, \\ E(0) = X(0) - X_{true}(0) = X_0 - X_{true,0} \end{cases}$$

$$\begin{cases} \dfrac{dE}{dt} = F(X - X_{true}) + KH(X_{true} - X) = (F - KH)E, \quad 0 < t < T, \\ E(0) = E_0. \end{cases}$$

Luenberger (asymptotic) observer

Luenberger observer, or asymptotic observer

[Luenberger (66)]

$$\frac{dE}{dt} = (F - KH)E$$

If $F - KH$ is a Hurwitz matrix, i.e. if its spectrum is strictly included in the half-plane $\{\lambda \in \mathbb{C}; Re(\lambda) < 0\}$, then $E \to 0$, i.e. $X \to X_{true}$ when $t \to +\infty$.

Indeed, the solution is

$$E(t) = \exp^{(F-KH)t} E(0)$$

and $E(t)$ decreases exponentially in time, at least with the following decay rate :

$$\max Re(Sp(F - KH)) < 0$$

Nudging : linear case example

Simple example : consider the following ODE in $\mathbb{R}^2$:

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x(t)$$

The characteristic polynomial of the model matrix F is :

$$det(\lambda I - F) = (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda$$

and the eigenvalues are 0 and 2.

We assume that the first component of x is observed : $H = (1 \quad 0)$. Then we are looking for a gain matrix

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

such that $F - KH$ has negative eigenvalues.

Nudging : linear case example

$$F - KH = \begin{pmatrix} 1-k_1 & 1 \\ 1-k_2 & 1 \end{pmatrix}$$

and the characteristic polynomial is :

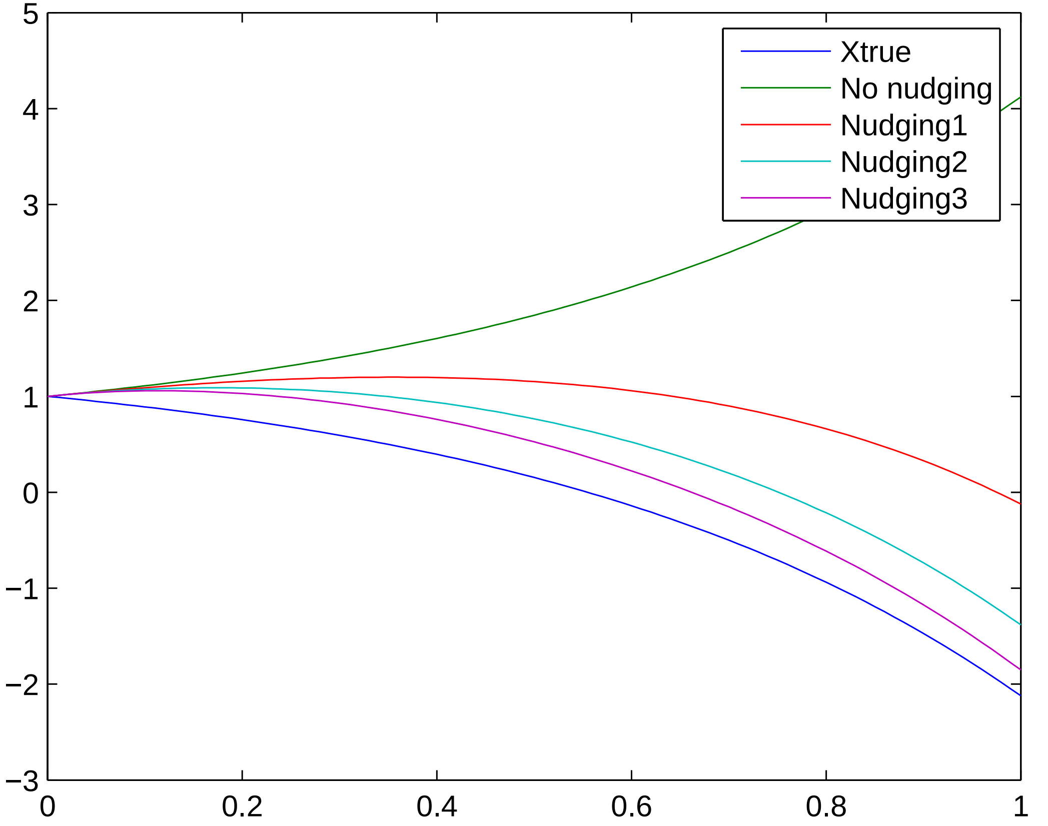
$$det(\lambda I - (F - KH)) = (\lambda - 1 + k_1)(\lambda - 1) - 1 + k_2 = \lambda^2 + (k_1 - 2)\lambda + (k_2 - k_1).$$

Choose for instance $k_1 = 4$ and $k_2 = 5$, which leads to the polynomial $\lambda^2 + 2\lambda + 1$, and the eigenvalues of $F - KH$ are now -1 and -1 : they are both strictly negative (or of strictly negative real part), and the nudging system is now stable.

In this case, the error asymptotically decreases in time : $E(t) = e^{-t} E_0$.

Nudging : linear case example

Numerical tests on this example : $X_{true,0} = [1; -2]$, $X_0 = [1; 0]$.



Case 1 : $k_1 = 2$ and $k_2 = 2 \Rightarrow$ eigenvalues 0 and 0 ;

Case 2 : $k_1 = 4$ and $k_2 = 5 \Rightarrow$ eigenvalues -1 and -1 ;

Case 3 : $k_1 = 6$ and $k_2 = 10 \Rightarrow$ eigenvalues -2 and -2.

Observability conditions

Conditions of existence of the stable gain matrix K :

The system is observable if, given two solutions X_1 and X_2 of the model system such that $HX_1 = HX_2$ in $[0, T]$, then $X_1(0) = X_2(0)$.

A system is observable if the initial state can be obtained ("observed") from the knowledge of the observations.

By linearity, the system is observable iff $He^{Ft}X_0 = 0$ in $[0, T]$ implies $X_0 = 0$, which is equivalent to the condition $Ker(H, HF, HF^2, \ldots, HF^{n-1}) = \{0\}$. A necessary and sufficient condition for the observability of the system (F, H) is :

$$rank[H; HF; \ldots; HF^{n-1}] = n.$$

Pole assignment

Pole assignment method : (or pole placement)

If a system (F, H) is observable, then there exists a matrix K such that $F - KH$ is stable, i.e. all eigenvalues have a strictly negative real part.

[Datta 1987, Arnold and Datta 1988].

$\Rightarrow$ in this case, it is possible to find a nudging matrix K that makes the nudging solution converge towards the true state.

1. Nudging
2. Linear case and Luenberger observer
3. ⇒ Non-linear case : Lorenz model
4. Observer design : shallow-water model

Nudging : non-linear case

Luenberger observer for nonlinear models :

We consider a reference trajectory of a nonlinear system given by

$$\frac{dX_{true}}{dt} = FX_{true} + G(X_{true}),$$

where F is the linear part of the model, and G is a nonlinear function, assumed to be differentiable and Lipschitz :

$$\|G(X_1) - G(X_2)\| \leq L\|X_1 - X_2\|, \quad \forall\ X_1, X_2,$$

where $L > 0$ is a Lipschitz constant.

We assume that the system is observed : $Y_{obs} = HX_{true}$, and that (F, H) is observable. Then we know that there exists a matrix K such that $F - KH$ is a stable matrix.

Error equation

We introduce the following observer equation :

$$\frac{dX}{dt} = FX + G(X) + K(Y_{obs} - HX) = FX + G(X) + KH(X_{true} - X).$$

This is the standard Luenberger observer (or nudging method).

Let $E = X - X_{true}$ be the error (difference between the observer and true trajectories). Then E satisfies the following equation :

$$\frac{dE}{dt} = FX + G(X) + KH(X_{true} - X) - FX_{true} - G(X_{true})$$

$$= (F - KH)E + G(X) - G(X_{true})$$

Error decrease

$$\frac{1}{2}\frac{d\|E\|^2}{dt} = E.\frac{dE}{dt} = E.\left((F-KH)E\right) + E.\left(G(X)-G(X_{true})\right)$$
$$\leq \lambda_{max}\|E\|^2 + \|E\|\,\|G(X)-G(X_{true})\| \leq (\lambda_{max}+L)\|E\|^2.$$

Then if K is chosen such that all the eigenvalues of $F-KH$ are (of real part) strictly smaller than $-L$, the opposite of the Lipschitz constant of G, then the square norm of the error decreases asymptotically in time.

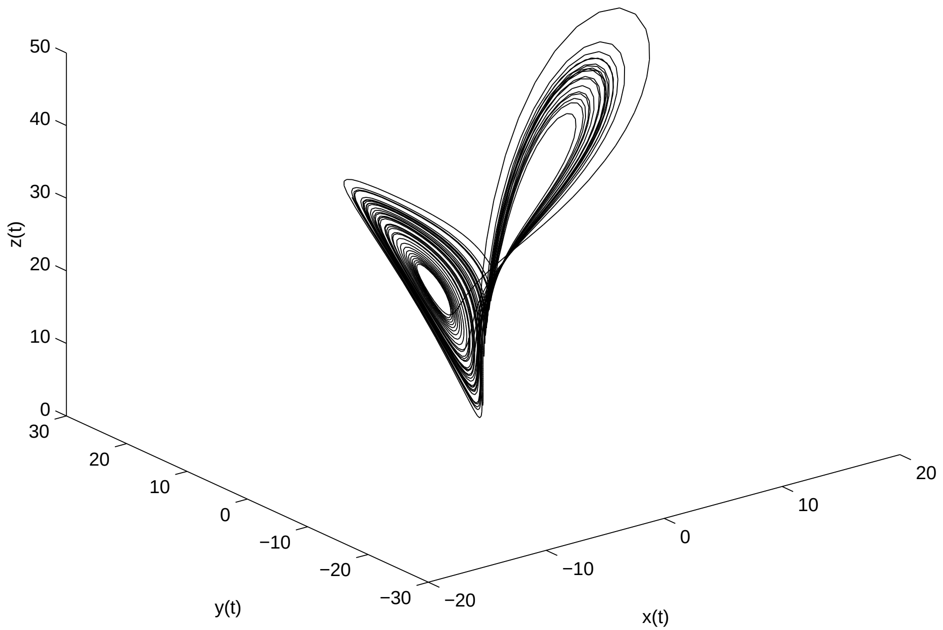
And then $X \to X_{true}$ when time goes to infinity.

Non-linear example

Example : Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = xy - \beta z, \end{cases}$$

with standard values of parameters $\sigma = 10$, $\rho = 28$ and $\beta = \frac{8}{3}$ for a chaotic behavior.



Lorenz : model decomposition

The model can be decomposed in a linear part, and a nonlinear part :

$$\frac{dX}{dt} = FX + G(X),$$

with

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad F = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}, \qquad G(X) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}.$$

Assuming that all considered trajectories are bounded (see previous figure), then the function G is Lipschitz :

$$\|G(X_2) - G(X_1)\|^2 = (x_2 z_2 - x_1 z_1)^2 + (x_2 y_2 - x_1 y_1)^2$$

$$= (x_2(z_2 - z_1) + z_1(x_2 - x_1))^2 + (x_2(y_2 - y_1) + y_1(x_2 - x_1))^2$$

$$\leq 2(x_2^2(z_2 - z_1)^2 + z_1^2(x_2 - x_1)^2 + x_2^2(y_2 - y_1)^2 + y_1^2(x_2 - x_1)^2) \leq L^2 \|X_2 - X_1\|^2$$

Lorenz : observability conditions

Then we can define the following Luenberger observer :

$$\frac{dX}{dt} = FX + G(X) + KH(X_{true} - X).$$

Assume that $H = (1\ ;\ 0\ ;\ 1)$.
Then $HF = (-\sigma\ ;\ \sigma\ ;\ -\beta)$, and $HF^2 = (\sigma^2 + \rho\sigma\ ;\ -\sigma^2 - \sigma\ ;\ \beta^2)$.

As the matrix $(H; HF; HF^2)$ is invertible, then (F, H) is observable, and we can place the poles of F : there exists K such that $F - KH$ is stable.

Lorenz : pole placement

Note that with $H = (1\ ;\ 0\ ;\ 0)$ or $H = (0\ ;\ 1\ ;\ 0)$, the system is not observable, and then it is not possible to place the poles anywhere, but it is still possible to find K such that $F - KH$ is stable. For instance with $H = (1\ ;\ 0\ ;\ 0)$ and $K = (0\ ;\ \rho\ ;\ 0)^T$,

$$F - KH = \begin{pmatrix} -\sigma & \sigma & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

3 negative eigenvalues : -1, $-\beta$ and $-\sigma$.

Lorenz : non-linear observer

Other possible observers :

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x_{true} - y - x_{true} z, \\ \dot{z} = x_{true} y - \beta z, \end{cases}$$

which can be rewritten as

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz + \rho(x_{true} - x) - z(x_{true} - x), \\ \dot{z} = xy - \beta z + y(x_{true} - x), \end{cases}$$

The nudging term is the same as in the previous slide (only x is observed, and $K = (0; \rho; 0)^T$). There is an additional term : $(0\ ;\ -z\ ;\ y)^T H(X_{true} - X)$

Lorenz : non-linear observer

Then the error $E = X - X_{true}$ is solution of the following ODE :

$$\frac{dE}{dt} = (F - KH)E + S(t)E,$$

with

$$F - KH = \begin{pmatrix} -\sigma & \sigma & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix},$$

and where $S(t)$ is the following matrix :

$$S(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_{true}(t) \\ 0 & x_{true}(t) & 0 \end{pmatrix}$$

Spectral decomposition

Let us compute the eigenvalues of $F-KH-S(t)$: the characteristic polynomial is

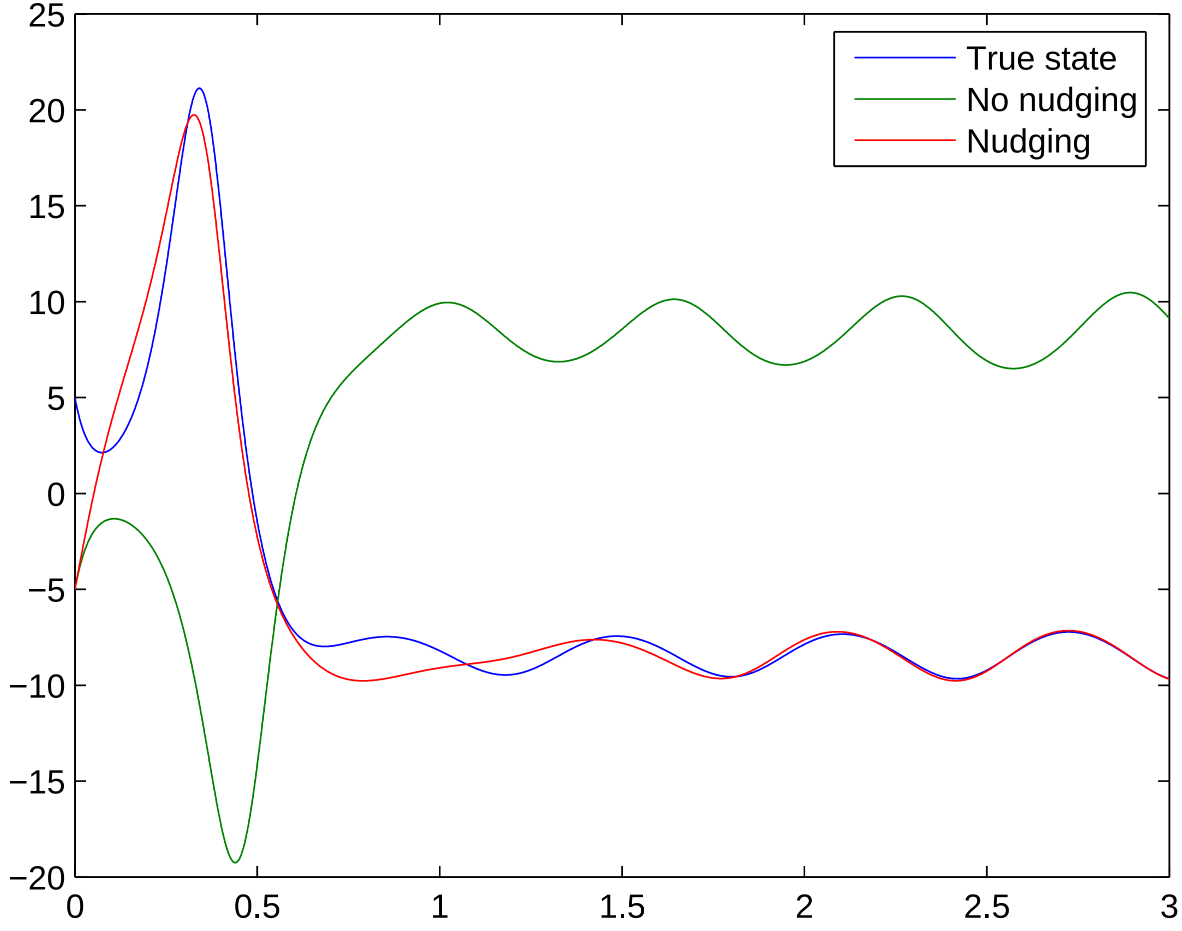
$$(\lambda+\sigma)(\lambda^2+\lambda(1+\beta)+(\beta+x_{true}(t)^2).$$

As $\beta>0$ and $\sigma>0$, for any value of $x_{true}(t)$, all three eigenvalues are strictly negative, or their real parts are strictly negative, and then the error decreases asymptotically in time.

Numerical tests

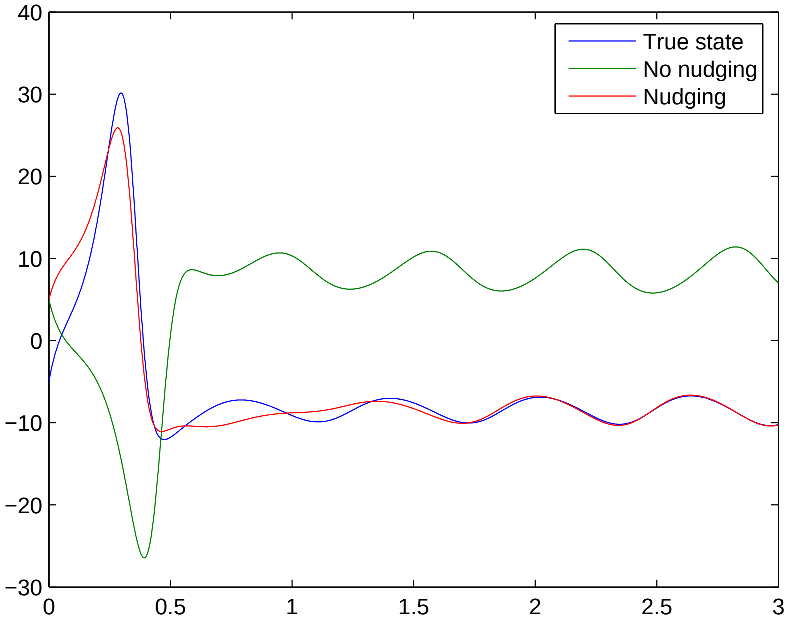
Numerical experiment on this example :

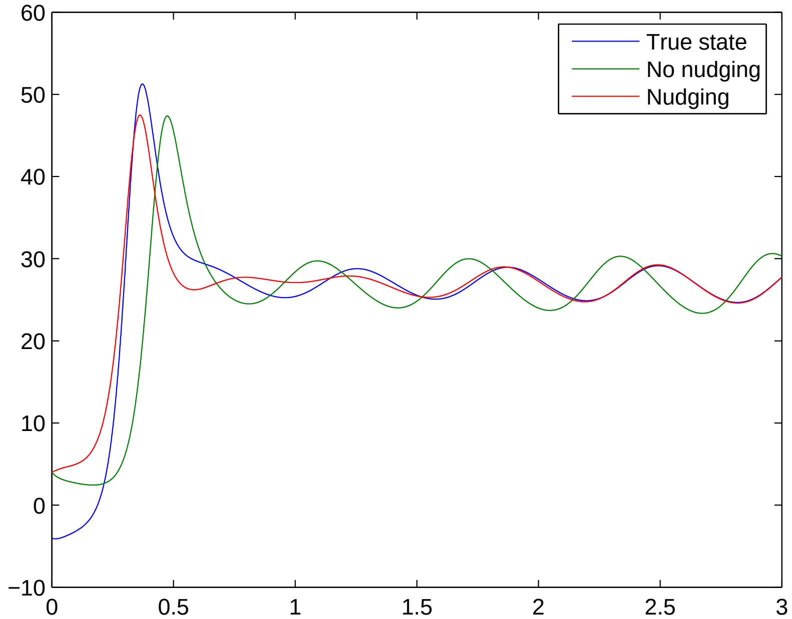
$X_{true}(0) = [5; -5; -4]$ and $X(0) = [-5; 5; 4]$.



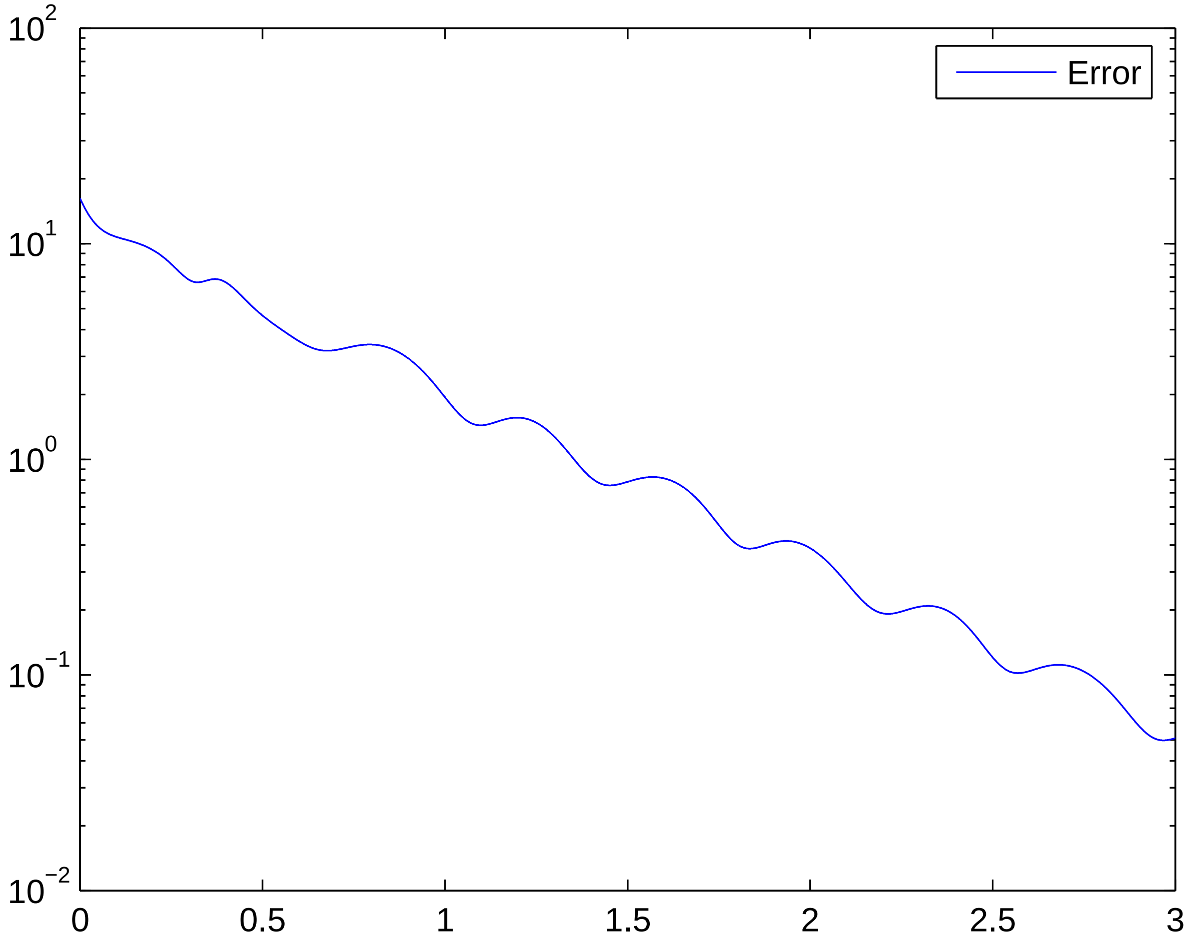
Numerical tests

Convergence of y and z also :





Numerical tests



Evolution of the norm of the error between $X_{nudging}$ and X_{true} versus time (largest eigenvalue $\simeq -1$).

1. Nudging

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⇒ 4. Observer design : shallow-water model

Observer design

2D shallow water model :

$$\begin{cases} \dfrac{\partial h}{\partial t} = -\nabla \cdot (hv), \\ \dfrac{\partial v}{\partial t} = -(v \cdot \nabla)v - g\nabla h \end{cases}$$

on a square domain with rigid boundaries and no-slip lateral boundary conditions. These equations are derived from Navier-Stokes equations, assuming the horizontal scale is much greater than the vertical one $\Rightarrow$ conservation of mass and of momentum.

We assume that the water height h is observed, and the question is : can we identify the velocity ? can we define an observer that corrects both variables (height and velocity) ?

Observer design

Any non-linear observer for this model writes :

$$\begin{cases} \dfrac{\partial h}{\partial t} = -\nabla \cdot (hv) + F_h(h_{obs}, v, h), \\ \dfrac{\partial v}{\partial t} = -(v \cdot \nabla)v - g\nabla h + F_v(h_{obs}, v, h), \end{cases}$$

where the correction terms vanish when the estimated height h is equal to the observed height h_{obs}.

Formal requirements :

- symmetry preservation : invariance to translations and rotations of the model, and then of the observer
- smoothing by convolution : reduce noise and smooth the measured output
- local stability : strong asymptotic convergence of the linearized error system

Requirements

Symmetry preservation : invariant scalar differential operators are polynomials of the Laplacian, and the dependence on v must also be invariant by translation and rotation $\Rightarrow |v|^2$:

$$F_h = Q_1(\Delta, h_{obs}, |v|^2, h - h_{obs}) + \nabla \left(Q_2(\Delta, h_{obs}, |v|^2, h - h_{obs}) \right) \cdot v$$

and its vectorial counterpart for the vectorial correction :

$$F_v = P_1(\Delta, h_{obs}, |v|^2, h - h_{obs}) v + \nabla \left(Q_2(\Delta, h_{obs}, |v|^2, h - h_{obs}) \right) .v$$

Requirements

Smoothing by convolution : the polynomials involve a differentiation process, and thus it must be coupled with a filtering process. We propose then to use a convolution between the previous invariant differential terms, and a kernel (that has to be invariant also $\Rightarrow$ isotropic gain) :

$$F_h = \varphi_h * \left[Q_1(\Delta, h_{obs}, |v|^2, h - h_{obs}) + \nabla\left(Q_2(\Delta, h_{obs}, |v|^2, h - h_{obs})\right) \cdot v\right]$$

$$= \iint \varphi_h(\xi^2 + \zeta^2)\left[Q_1(\Delta, h_{obs}, |v|^2, h - h_{obs})\right.$$

$$\left. + \nabla\left(Q_2(\Delta, h_{obs}, |v|^2, h - h_{obs})\right) \cdot v\right]_{(x-\xi, y-\zeta, t)} d\xi d\zeta$$

Example of observer

Most simple observer that should work : no derivatives (or smallest order of derivative) in order to deal with noisy data.

$$F_h = \iint \varphi_h(\xi^2 + \zeta^2)\,[h - h_{obs}]_{(x-\xi, y-\zeta, t)}\; d\xi d\zeta,$$

$$F_v = \iint \varphi_v(\xi^2 + \zeta^2)\,[\nabla(h - h_{obs})]_{(x-\xi, y-\zeta, t)}\; d\xi d\zeta,$$

with simple invariant kernels :

$$\varphi(x, y) = \beta \exp(-\alpha(x^2 + y^2)).$$

Such an observer preserves the symmetries of the system : convolution product with an isotropic kernel. Such correction terms make the observer very robust to noise, as they operate a smoothing of the measured image. The high frequencies in the signal are thus efficiently filtered.

Convergence study

Convergence on the linearized system : linearization around the equilibrium state : $\bar{h}$ =constant, and $\bar{v} = 0$. We consider only small velocities, and small variations of the height. Then the linearized system is :

$$\frac{\partial(\delta h)}{\partial t} = -\bar{h}\;\nabla \cdot \delta v,$$

$$\frac{\partial(\delta v)}{\partial t} = -g\nabla \delta h,$$

and the estimation errors, $\tilde{h} = \delta h - \delta h_{true}$ and $\tilde{v} = \delta v - \delta v_{true}$, are solution of the following linear equations :

$$\frac{\partial \tilde{h}}{\partial t} = -\bar{h}\;\nabla \cdot \tilde{v} - \varphi_h * \tilde{h},$$

$$\frac{\partial \tilde{v}}{\partial t} = -g\nabla \tilde{h} - \varphi_v * \nabla \tilde{h}.$$

Convergence study

Eliminating $\tilde{v}$ and using $\nabla(\varphi_v * \nabla h) = \varphi_v * \Delta h$ yields a modified damped wave equation with external viscous damping :

$$\frac{\partial^2 \tilde{h}}{\partial t^2} = g\bar{h}\Delta\tilde{h} + \bar{h}\,\varphi_v * \Delta\tilde{h} - \varphi_h * \frac{\partial \tilde{h}}{\partial t}.$$

Convergence study

Theorem : If φ_v and φ_h are defined by $\varphi(x,y) = \beta \exp(-\alpha(x^2+y^2))$ with $\beta_v, \beta_h, \alpha_v, \alpha_h > 0$, then the first order approximation of the error system around the equilibrium $(h,v) = (\bar{h}, 0)$ is strongly asymptotically convergent. Indeed if we consider the following Hilbert space and norm : $\mathcal{H} = H^1(\Omega) \times L^2(\Omega)$,

$$\|(u,w)\|_{\mathcal{H}} = \left(\int_\Omega \|\nabla u\|^2 + |w|^2 \right)^{1/2},$$

then

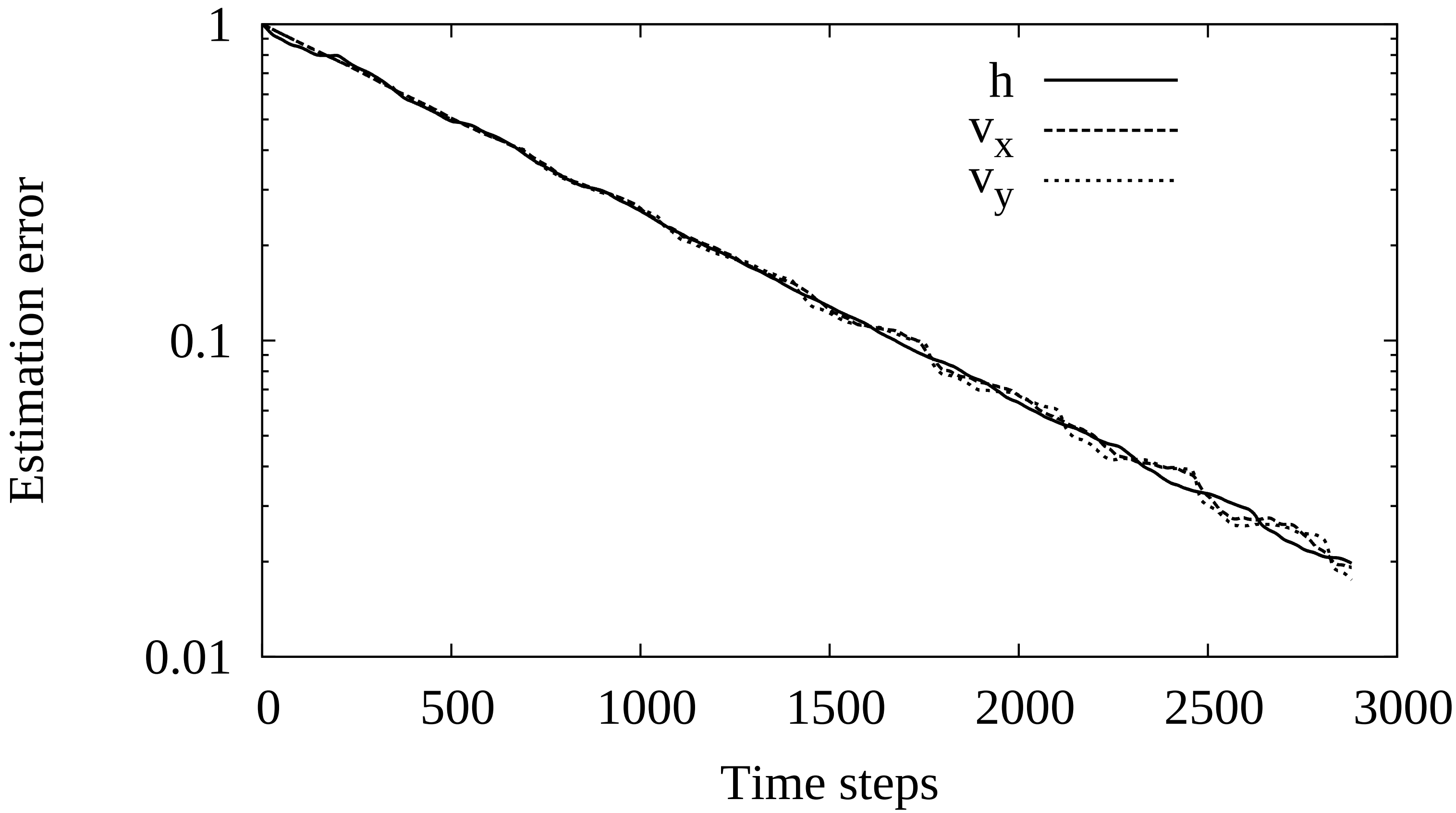
$$\lim_{t\to\infty} \left\| \left(\tilde{h}(t), \frac{\partial \tilde{h}}{\partial t}(t) \right) \right\|_{\mathcal{H}} = 0\,.$$

This theorem proves the strong and asymptotic convergence of the error $\tilde{h}$ towards 0, and then it also gives the same convergence for $\tilde{v}$. We deduce that the observer tends to the true state when time goes to infinity.

Proof : based on Fourier decomposition of the solution.

[Auroux-Bonnabel, IEEE TAC 2011]

Numerical tests



Evolution of the estimation error in relative norm versus the number of time steps, in the case of perfect observations, with $\alpha_h = \alpha_v = 1\ m^{-2}$ and $\beta_h = 5.10^{-7}\ s^{-1}$, and with a 100% error on the initial conditions, for the height h, longitudinal velocity v_x and transversal velocity v_y.

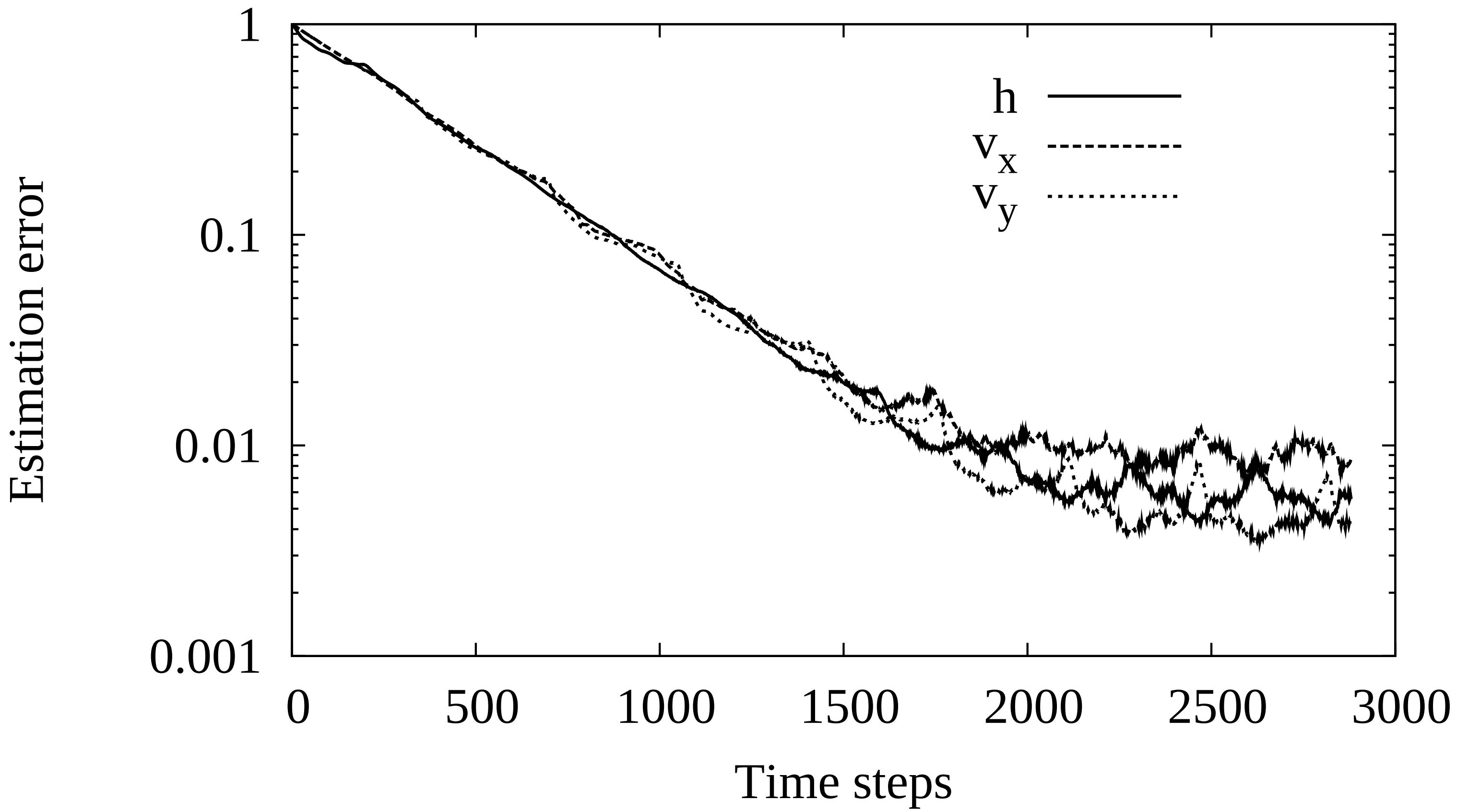
Numerical tests

In the case of discrete observations, we can assume that the height is available everywhere, but not at every time. If for instance we add the correction term to the observer equations only every 12 time steps, the evolution of the estimation error is similar to what is shown on previous figure, with a smaller convergence rate. The convergence rate is approximately 11.7 times smaller than the convergence rate in the full observation case.

$\Rightarrow$ from the numerical point of view, discrete observations in time do not degrade the method.

Note also that the both components of the velocity are corrected, with similar convergence rates, as predicted by the theory.

Numerical tests



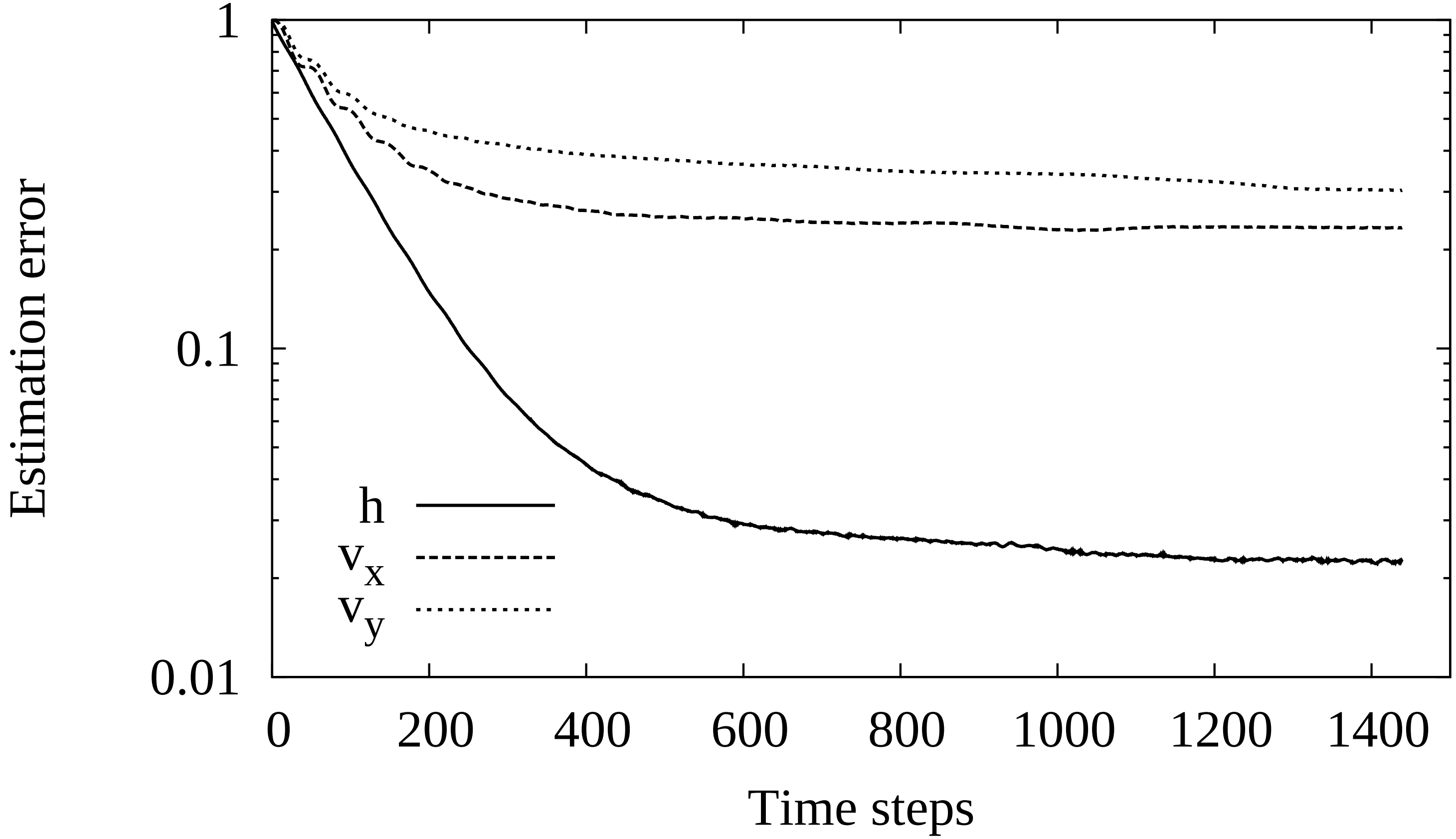
Evolution of the estimation error in relative norm versus the number of time steps, in the case of noisy observations (20% noise), with $\alpha_h = \alpha_v = 1\ m^{-2}$ and $\beta_h = 2.10^{-7}\ s^{-1}$, and with a 100% error on the initial conditions, for the height h, longitudinal velocity v_x and transversal velocity v_y.

Numerical tests

We can compare with standard nudging by taking very large values of α (such that the Gaussian is close to a Dirac function). The estimation error at convergence is nearly 3 times larger with standard nudging than with this estimator.

All variables are identified with less than 1% of error, whereas only h is observed, with a 20% error. Without the term $\nabla(h - h_{obs})$ in the velocity equation, it would not have been possible to estimate precisely v, and neither h. Without the convolution term, the estimation error would have also been much larger, as h_{obs} are noisy, and differentiating noise is not a good idea...

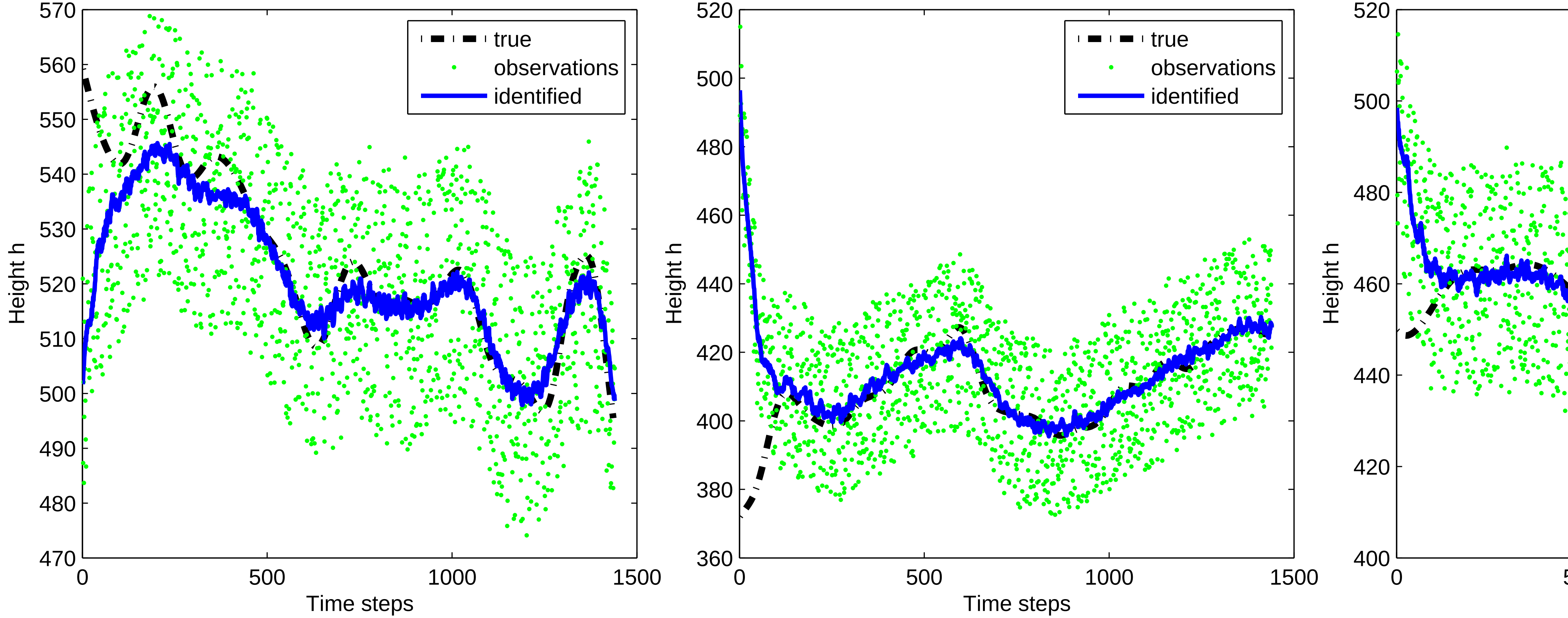
Numerical tests : non-linear model



Full non-linear model : evolution of the estimation error in relative norm versus the number of time steps, in the case of noisy observations (20% noise), for the standard nudging.

Also works for the full non-linear shallow water model (even if the identification is less accurate). The estimation errors are nearly 1.5 smaller with the new observer than with standard nudging.

Numerical tests : non-linear model



Evolution of the true height, the observed (noisy) height, and the identified (observer) height versus time, for three different points of the domain, located along the energetic current in the middle of the domain.

Conclusions and perspectives

Conclusions :

- Nudging is a very simple four-dimensional data assimilation scheme
- Continuous forcing and correction of the model
- Weak constraint : does not assume the model to be perfect
- Physical interpretations and link with other DA methods
- Observer design for non-linear PDE models

Conclusions and perspectives

Perspectives :

- Study of full primitive ocean models : correction of non-observed variables ?
- Observer design for more complex non-linear models
- Real-time data assimilation and weather forecast
- Theoretical convergence results for non-linear PDEs