Laplacians and wave equation on polyhedral surfaces.

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Outline

Spherical polyhedron.

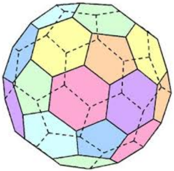

Figure: Polyhedron

Non-spherical polyhedron.

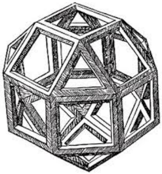

Figure: Polyhedron

Polyhedral surface M. Arbitrary number of vertices with total angles β_j. Gauss-Bonnet relation

$$\sum_j (1 - \frac{\beta_j}{2\pi}) = \chi(M).$$

Complex structure. On a flat face complex coordinate $z = x_1 + ix_2$. Near a point of an edge — unfolding.

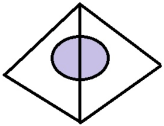

Figure: Coordinate near a point of an edge

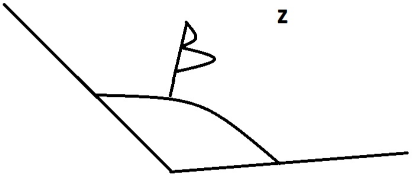

Figure: Coordinate near a vertex

Near the vertex $\zeta = z^{\frac{2\pi}{\beta}}$.

Flat metric with singularities at the vertices

$$ds^2 = (\frac{\beta}{2\pi})^2 |\zeta|^{2(\frac{\beta}{2\pi}-1)} d\zeta d\bar{\zeta}.$$

Near a vertex Laplacian has the form

$$\Delta = c^2(y)(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}),$$

$$c(y) = (\frac{2\pi}{\beta})(y_1^2 + y_2^2)^{\frac{1}{2}(1-\frac{\beta}{2\pi})}.$$

Velocity of waves has point singularities.

Self-adjoint Laplacian

Definition of the Hermitian Laplace operator Δ: 2 conditions.

- Δ is self-adjont;
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Laplacian

Formal definition. Smooth Riemannian manifold $M_0 = M \setminus \{P_1, \dots P_N\}$, P_j — vertices. On $C_0^\infty(M_0)$ — usual Δ_0. $\tilde{\Delta}_0$ — closure with respect to the graph norm

$$||u||_\Delta^2 = ||u||^2 + ||\Delta_0 u||^2.$$

Definition

Δ is a self-adjoint extension of $\tilde{\Delta}_0$.

Coupling conditions

Boundary conditions at the vertices:

$$F_0^+ = 1, \quad F_0^- = \log r,$$

$$F_k^\pm = (\frac{2\pi|k|}{\beta})^{-1/2} r^{\pm(\frac{2\pi|k|}{\beta})} \mathrm{e}^{\frac{2\pi i k\theta}{\beta}}, \quad |k| < \frac{\beta}{2\pi}.$$

Functions have singularities

$$\psi = \sum_k (\alpha_k^+ F_k^+ + \alpha_k^- F_k^-) + \psi_0, \quad \psi_0 = O(r).$$

Coupling conditions

Vector $\xi = (u, v)$, $u = (\alpha_1^-, \dots \alpha_s^-)$, $v = (\alpha_1^+, \dots \alpha_s^+)$.
In $\mathbb{C}^s \oplus \mathbb{C}^s$ consider standard skew-Hermitian form

$$< \xi^1, \xi^2 >= \sum_{j=1}^{s} (u_j^1 \bar{v}_j^2 - v_j^1 \bar{u}_j^2).$$

and fix the Lagrangian (s-dimensional isotropic) plane L.
Coupling conditions

$$\xi \in L, \quad i(E + U)u + (E - U)v = 0,$$

U is unitary matrix.

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Local coupling conditions — for each vertex separately:

$$L = \oplus_q L_q$$

.

For each vertex P $\xi_P = (u_P, v_P)$, $u_P = (\alpha_1^-(P), \dots \alpha_N^-(P))$, $v_P = (\alpha_1^+(P), \dots \alpha_N^+(P))$,

$$i(E + U_q)u_q + (E - U_q)v_q = 0,$$

U_P is a unitary $N \times N$-matrix.

Non-Hermitian Laplacians

Non-Hermitian case: plane L is not Lagrangian (matrix U is not unitary). Then $\xi \in L$ for $\psi \in \mathrm{Dom}(\Delta)$ and $\xi \in L^{\angle}$ for $\psi \in \mathrm{Dom}(\Delta^*)$.

Examples:

- Real Δ (commutes with complex conjugation) $\Leftrightarrow$ real plane L (invariant with respect to complex conjugation)
- Pseudo-Hermitian with respect to complex conjugation $\Leftrightarrow$ plane L is Lagrangian with respect to the skew symmetric form

$$[\xi^1, \xi^2] = \sum_j (u_j^1 v_j^2 - v_j^1 u_j^2).$$

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Harmonic functions

$$\Delta\psi = 0.$$

Theorem

The space of harmonic functions is isomorphic to the intersection $L \cap L_0$, where L_0 is a fixed Lagrangian plane defined by M only. It can me expressed in terms of Mittag-Leffler problem on M.

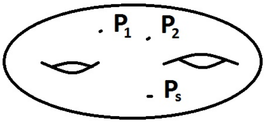

Figure: Mittag-Leffler problem

Find meromorphic 1-form ω with prescribed Laurent part

$$\omega = (\sum_j \frac{a_j}{(z - P_l)^j})dz + \omega_0$$

Convex polyhedron

Example: convex polyhedron in $\mathbb{R}^3$ with local boundary conditions.

$$f = c_0 + \sum_{j=1}^{N} c_j \log|z - z_j|,$$

$$\cos\theta_j (c_0 + \sum_{i \neq j} c_i \log|z_i - z_j|) + \sin\theta_j c_j = 0,$$

$$\sum_j c_j = 0.$$

Example: one vertex with angle $\beta \in (2\pi, 4\pi)$ and $N-1$ angles less then 2π. Harmonic functions

$$u = c_0 + \left(\frac{\beta}{2\pi}\right)^{1/2}\left(\frac{a_1}{z - z_1} + \frac{a_2}{\bar{z} - \bar{z}_1}\right) + a_3 \log|z - z_1| + \sum_{j=2}^{N} c_j \log|z - z_j|.$$

Vectors $\alpha^- = (a_1, a_2, a_3, c_2, \dots, c_N)$,
$\alpha^- = (b_1, b_2, b_3, d_2, \dots, d_N)$.

$$b_1 = \frac{1}{2}\left(\frac{\beta}{2\pi}\right)^{-1/2} \sum_{j=2}^{N} \frac{c_j}{z_1 - z_j}, \quad b_2 = \frac{1}{2}\left(\frac{\beta}{2\pi}\right)^{-1/2} \sum_{j=2}^{N} \frac{c_j}{\bar{z}_1 - \bar{z}_j},$$

$$b_3 = c_0 + \sum_{j=2}^{N} c_j \log|z_1 - z_j|,$$

$$d_j = c_0 + (\frac{\beta}{2\pi})^{1/2}(\frac{a_1}{z_j - z_1} + \frac{a_2}{\bar{z}_j - \bar{z}_1}) + \sum_{k \neq j} c_k \log|z_k - z_j|.$$

Coupling conditions

$$c_j \cos\theta_j + d_j \sin\theta_j, \quad j = 2, \ldots, N, \quad \sum_{j=2}^{N} + a_3 = 0,$$

$$i(E + U)a + (E - U)b = 0,$$

U is a unitary 3×3-matrix.

McKean-Singer formula

$$(4\pi t)^{d/2}\mathrm{tr}(\mathrm{e}^{t\Delta}) = vol(M) + \frac{t}{3}\int_M K d\sigma + \frac{\pi t^2}{180}\int_M P_2(R) d\sigma + \dots$$

For smooth compact 2D surface M

$$\mathrm{tr}(\mathrm{e}^{t\Delta}) = \frac{Area(M)}{4\pi t} + \frac{1}{6}\chi(M) + O(t).$$

Theorem

Let $\alpha^- = 0$ (Friedrichs Laplacian). Then

$$\mathrm{tr}(\mathrm{e}^{t\Delta}) = \frac{Area(M)}{4\pi t} + \frac{1}{12}\sum_k (\frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi}) + O(\mathrm{e}^{-c/t}).$$

For arbitrary Hermitian Laplacian

$$\mathrm{tr}(\mathrm{e}^{t\Delta}) = \frac{Area(M)}{4\pi t} + \frac{1}{12}\sum_k \left(\frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi}\right) + \sum_{j=1}^{\infty} t^{j/2} q_j(\log t),$$

$$q_j(\log t) = \sum_s \frac{g_{js}}{(\log t)^s}.$$

Cauchy problem for the wave equation

$$u_{tt} = \Delta u, \quad u\big|_{t=0} = u_0, \quad u_t\big|_{t=0} = u_1.$$

For a pyramide — exact solution in terms of special integral transform.

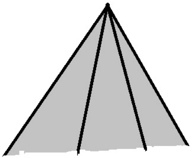

Figure: Pyramide

Integral transform

$$u(r) = \alpha K_0(\varkappa r) + \int_0^\infty W(r,p)\hat{u}(p)pdp,$$

$$W(r,p) = \sin\theta\, Y_0(pr) - 2(\cos\theta + \sin\theta(\log\frac{p}{2} + \gamma))J_0(rp)$$

$$\varkappa = 2\mathrm{e}^{-(\gamma + \cot\theta)}.$$

δ-type initial function.

$$u_{tt} = \Delta u, \quad u|_{t=0} = u_0(\frac{z - z_0}{\varepsilon}), \quad u_t|_{t=0} = 0, \quad \varepsilon \to 0$$

z_0 is a point of a face, $u_0(y) \to 0$ as $|y| \to \infty$ rather rapidly.

Definition

The asymptotic support of the solution u is the set Q_t:

$$u(x, t) = O(\varepsilon^2), \quad x \notin Q_t.$$

Assertion

For sufficiently small t

$$Q_t : |z - z_0| = t.$$

Scattering on a vertex

Theorem

Let M be an infinite pyramid. Then for sufficiently large t

$$Q_t = Q_1 \cup Q_2$$

Q_1 is formed by the endpoints of geodesics, starting from z_0, Q_2 is formed by the endpoints of geodesics, starting from the vertex.

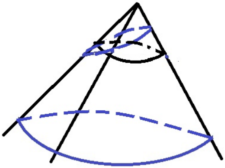

Figure: Pyramid

Outline

General result

Let $m(t)$ be the maximal number of geodesics, connecting pairs of vertices, s.t. their lengths are at most t. **Case 1: suppose that $m(t) = \alpha t^{\gamma}(1 + t^{-\epsilon})$.**

Theorem

$$\log N(t) = (\gamma + 1)\left(\frac{\alpha\Gamma(\gamma+1)\zeta(\gamma+1)}{\gamma^{\gamma}}\right)^{\frac{1}{\gamma+1}} t^{\frac{\gamma}{\gamma+1}}(1 + o(1))$$

Outline

Case 2: suppose that $m(t) = \mathrm{e}^{Ht(1+O(t^{-\epsilon}))}$ (usually — positive topological entropy).

Theorem

$$\log N(t) = Ht(1 + o(1)).$$

Key point of the proof — reduction to the certain problem of analytic number theory — namely, the problem of abstract primes.

Abstract primes

Arithmetical semigroup $G = \oplus_{j \in J} \mathbb{Z}_+$, J - countable.
Homomorphism $\rho : G \to \mathbb{R}_+$, we identify j with the generator of $\mathbb{Z}_+$.

$$m(t) = \sharp\{j \in J | \rho(j) \leq t\}, \quad N(t) = \sharp\{g \in G, \rho(g) \leq t\}.$$

If one knows the asymptotics of m (N), how to compute the asymptotics of N (m)?
j—primes,$\rho(j)$=log j: $m(\log t)$ — distribution of primes,
j—integers, $\rho(j) = j$: $N(t)$ — number of partitions of integer t.

THANK YOU
FOR YOUR
ATTENTION!