Localized Eigenfunctions of Hermitian and Non-Hermitian Schroedinger Operators on Geometric Graphs

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Outline

Metric graph — graph with parametrization and metric on edges.

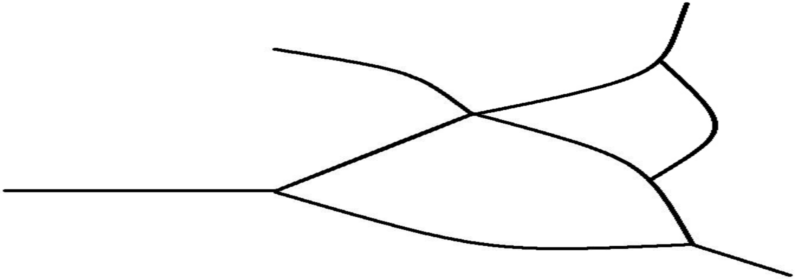

Рис.: Graph

Self-adjoint Schrödinger operator

Schrödinger operator

$$\hat{H} = -\frac{h^2}{2}\Delta + V(x),$$

V is smooth on edges.
Definition of the Laplace operator $\frac{h^2}{2}\Delta$: 2 conditions.

- Δ is self-adjont;
- If M is a disconnected then

$$\Delta = \oplus_j \frac{h^2}{2}\frac{d^2}{dz_j^2}$$

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Laplacian

Formal definition. Consider the direct sum

$$\Delta_0 = \oplus_j \frac{h^2}{2} \frac{d^2}{dz_j^2}$$

with Neumann boundary conditions.

Definition

Δ is a self-adjoint extension of the restriction $\Delta_0|_W$, where

$$W = \{\psi \in Dom(\Delta_0), \quad \psi(q_s) = 0\}.$$

Coupling conditions

Vector $\xi = (u, v)$, $u = (h\psi'(q_1), \dots, h\psi'(q_N))$,
$v = (\psi(q_1), \dots, \psi(q_N))$, q_j — endpoints of the edges.
In $\mathbb{C}^N \oplus \mathbb{C}^N$ consider standard skew-Hermitian form

$$< \xi^1, \xi^2 > = \sum_{j=1}^{N} (u_j^1 \bar{v}_j^2 - v_j^1 \bar{u}_j^2).$$

and fix the Lagrangian (N-dimensional isotropic) plane L.
Coupling conditions

$$\xi \in L, \quad -i(E+U)u + (E-U)v = 0,$$

U is unitary matrix.

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$$\xi \in L, \quad -i(E+U)u + (E-U)v = 0,$$

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Local coupling conditions — for each vertex separately:

$$L = \oplus_q L_q$$

.

For each vertex q $\xi_q = (u_q, v_q)$, $u = (h\psi'(q_1), \dots, h\psi'(q_m))$, $v = (\psi(q_1), \dots, \psi(q_m))$,

$$i(E + U_q)u_q + (E - U_q)v_q = 0,$$

U_q is a unitary $m \times m$-matrix.

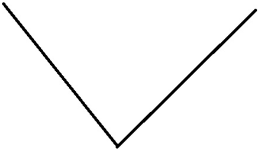
Non-Hermitian Laplacians

Non-Hermitian case: plane L is not Lagrangian (matrix U is not unitary). Then $\xi \in L$ for $\psi \in \mathrm{Dom}(\Delta)$ and $\xi \in L^{\angle}$ for $\psi \in \mathrm{Dom}(\Delta^*)$.

Examples:

- Real Δ (commutes with complex conjugation) $\Leftrightarrow$ real plane L (invariant with respect to complex conjugation)
- Pseudo-Hermitian with respect to complex conjugation $\Leftrightarrow$ plane L is Lagrangian with respect to the skew symmetric form
$$[\xi^1, \xi^2] = \sum_j (u_j^1 v_j^2 - v_j^1 u_j^2).$$

Exotic spectral properties for non-Hermitian case. Example:



If $\psi_1 = \psi_2$, $h\psi_1' = -h\psi_2'$ (self-adjoint case) then $E_n = -(\pi n/l)^2$.
If $\psi_1 = \psi_2$, $h\psi_1' = h\psi_2'$ (real non-Hermitian case) then $E \in \mathbb{C}$.
Further we discuss Hermitian case only.

Outline

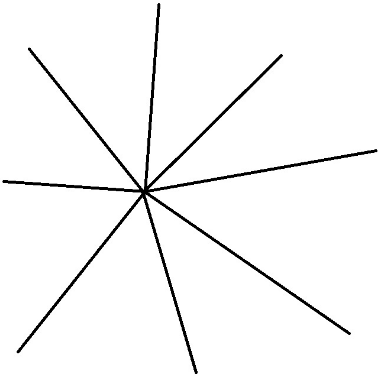
2 sources

2 sources of localized eigenfunctions

- Non-trivial coupling conditions.
- Jumps of the potential in vertices.

vertex

Let M be a star graph, potential is constant on each edge $V|_{e_j} = c_j$, $\varkappa_j = \sqrt{2(c_j - \lambda)}$.



Assertion

Let λ be a solution of equation

$$\det(i(E+U)\mathcal{K}+(E-U))=0,$$

$\mathcal{K}=\mathrm{diag}(\varkappa_1,\dots\varkappa_m)$, φ is an eigenvector

$$(i(E+U)\mathcal{K}+(E-U))\varphi=0.$$

Then λ is an eigenvalue of $\hat{H}$; eigenfunction coincides with $\varphi_j e^{-\varkappa_j x_j/h}$ on j-th edge.

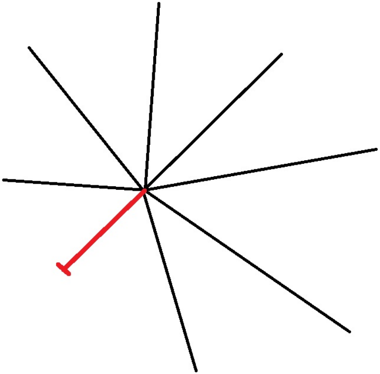
1. As $h \to 0$ eigenfunctions are localized near the vertex.
2. If $c_j = 0$ and U is nonsingular (i.e. -1 is not an eigenvalue), then $\lambda = -\frac{\varkappa^2}{2}$,where $\varkappa$ is a positive eigenvalue of the matrix

$$A = i\frac{E - U}{E + U}.$$

Outline

Segment

Let M be a finite interval, coupled to a star; Dirichlet condition in the vertex of degree 1. Let $V = c_0$ on the finite einterval, $V = c_j > c_0$ on j-th infinite edge, $k = \sqrt{2(\lambda - c_0)}$, $\varkappa_j = \sqrt{2(c_j - \lambda)}$, $j \geq 1$.



Assertion

Let λ be solution of the equation

$$\det(i(E+U)\mathcal{K}_1 + (E-U)\mathcal{K}_2) = 0,$$

$\mathcal{K}_1 = \mathrm{diag}(\cos(kl/h), \varkappa_1, \ldots, \varkappa_{m-1})$,
$\mathcal{K}_2 = \mathrm{diag}(\sin(kl/h), 1, \ldots, 1)$, φ is an eigenvector. Then λ is an eigenvalue of $\hat{H}$, the eigenfunction equals $\varphi_j e^{-\varkappa_j x_j/h}$ on j-th infinite edge and equals $\varphi_0 \sin(kx_0/h)$ on the segment.

1. As $h \to 0$, eigenfunctions are localized near a finite interval.
2. Localized eigenfunctions are produced by the jumps of V in the vertex - these jumps form a potential barrier.

Outline

Let M be arbitrary graph and V be arbitrary potential. Let P be a vertex of degree m and c_j be limits of V in the vertex P. Let $\varkappa_j = \sqrt{2(c_j - \lambda_0)}$, $\lambda_0 < \min\{c_1, \ldots, c_m\}$.

Theorem

Let λ_0 be solution of the equation

$$\det(i(E+U)\mathcal{K}+(E-U))=0,$$

$\mathcal{K}=\mathrm{diag}(\varkappa_1,\ldots,\varkappa_m)$, ϕ_0 is an eigenvector. There exists a point λ of the spectrum of $\hat{H}$, which is represented by the asymptotic serie

$$\lambda \sim \sum_{k=0}^{\infty} h^k \lambda_k.$$

For each K there exists a function

$$\psi^j_{(K)}(x_j) = e^{\frac{-S_j(x_j)}{h}} \sum_{k=0}^{K} h^k \varphi^j_k(x_j), \quad \varphi^j_0(0) = \phi^j_0,$$

with the following properties.

1. S_j, φ^j_k are polynomials, $\Re S_j > 0$.
2. There exists a function $\psi^{(K)}$ from the domain of $\hat{H}$, such that on j-th edge $\psi^{(K)} = \psi^j_{(K)}(x) + o(h^K)$, vanishing on the remaining part of M and satisfying the spectral equation

$$\hat{H}\psi^{(K)} = \lambda^{(K)}\psi^{(K)} + o(h^K), \quad \lambda^{(K)} = \sum_{j=0}^{K} h^j \lambda_j$$

Outline

Let M contains a vertex of degree 1; let e be the edge, starting from this vertex and let P be the second enpoint of e . Let $x \in [0, l]$ be coordinate on e, x_j be coordinates on the other edges, coming to P, $x_j(P) = 0$. Let c_j be the limits of V in P, and let $c_j > \max_{x \in e} V(x)$. Let $\mathcal{K}_1 = \mathrm{diag}(\cos\Phi, \varkappa_1, \ldots, \varkappa_{m-1})$, $\mathrm{K}_2 = \mathrm{diag}(\sin\Phi, 1, \ldots, 1)$, $\mathcal{K}_0 = \mathrm{diag}(0, \varkappa_1, \ldots, \varkappa_{m-1})$.

$$\Phi = \frac{1}{h}\int_0^l \sqrt{2(\lambda_0 - V(x))}dx + \lambda_1 \int_0^l \frac{dx}{\sqrt{2(\lambda_0 - V(x))}},$$

$$\varkappa_j = \sqrt{2(c_j - \lambda_0)}, \quad j = 1, \ldots, m-1, \quad \max_{x \in e} V(x) < \lambda_0 < \min_j c_j,$$

Theorem

Let the matrix $i(E+U)\mathcal{K}_0 + (E-U)$ be nondegenerate. Let λ_1 satisfy quantization condition

$$\det(i(E+U)\mathcal{K}_1 + (E-U)\mathcal{K}_2) = 0.$$

Then there exists a poin λ from the spectrum of $\hat{H}$, such that $|\lambda - (\lambda_0 + h\lambda_1)| = O(h^2)$.

Quantization condition

$$F_1 \cos \Phi + F_2 \sin \Phi = 0,$$

$F_j(\lambda_0)$ are smooth functions. Alternatively,

$$\frac{1}{\pi h} \int_0^l \sqrt{2(\lambda_0 - V(x))} dx + \frac{\lambda_1}{\pi} \int_0^l \frac{dx}{\sqrt{2(\lambda_0 - V(x))}} + \frac{1}{\pi} \arctan \frac{F_1}{F_2} = M,$$

$M \in \mathbb{Z}$, $\quad M = O(\frac{1}{h})$. Analog of the Maslov index

$$\frac{4}{\pi} \arctan \frac{F_1}{F_2} \in [-2, 2]$$

Explicit formula for λ_1:

$$M_0 = [\frac{1}{\pi h}\int_0^l \sqrt{2(\lambda_0 - V(x))}dx],$$

$$\lambda_1 = \frac{\left(- \arctan\frac{F_1}{F_2} - \pi\{\frac{1}{\pi h}\int_0^l \sqrt{2(\lambda_0 - V(x))}dx\} + \pi M_1 \right)}{\int_0^l \frac{dx}{\sqrt{2(\lambda_0 - V(x))}}},$$

where $M_1 = M - M_0 = O(1)$ (here $[\circ]$ and $\{\circ\}$ are integral and fractional parts).

Outline

Let M_0 be a subgraph of a graph M, containing finite edges only; a vertex of M_0 is said to be interior if all edges incident to the vertex belong to M_0, and a vertex is said to be boundary otherwise. Consider the boundary vertices; for each of them, we compute the limits c_j of the potential V along all edges not belonging to M_0. Suppose that the following condition is satisfied:

$$\max_{M_0} V(x) < \min_j c_j.$$

This condition is central for the existence of semiclassical eigenfunctions localized near M_0; the "barriers" formed by the potential at the boundary vertices, prevent the quantum particle from leaving the subgraph.

Algorithm of construction of semi-classical eigenfunctions.
Step 1.
Consider an edge of the graph M that does not belong to M_0 and is incident to one of the boundary vertices of the subgraph. Construct a semiclassical $\mathrm{mod}O(h^2)$ solution (of the Schrödinger equation) which is localized near this vertex; the corresponding function has the form $\varphi(x)\mathrm{e}^{-S(x)/h}$, where x is the coordinate on the corresponding edge. Denote by b the value of the function φ at the vertex.

Step 2.
On every edge of the subgraph M_0, we construct the WKB-asymptotics $\mathrm{mod}O(h^2)$ of the solution of the Schrödinger equation of the form

$$\psi_j(x) = \frac{1}{(\lambda_0 - V(x))^{1/4}} \left(b_1 \cos \left(\frac{S(x)}{h} + \lambda_1 \int_{x_0}^{x} \frac{dx}{S'(x)} \right) + b_2 \sin \left(\frac{S(x)}{h} + \lambda_1 \int_{x_0}^{x} \frac{dx}{S'(x)} \right) \right).$$

Step 3.
Let us substitute all these functions into the boundary conditions at the vertices of the subgraph M_0; we obtain a homogeneous system of linear equations for a vector of the coefficients b, b_1, b_2 collected over all edges incident to the vertices of M_0 (it is easy to show that this is a square system). Equating the determinant of this system to zero, we obtain the quantization condition, from which, similarly to the previous section, we obtain a correction for the eigenvalue λ_1 (for a fixed λ_0).

THANK YOU
FOR YOUR ATTENTION!