The Lyapunov equation and non-Hermitian Hamiltonians in open quantum systems

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Open quantum systems-introduction

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- Integrate out the bath degrees of freedom to obtain effective non-unitary dynamics of the system.

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Yes!
But there will be 'noise' (quantum/thermal fluctuations).

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The Markovian non-unitary QME (Gorini, Kossakowski, Sudarshan, Lindblad):
$\frac{\partial \rho}{\partial t} = i[\rho, \hat{H}] + \sum_{\alpha,\nu} \mathbf{m}_{\alpha\nu}\big(\hat{V}_\alpha \rho \hat{V}_\nu^\dagger - \frac{1}{2}\{\hat{V}_\nu^\dagger \hat{V}_\alpha, \rho\}\big)$.

$\mathbf{m} \rightarrow$ Hermitian and positive semi-definite matrix.
$\hat{H} \rightarrow$ Hermitian system operator.
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V. Gorini, A. Kossakowski, E. C. G. Sudarshan, Journal of Mathematical Physics, **17**, 821, 1976,
G. Lindblad, Communications in Mathematical Physics, **48**, 119, 1976.

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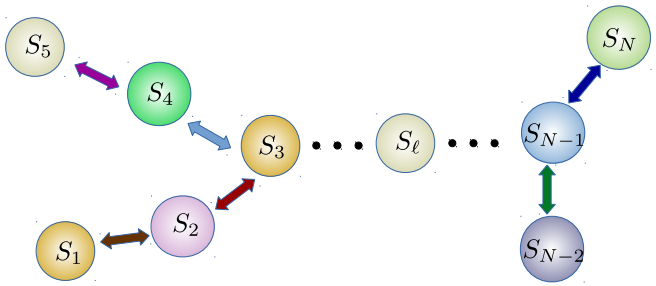

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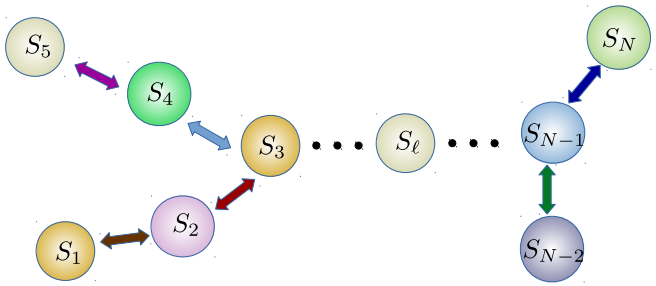

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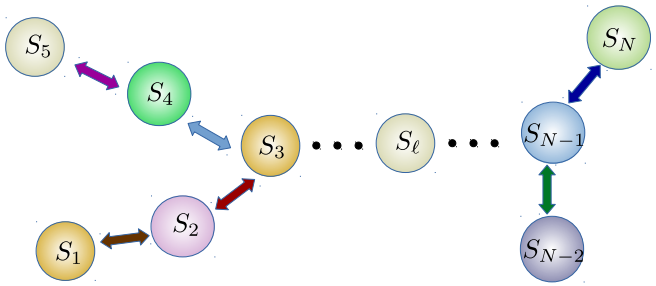

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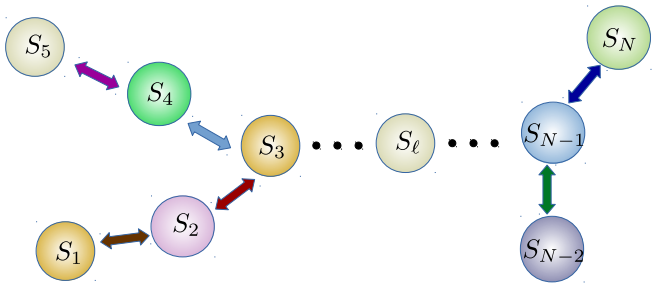

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$$\frac{d\mathbf{C}}{dt} = -i[\mathbf{C}, \mathbf{D}]$$

General non-interacting system (open, 1 bath)

(Bath attached at ℓth site)

$$\hat{\mathcal{H}}_B = \sum_{r=1}^{\infty} \Omega_{\ell r} \hat{B}^\dagger_{\ell r} \hat{B}_{\ell r}, \quad \hat{\mathcal{H}}_{SB} = \varepsilon \sum_r \kappa_{\ell r} (\hat{B}^\dagger_{\ell r} \hat{a}_\ell + \hat{a}^\dagger_\ell \hat{B}_{\ell r}), \quad \varepsilon \ll 1$$

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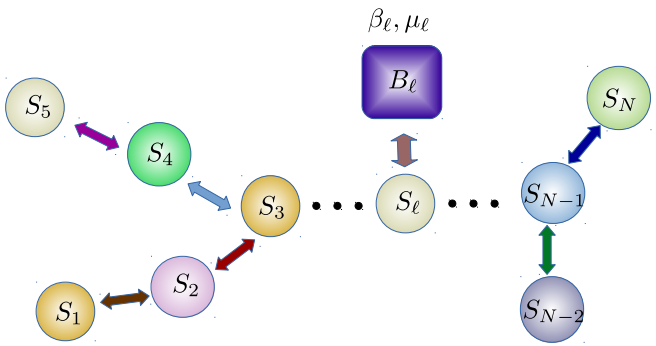

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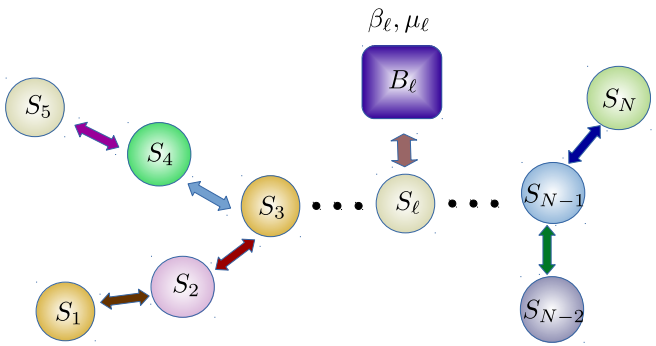

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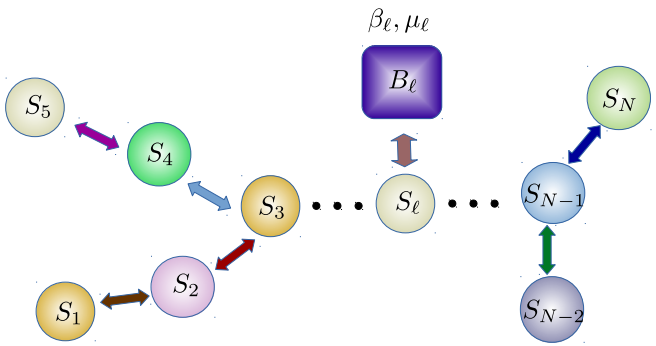

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Under Born-Markov approximation:

- Born: Up to $O(\varepsilon^2)$.
- Markov:
  Bath 'relaxation time' is much faster than system time scales.

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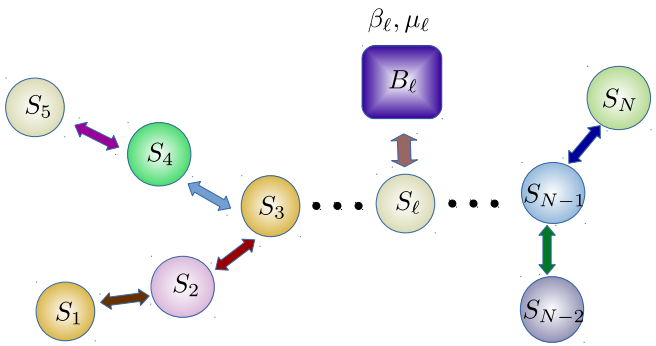

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$$\mathfrak{J}_\ell(\omega) = 2\pi \sum_r \mid \kappa_{\ell r} \mid^2 \delta(\omega - \Omega_{\ell r}), \ \mathfrak{n}_\ell(\omega) = (exp(\beta_\ell(\omega - \mu_\ell) \pm 1)^{-1}$$

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$$\frac{\partial \rho}{\partial t} =$$
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$$\mathbf{v}^\ell = [\Phi_\ell]^\dagger [\Phi_\ell] \left([\mathfrak{J}_\ell] + i[\mathfrak{J}^\Delta_\ell] \right)$$

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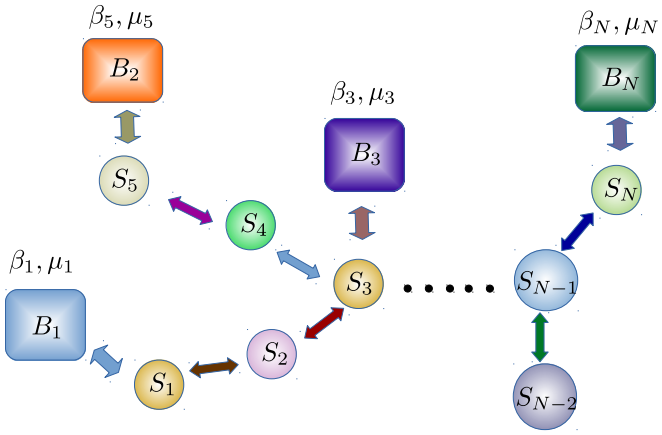

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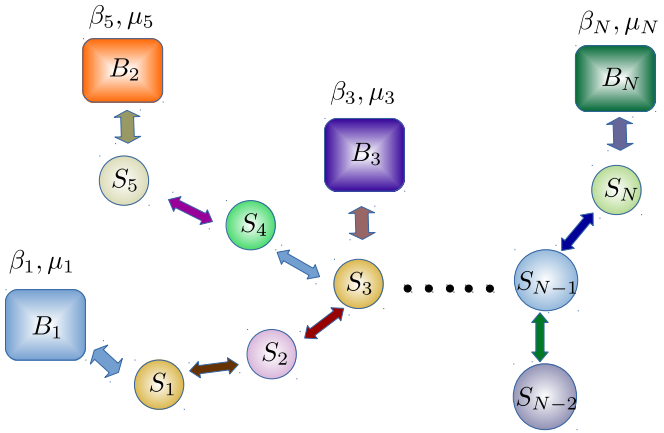

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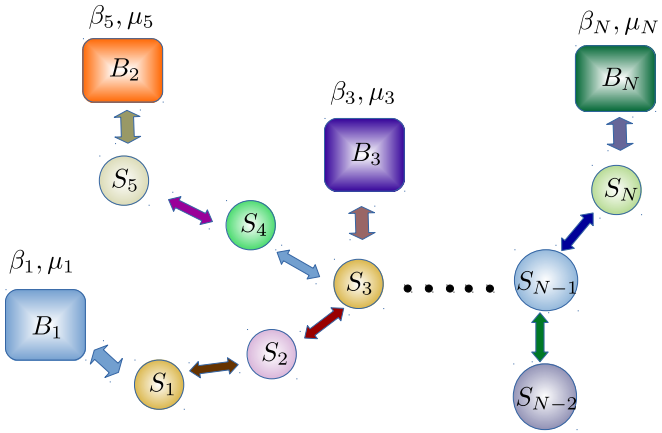

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Arbitrary dimension
and geometry of lattice, arbitrary number of sites attached to baths, under Born-Markov approximation.

The Lyapunov equation: solution and unique NESS

Solution of the Lyapunov equation:

$$\mathbf{C}(t) = e^{-\mathcal{G}t}\mathbf{C}(0)e^{-\mathcal{G}^\dagger t} + \varepsilon^2 \int_0^t dt' e^{-\mathcal{G}t'}\mathbf{Q}e^{-\mathcal{G}^\dagger t'}$$

$\mathcal{G} = -i\hat{H}_{eff}$, $\hat{H}_{eff} = \mathbf{D} - i\varepsilon^2\mathbf{v}$
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sufficient condition: $\mathbf{v} + \mathbf{v}^\dagger \rightarrow$ positive definite.

Accuracy of the Lyapunov equation

$C_{\alpha\nu}(t)$ is obtained correct to the leading order contribution from the baths, i.e,
$C_{\alpha\alpha}(t) \rightarrow$ correct to $O(\varepsilon^0)$
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Checked against exact (i.e, without Born-Makrov approximation) results for a simple system in

Archak Purkayastha, Abhishek Dhar, Manas Kulkarni, Phys. Rev. A, **93**, 062114, (2016).

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(C. H. Fleming, N. I. Cummings, Phys. Rev. E, **83**, 031117, 2011)

Checked against exact (i.e, without Born-Makrov approximation) results for a simple system in

Archak Purkayastha, Abhishek Dhar, Manas Kulkarni, Phys. Rev. A, **93**, 062114, (2016).

Works very well!

An extension of the Lyapunov equation

Simple extension: AC drive from the baths:

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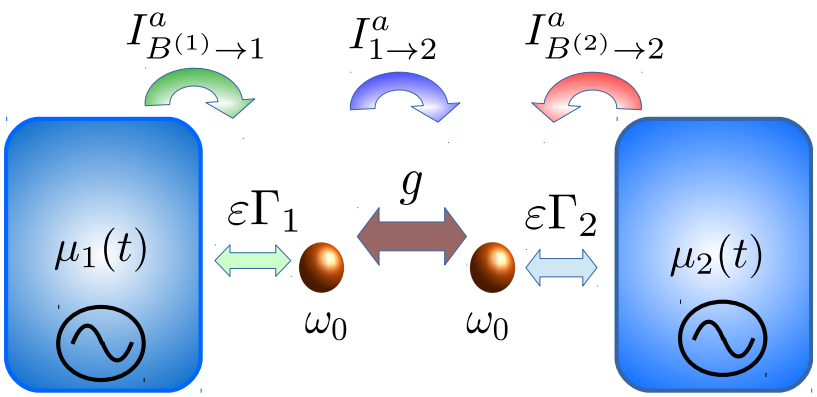
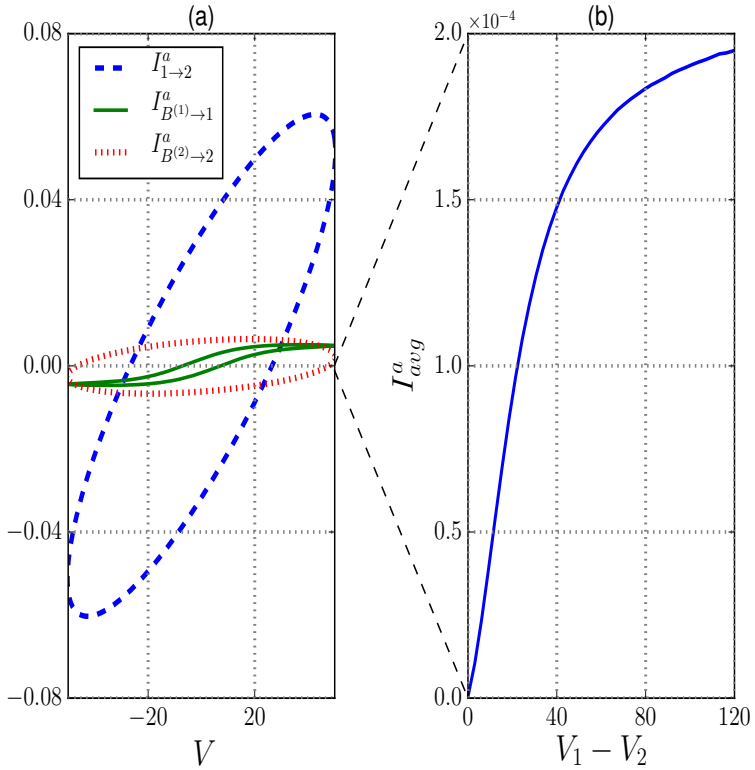
Time dependence is in the inhomogeneous part $\rightarrow$ tremendous simplification!

Two-sites AC drive

Archak Purkayastha, Yonatan Dubi, Phys. Rev. B **96**, 085425 (2017).



$$\mu_1(t) = V_1 \cos(2gt),$$
$$\mu_2(t) = V_2 \sin(2gt)$$



$$V(t) = \mu_1(t) - \mu_2(t)$$

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- Can we make the 'non-Hermitian Hamiltonian' $\mathcal{PT}$-symmetric?

Thank You