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Keldysh Technique. June - 2018, ICTS

OUTLINE.

1. FORMALISM. Connection with equilibrium.

2. Application: Non-interacting bosons.

3. Application: Two level system coupled to bath

4. Application: Spatially extended system coupled to bath. a) Voltage induced ferromagnetic - paramagnetic phase transition.
b). Mapping to Langevin equations in the high temperature and/or high voltage limit.

Refs:
1. A. Kamenev, Nonequilibrium Field Theory.
2. A. Mitra, S. Takei, Y. B. Kim, A.J. Millis, PRL, 97, 236808 (2006).

Suppose at $t=0$, system is described by the wave-function $|\phi_H\rangle$. At $t>0$, the system evolves according to H.

Then,

$$\langle F(t)\rangle = \langle \phi_H | e^{iHt} F e^{-iHt} | \phi_H\rangle$$

In the Schrödinger representation

$$|\phi_s(t)\rangle = e^{-iHt} |\phi_H\rangle$$

$$\therefore \langle F(t)\rangle = \langle \phi_s(t) | F | \phi_s(t)\rangle .$$

If, $H = H_0 + V$ where H_0 is exactly solvable, and V something complicated, it is convenient to go into the interaction representation

$$|\phi_i(t)\rangle = e^{iH_0 t} |\phi_s(t)\rangle = e^{iH_0 t} e^{-iHt} |\phi_H\rangle$$

$$\therefore \frac{d}{dt}|\phi_i(t)\rangle = i e^{iH_0 t}(H_0 - H) e^{-iHt} |\phi_H\rangle$$

$$= -i V_{int}(t) |\phi_i(t)\rangle$$

where $V_{int}(t) = e^{iH_0 t} V e^{-iH_0 t}$

$$\therefore \quad |\phi_i(t)\rangle = S(t,t_0)\,|\phi_i(t_0)\rangle$$

where

$$S(t,t_0) = T\, e^{-i\int_{t_0}^{t} dt'\, V_{int}(t')}$$

Note: $S(t,t')S(t',t'') = S(t,t'')$

$$\Rightarrow \langle F(t)\rangle = \langle \phi_H|\, e^{iHt}\, \underbrace{e^{-iH_0t}\, e^{iH_0t}}_{1}\, F\, \underbrace{e^{-iH_0t}\, e^{iH_0t}}_{1}\, e^{-iHt}\,|\phi_H\rangle$$

$$= \langle \phi_i(t)|\, F_{int}(t)\,|\phi_i(t)\rangle \quad \text{where } F_{int}(t) = e^{iH_0t}\, F\, e^{-iH_0t}$$

$$\boxed{\text{Since } S(t,t_0)\,S(t_0,t) = 1 \Rightarrow S(t_0,t) \text{ is inverse of } S(t,t_0)}$$

$$\Rightarrow \langle F(t)\rangle = \langle \phi_i(t_0)|\, \underbrace{S(t_0,t)}_{\text{backward evolution}}\, F_{int}(t)\, \underbrace{S(t,t_0)}_{\text{forward evolution}}\,|\phi_i(t_0)\rangle$$

Let us suppose $|\phi_i(t_0)\rangle$ is an eigenstate of H_0, denoted by $|\phi_0\rangle$

$$\Rightarrow |\phi_i(t_0)\rangle = e^{iH_0t}\,|\phi_s(t)\rangle = e^{i\mathcal{E}_0 t}\,|\phi_0\rangle$$

where $H_0|\phi_0\rangle = \mathcal{E}_0|\phi_0\rangle$

This phase factor cancels with the one coming from $\langle \phi_i(t_0)|$, so that

$$\boxed{\langle F(t)\rangle = \langle \phi_0|\, S(t_0,t)\, F_{int}(t)\, S(t,t_0)\,|\phi_0\rangle}$$

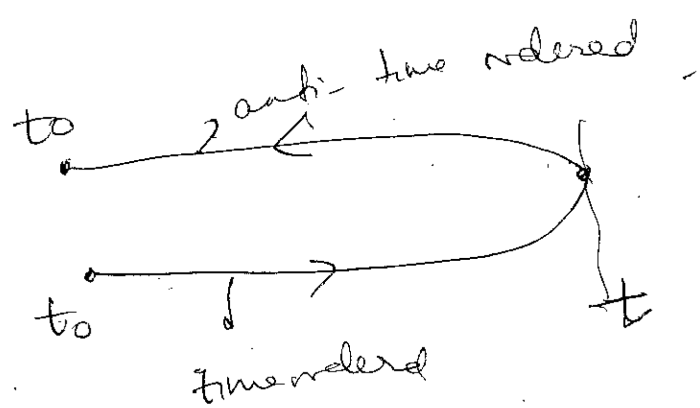
Even if $|\phi_i(t_0)\rangle$ is not an eigenstate of H_0, but some known initial state: $e^{iH_0t}|\phi_0\rangle$, then we write

$$\boxed{\langle F(t)\rangle = \langle \phi_0|\, e^{-iH_0 t_0}\, S(t_0,t)\, F_{int}(t)\, S(t,t_0)\, e^{iH_0 t}\, |\phi_0\rangle}$$

Since H_0 is exactly solvable

$e^{iH_0t}|\phi_0\rangle$ can be exactly computed even when $|\phi_0\rangle$ is not an eigenstate of H_0.

Thus in Keldysh:



CONNECTION WITH EQUILIBRIUM

Suppose V is 0 at $t = -\infty$, and gradually switched on so that it reaches its full magnitude at $t=0$.

$$\therefore H = \hat{H}_0 + \hat{V} e^{-\epsilon|t|}$$

Then if $H_0|\phi_0\rangle = W_0|\phi_0\rangle$, $|\phi_0\rangle$: is ground state

and $\left((\hat{H}_0 + \hat{V})|\psi_0\rangle = E_0|\psi_0\rangle\right)$ where $|\psi_0\rangle$ is full interacting ground state, then,

~~theor~~ Gell-Mann-Low Adiabaticity theorem states

$$\frac{|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = \lim_{\epsilon\to 0} \frac{|\psi_\epsilon\rangle}{\langle\phi_0|\psi_\epsilon\rangle} = \lim_{\epsilon\to 0} \frac{S_\epsilon(0,-\infty)|\phi_0\rangle}{\langle\phi_0|S_\epsilon(0,-\infty)|\phi_0\rangle}$$

The above is true only if r.h.s. exists to all orders in perturbation theory. Then as V is slowly turned on, $|\phi_0\rangle$ adiabatically evolves to $|\psi_0\rangle$.

On can easily see that if V and H_0 are such that $\langle \phi_0 | \psi_0 \rangle = 0$, then

$$\lim_{\epsilon \to 0} \frac{S_\epsilon(0,-\infty)\,|\phi_0\rangle}{\langle \phi_0 | S_\epsilon(0,-\infty) | \phi_0\rangle} \quad \text{will diverge.}$$

This excludes many system such as

H_0 : band insulator,
$H_0 + V$; Topological insulator

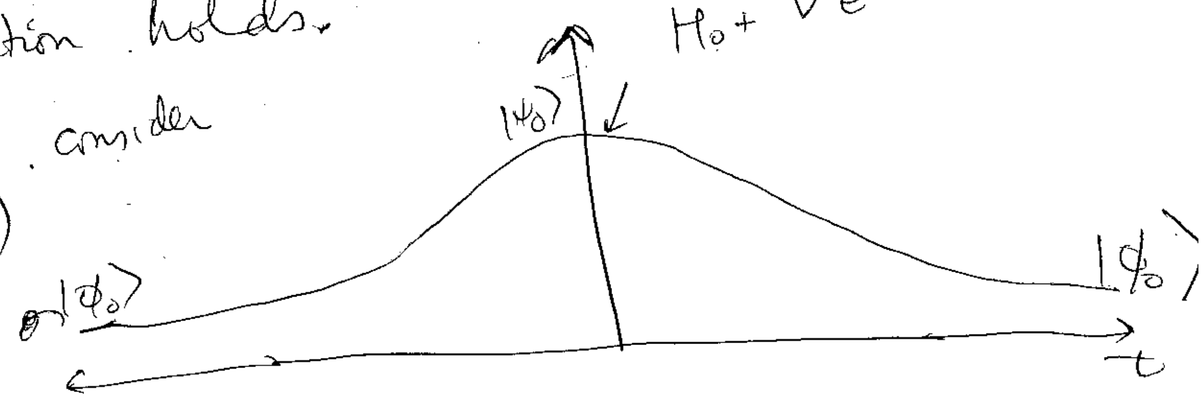
or

H_0 ! disordered (short-ranged correlations)
$H_0 + V$; Long-range order.

etc.

However, let us say, that adiabaticity condition holds.

Then consider $S(-\infty,\infty)$



Then

$$\langle \phi_0 | S(-\infty, \infty) = \langle \phi_0 | e^{iL} \quad \text{where } e^{iL} \text{ is a phase.}$$

$$e^{iL} = \langle \phi_0 | S(-\infty, \infty) | \phi_0 \rangle \Big| \cdot \bar{e}^{iL} = \langle \phi_0 | S(\infty, -\infty) | \phi_0 \rangle$$

The way Keldysh reduces to equilibrium zero-temperature formalism is as follows:

In Keldysh the chosen time dependence gives

$$\langle F(t) \rangle = \langle \phi_0 | S(-\infty, t) \, F_{int}(t) \, S(t, -\infty) | \phi_0 \rangle$$

Write $S(-\infty, t) = S(-\infty, \infty) \, S(\infty, t)$

$$\langle F(t) \rangle = \underbrace{\langle \phi_0 | S(-\infty, \infty)} \, S(\infty, t) \, F_{int}(t) \, S(t, -\infty) | \phi_0 \rangle$$

$$= \langle \phi_0 | e^{iL} \, S(\infty, t) \, F_{int}(t) \, S(t, -\infty) | \phi_0 \rangle$$

$$= \frac{\langle \phi_0 | S(\infty, t) \, F_{int}(t) \, S(t, -\infty) | \phi_0 \rangle}{\bar{e}^{iL}}$$

$\Rightarrow$ Zero temperature equilibrium:

$$\langle F(t) \rangle = \frac{\langle \phi_0 | S(\infty, t) \, F_{int}(t) \, S(t, -\infty) | \phi_0 \rangle}{\langle \phi_0 | S(\infty, -\infty) | \phi_0 \rangle}$$

Two time contour replaced by one time contour if adiabaticity theorem holds.

For pure state

$$|\phi_0\rangle\langle\phi_0| = \text{density matrix } \hat{\rho}.$$

Can easily generalize to mixed state

where $\hat{\rho} = \sum_n e^{-\beta E_n} |n\rangle\langle n|$

Note, not necessary to be in an equilibrium initial state.

$$\langle F(t)\rangle = \mathrm{Tr}\left[S(t_0,t)\, F_{int}(t)\, S(t,t_0)\, \hat{\rho} \right]$$

2. Non-interacting bosons.

$$H = \omega_0 b^\dagger b$$

$b, b^\dagger \rightarrow$ Boson annihilation and creation operators.

$$1 = \mathrm{Tr}\, \rho(t) = \mathrm{Tr}\, e^{-iHt} \rho_0 e^{iHt}$$

Even though $\mathrm{Tr}\,\rho = 1$, it is still helpful to write $\mathrm{Tr}\,\rho$ as a path integral.

Use coherent state $|\phi\rangle$

$$\left.\begin{aligned} b|\phi\rangle &= \phi|\phi\rangle \\ b^\dagger|\phi\rangle &= \frac{\partial}{\partial\phi}|\phi\rangle \end{aligned}\right\}$$

$$\therefore \langle\phi|b^\dagger b|\phi\rangle = \phi^*\phi$$

$$\langle\phi_1|\phi_2\rangle = e^{\phi_1^*\phi_2}$$

See for example: Negle & Orland: Quantum Many Particle Systems.

$$Z_K = \mathrm{Tr}\,\rho(t) = \int \mathcal{D}[\phi, \phi^*] \; e^{iS_K(\phi, \phi^*)}$$

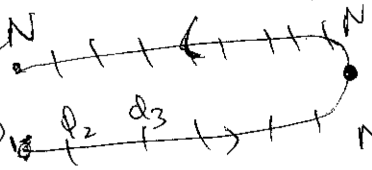
$$S_K = \int_C dt \; [\phi^* \vec{G} \phi]$$

$$= \int_{-\infty}^{+\infty} dt \left[\phi^*_-(i\partial_t - \omega_0)\phi_- - \phi^*_+(i\partial_t - \omega_0)\phi_+ \right]$$

$-$: Time ordered

$+$: Anti-time ordered.

This form might give the impression $\langle \phi_- \phi^*_+ \rangle = 0$, however this is incorrect.



$\longrightarrow$ Consider Suzuki-Trotter decompos

ϕ_1 lies on "$-$" contour

ϕ_{2N} lies on "$+$" contour

Because $\langle \phi_1 | \rho_0 | \phi_{2N} \rangle \neq 0$

The initial density matrix couples $-$ & $+$ field

Since $\langle \hat{O}(t) \rangle = \langle \psi | \underbrace{e^{iHt}}_{\text{Anti-time-ordered}} \hat{O} \underbrace{e^{-iHt}}_{\text{time ordered}} | \psi \rangle$

Convention is that in all expectation values anti-time ordered field is placed on the left of the time ordered fields.

- $$iG_{-+}(t,t') = \langle \phi_-(t)\,\phi_+^\dagger(t')\rangle$$
$$= \langle \phi_+^\dagger(t')\,\phi_-(t)\rangle = e^{-i\omega_0(t-t')}\, n_B$$

where $n_B = \mathrm{Tr}\left[b^\dagger b\,\rho_0\right]$: Boson occupation.

- $$iG_{+-}(t,t') = \langle \phi_+(t)\,\phi_-^\dagger(t')\rangle = \langle \phi(t)\,\phi^\dagger(t')\rangle$$
$$= (1+n_B)\, e^{-i\omega_0(t-t')}$$

- $$iG_{--}(t,t') = T\langle \phi_-(t)\,\phi_-^\dagger(t')\rangle$$
$$= \Theta(t-t')\langle \phi(t)\,\phi^\dagger(t')\rangle + \Theta(t'-t)\langle \phi^*(t')\,\phi(t)\rangle$$
$$= \Theta(t-t')\,(1+n_B)\, e^{-i\omega_0(t-t')} + \Theta(t'-t)\, n_B\, e^{-i\omega_0(t-t')}$$

- $$iG_{++}(t,t') = \tilde{T}\langle \phi_+(t)\,\phi_+^\dagger(t')\rangle$$
$$= \Theta(t-t')\, n_B\, e^{-i(t-t')\omega_0} + \Theta(t'-t)\,(1+n_B)\, e^{-i(t-t')\omega_0}$$

Convention is to rotate fields:-

Define $\phi_q = \frac{\phi_- + \phi_+}{\sqrt{2}}$, $\phi_{cl} = \frac{(\phi_- - \phi_+)}{\sqrt{2}}$.

Then $G^K(t,t') = -i\langle \phi_{cl}(t)\,\phi^*_{cl}(t')\rangle = G_{--} + G_{++} = G_{-+} + G_{+-}$

$G^R = -i\langle \phi_{cl}(t)\,\phi^*_q(t')\rangle = G_{--} - G_{-+} = G_{+-} - G_{++}$

$G^A = i\langle \phi_q(t)\,\phi^*_{cl}(t')\rangle = G_{--} - G_{+-} = G_{-+} - G_{++}$

Note $G_{--} + G_{++} = G_{-+} + G_{+-}$.

Physical meaning of quantum (ϕ_q) & classical (ϕ_c) fields:

suppose $H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2 = \omega_0 b^\dagger b$.
$= (n+\frac{1}{2})\hbar\omega_0$

Write density matrix ρ_0 ~~for~~ in the position basis. Assume ρ_0 is thermal.

$$\langle q_{cl}+q_q|\rho_0|q_{cl}-q_q\rangle = \begin{cases} e^{-\frac{m\omega_0}{\hbar}(q_{cl}^2+q_q^2)} & (T=0) \\ e^{-\frac{2m\omega_0 T q_q^2}{\hbar}} & (T \gg \omega_0) \end{cases}$$

q_q measures how off-diagonal the density matrix is.

Therefore when $T \gg \omega_0$, the system is "classical" as quantum fluctuation are exponentially suppressed, and density matrix is diagonal.
For $T = 0$, fluctuations in quantum and ~~RAA~~ classical fields are equally likely.

Important property of thermal equilibrium state is the Fluctuation-Dissipation-theorem (FDT).

<u>Fermions:</u>

$$\frac{G_K(\omega)}{2\,\mathrm{Im}\,G^R(\omega)} = \tanh\left(\frac{\omega}{2T}\right) \rightarrow$$ Bounded between -1 and 1

<u>Bosons</u>

$$\frac{D_K(\omega)}{2\,\mathrm{Im}\,D^R(\omega)} = \coth\left(\frac{\omega}{2T}\right) \rightarrow$$ divergent $\omega \to 0$ reflecting boson occupation not bounded.

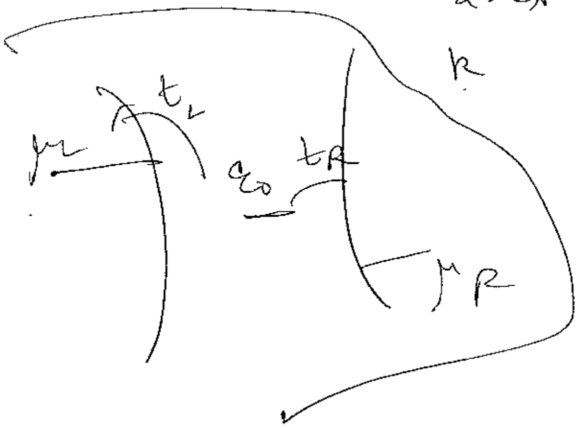
3. Exactly solvable non-equilibrium problem

Two level system coupled to a reservoir.

$d, d^\dagger, c_{k\alpha}, c^\dagger_{k\alpha}$ → Spinless fermions,

$$H = \varepsilon_0 d^\dagger d + \sum_{\substack{\alpha = L,R \\ k}} \left(t_{k\alpha} c^\dagger_{k\alpha} d + h.c. \right)$$

$$+ \sum_{\substack{\alpha = L,R \\ k}} \varepsilon_{k\alpha} c^\dagger_{k\alpha} c_{k\alpha}$$



Ideal reservoirs where

$\langle c^\dagger_{kL} c_{kL} \rangle = f(\varepsilon_k - \mu_L)$ always.

$\langle c^\dagger_{kR} c_{kR} \rangle = f(\varepsilon_k - \mu_R)$ always.

$$f(x) = \frac{1}{e^{\beta x} + 1} \text{ : Fermi-function.}$$

$d, c, d^\dagger, c^\dagger$ → fermionic operators.

Non-equilibrium because $\mu_L \neq \mu_R$.

We will assume $t_{k\alpha} \simeq t_\alpha$ independent of k.

Consider isolated level ($t_\alpha = 0$)

$\varepsilon_0 d^\dagger d \rightarrow$ Note $d^\dagger d$ takes only two values (0 and 1). Hence "two-level" system.

$$g_R(t,t') = -i\theta(t-t')\langle\{d(t), d^\dagger(t')\}\rangle$$

$$g_K(t,t') = -i\langle[d(t), d^\dagger(t')]\rangle$$

$$g_R(t,t') = -i\theta(t-t')\, e^{-i\varepsilon_0(t-t')}$$

$$g_K(t,t') = -i\, e^{-i\varepsilon_0(t-t')}\left[1 - 2\langle d^\dagger d\rangle\right]$$

$$\therefore g_{R,K}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} g_{R,K}(t)$$

$$g_R(\omega) = \frac{1}{\omega - \varepsilon_0 + i\delta}$$

$$g_K(\omega) = -2\pi i\,\delta(\omega - \varepsilon_0)\left[1 - 2\langle d^\dagger d\rangle\right]$$

If we assume occupation in thermal the $\langle d^\dagger d\rangle = \frac{1}{e^{\beta\varepsilon_0}+1}$ and $\left[1 - 2\langle d^\dagger d\rangle\right] = \tanh\left(\frac{\varepsilon_0}{2T}\right)$.

$\therefore \; \mathrm{Im}\, g_R(\omega) = -\pi i\, \delta(\omega - \varepsilon_0)$.

$$g_K(\omega) = -2\pi i\, \delta(\omega-\varepsilon_0) \tanh\left(\frac{\varepsilon_0}{2T}\right)$$
$$= -2\pi i\, \delta(\omega-\varepsilon_0) \tanh\left(\frac{\omega}{2T}\right).$$

$\therefore$ FDT is obeyed for a thermal occupation of dot level.

Now $t_\alpha \neq 0$

Keldysh path integral

$$Z_K = \int \mathcal{D}[\bar{d}, d, \bar{c}, c]\, e^{i S_K}$$

where

$$S_K = \int_{-\infty}^{+\infty} dt \Big[(\bar{d}_- \; \bar{d}_+)\, g_d^{-1} \begin{pmatrix} d_- \\ d_+ \end{pmatrix} + \sum_{\substack{k \\ \alpha = L,R}} (\bar{c}_{k\alpha-} \; \bar{c}_{k\alpha+})\, g_{k\alpha}^{-1} \begin{pmatrix} c_{k\alpha-} \\ c_{k\alpha+} \end{pmatrix}$$
$$+ \sum_{k\alpha} \Big\{ (\bar{d}_- \; \bar{d}_+) \begin{pmatrix} -t_\alpha & 0 \\ 0 & t_\alpha \end{pmatrix} \begin{pmatrix} c_{k\alpha-} \\ c_{k\alpha+} \end{pmatrix} + (\bar{c}_{k\alpha-} \; \bar{c}_{k\alpha+}) \begin{pmatrix} -t_\alpha & 0 \\ 0 & t_\alpha \end{pmatrix} \begin{pmatrix} d_- \\ d_+ \end{pmatrix} \Big\} \Big]$$

Acido:

Here some assumption have been made.

We have assumed dot level was coupled to leads at $t \to -\infty$. Then since dot is small as compared to leads, it reaches a steady state in a time $t \sim \frac{1}{t_{\alpha}^2 + \text{density of states}}$

We are now probing the steady state properties.

Next step, integrate out leads.

Define leads green's function

$$g^{\alpha}_{ab}(t,t') = -i \sum_{k} \langle c_{k\alpha a}(t) \bar{c}_{k\alpha b}(t') \rangle$$

$a,b = \pm$. $\boxed{\text{Assume } t_{k\alpha} = t_\alpha, \text{ independent of } k}$

Gaussian integral can be done (18)

$$S_K = \int_{-\infty}^{\infty} dt \, (\bar{d}_- \; \bar{d}_+)_t \, g_d^{-1}(t,t') \begin{pmatrix} d_- \\ d_+ \end{pmatrix}_{t'}$$

$$- \sum_\alpha (\bar{d}_- \; \bar{d}_+)_t \, t_\alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} g^\alpha_{--} & g^\alpha_{-+} \\ g^\alpha_{+-} & g^\alpha_{++} \end{pmatrix}_{t t'} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d_- \\ d_+ \end{pmatrix}_{t'}$$

The above may be written in the compact form

$$S_K = \int_{-\infty}^{+\infty} dt \, (\bar{d}_- \; \bar{d}_+) \left[g_d^{-1} - \Sigma \right] \begin{pmatrix} d_- \\ d_+ \end{pmatrix}$$

where

$$\boxed{\Sigma = \sum_\alpha t_\alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} g^\alpha_{--} & g^\alpha_{-+} \\ g^\alpha_{+-} & g^\alpha_{++} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Full dot Green's function

$G^{-1} = g^{-1} - \Sigma \rightarrow$ Traditional Dyson equation, but in Keldysh space.

Let us rotate to q, cl space

$$\begin{pmatrix} d_{cl} \\ d_q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} d_- \\ d_+ \end{pmatrix}$$

$$(\bar{d}_q \ \bar{d}_{cl}) = \frac{1}{\sqrt{2}} (\bar{d}_- \ \bar{d}_+) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\therefore S_K = \int_{-\infty}^{+\infty} dt \, (\bar{d}_q \ \bar{d}_{cl}) \left(g^{-1} - \Sigma \right) \begin{pmatrix} d_{cl} \\ d_q \end{pmatrix}$$

$$\hat{\Sigma} = \begin{pmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{pmatrix}.$$

$$G_d = \begin{pmatrix} G_R & G_K \\ 0 & G_A \end{pmatrix}$$

$$\Rightarrow G_d^{-1} = \left(\begin{array}{c|c} G_R^{-1} & -G_R^{-1} G_K G_A^{-1} \\ \hline 0 & G_A^{-1} \end{array} \right)$$

$$\therefore G_d^{-1} = g_d^{-1} - \hat{\Sigma}$$

$$\Rightarrow \boxed{G_R^{-1} = g_R^{-1} - \Sigma_R \ ; \quad G_A^{-1} = g_A^{-1} - \Sigma_A}$$

$$-\vec{G}_R\, G_K\, \vec{G}_A = \underbrace{-\vec{g}_R\, g_K\, \vec{g}_A}_{\approx 0 \text{ as}} - \Sigma^K$$

No broadening for isolated level.

$$\therefore \boxed{G_K = G_R\, \Sigma_K\, G_A}$$

Now let us evaluate $\hat{\Sigma}$.

$$\Sigma^R(t,t') = -i\theta(t-t') \sum_{\substack{k \\ \alpha = L,R}} t_\alpha^2\, e^{-i\varepsilon_{k\alpha}(t-t')}$$

$$\therefore \Sigma_R(\omega) = \sum_{\substack{k \\ \alpha = L,R}} t_\alpha^2 \frac{1}{\omega - \varepsilon_{k\alpha} + i\delta}$$

$$\cong -i\pi \sum_{\substack{k \\ \alpha = L,R}} t_\alpha^2\, \delta(\omega - \varepsilon_{k\alpha})$$

(We drop real part as this only renormalized level energy ε_0)

Now introduce density of states $\nu(\varepsilon)$,

$$\Rightarrow \Sigma^R(\omega) = -i\pi \int d\varepsilon\, \nu(\varepsilon)\, \delta(\omega - \varepsilon) \sum_{\alpha = L,R} t_\alpha^2$$

$$= -i\pi\, \nu(\omega) \left(t_L^2 + t_R^2 \right)$$

Assume density of states are frequency independe

$$\Rightarrow \Sigma^R(\omega) = -i(\Gamma_L + \Gamma_R) = -i\Gamma$$

where $\Gamma_{\alpha = L,R} = \pi \nu\, t_\alpha^2$.

$$\therefore \quad G_R(\omega) = \frac{1}{\omega - \varepsilon_0 + i\Gamma}$$

$\therefore$ Leads braden the level.

$$G_R(t) = -i\theta(t-t')\, e^{-i\varepsilon_0(t-t') - \Gamma(t-t')}$$

Life-time of Γ^{-1} is the time that an electron escapes into the leads or occupies an empty level.

$$\Sigma^K(t,t') = \sum_\alpha t_\alpha^2 \, e^{-i\varepsilon_k(t-t')} \tanh\left(\frac{\varepsilon_k-\mu_\alpha}{2T}\right).$$

In frequency space

$$\Sigma^K(\omega) = -2\pi i \sum_{k,\alpha} t_\alpha^2 \, \delta(\omega-\varepsilon_k) \tanh\left(\frac{\varepsilon_k-\mu_\alpha}{2T}\right)$$

Converting $\sum_k = \int d\varepsilon\, \nu$

$$= -2i \left[\Gamma_L \tanh\left(\frac{\omega-\mu_L}{2T}\right) + \Gamma_R \tanh\left(\frac{\omega-\mu_R}{2T}\right)\right]$$

$$\Sigma^K(\omega) = -2i\Gamma \left[\frac{\Gamma_L}{\Gamma_L+\Gamma_R} \tanh\left(\frac{\omega-\mu_L}{2T}\right) + \frac{\Gamma_R}{\Gamma_L+\Gamma_R} \tanh\left(\frac{\omega-\mu_R}{2T}\right)\right]$$

$$\Rightarrow \boxed{G_K = \frac{-2\pi i \Gamma}{(\omega-\varepsilon_0)^2+\Gamma^2} \left[\frac{\Gamma_L}{\Gamma} \tanh\left(\frac{\omega-\mu_L}{2T}\right) + \frac{\Gamma_R}{\Gamma} \tanh\left(\frac{\omega-\mu_R}{2T}\right)\right]}$$

When $\mu_L = \mu_R = 0$, FDT is obeyed.

where $G_K(\omega) = \left(2\, \mathrm{Im}\, G_R(\omega)\right) \tanh\left(\frac{\omega}{2T}\right)$

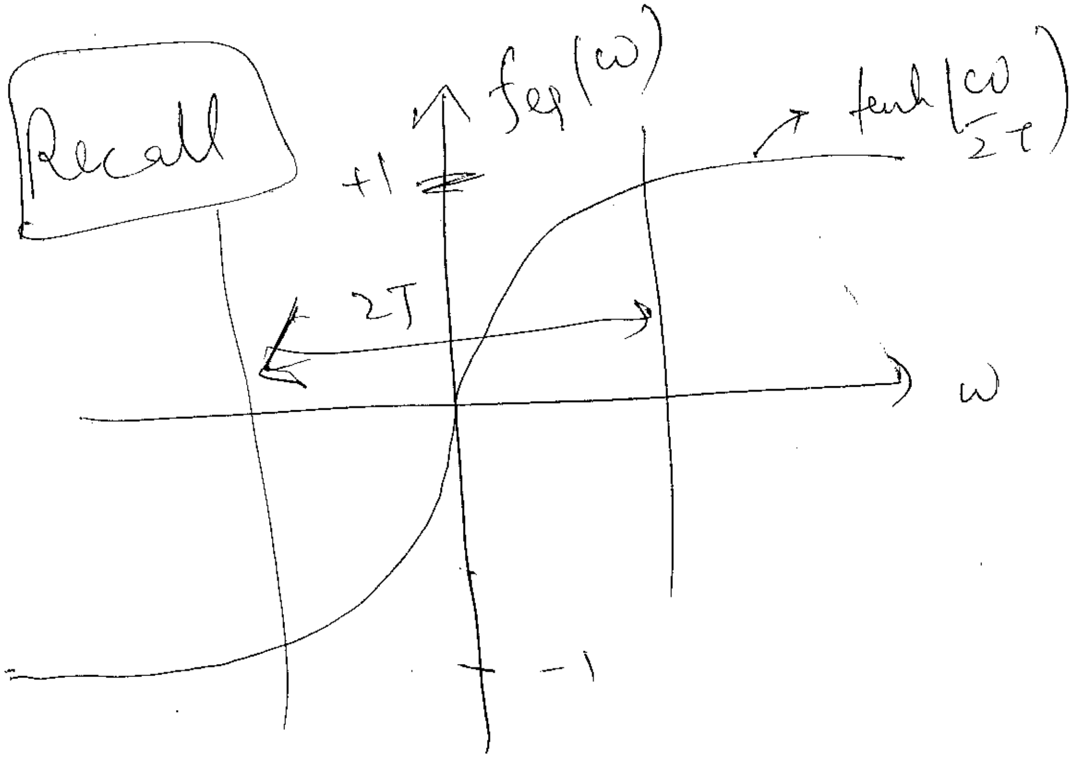
But $\mu_L \neq \mu_R \Rightarrow$

$$G^K(\omega) = (G_R - G_A)(\omega) \cdot \underbrace{f_{neq}(\omega)}_{\text{Non-equilibrium distribution function}}$$

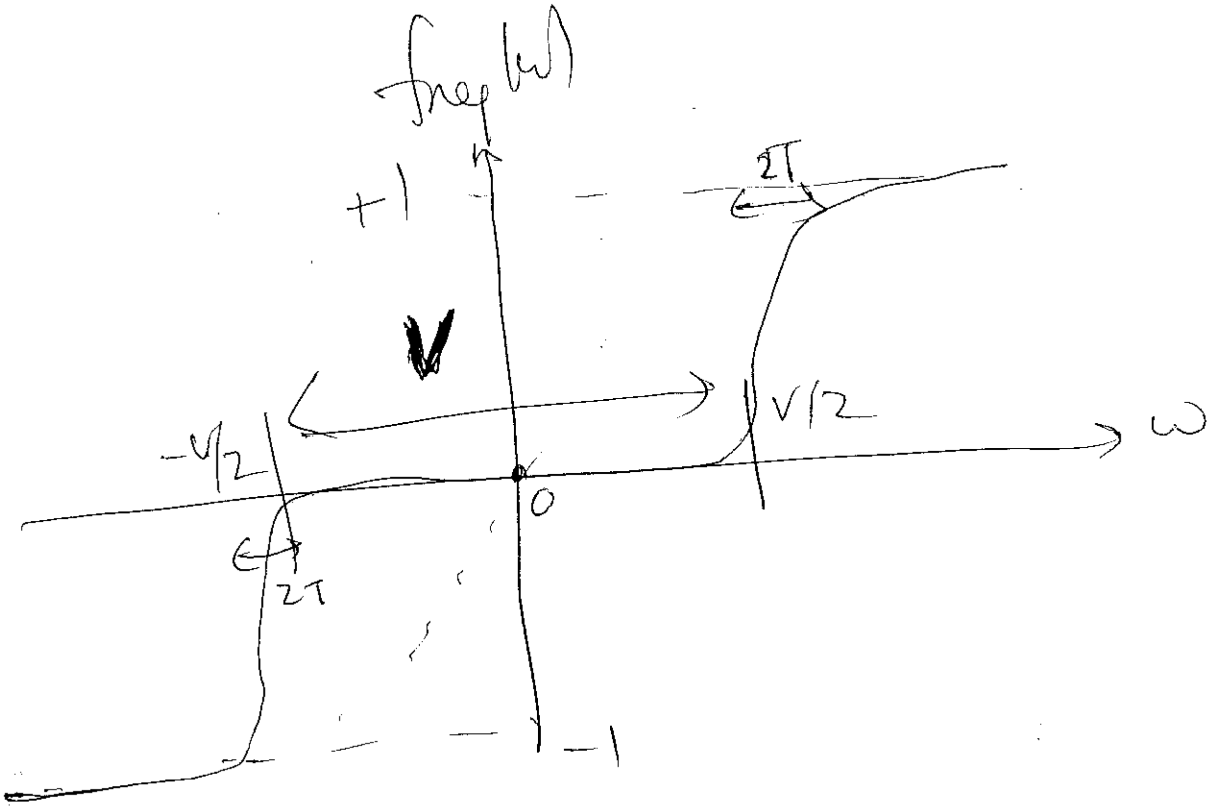
$$f_{eq}(\omega) = \tanh\left(\frac{\omega}{2T}\right) \rightarrow \text{Equilibrium distribution function}$$

$$f_{neq}(\omega) = \frac{\Gamma_L}{\Gamma_L + \Gamma_R} \tanh\left(\frac{\omega - \mu_L}{2T}\right) + \frac{\Gamma_R}{\Gamma_L + \Gamma_R} \tanh\left(\frac{\omega - \mu_R}{2T}\right)$$

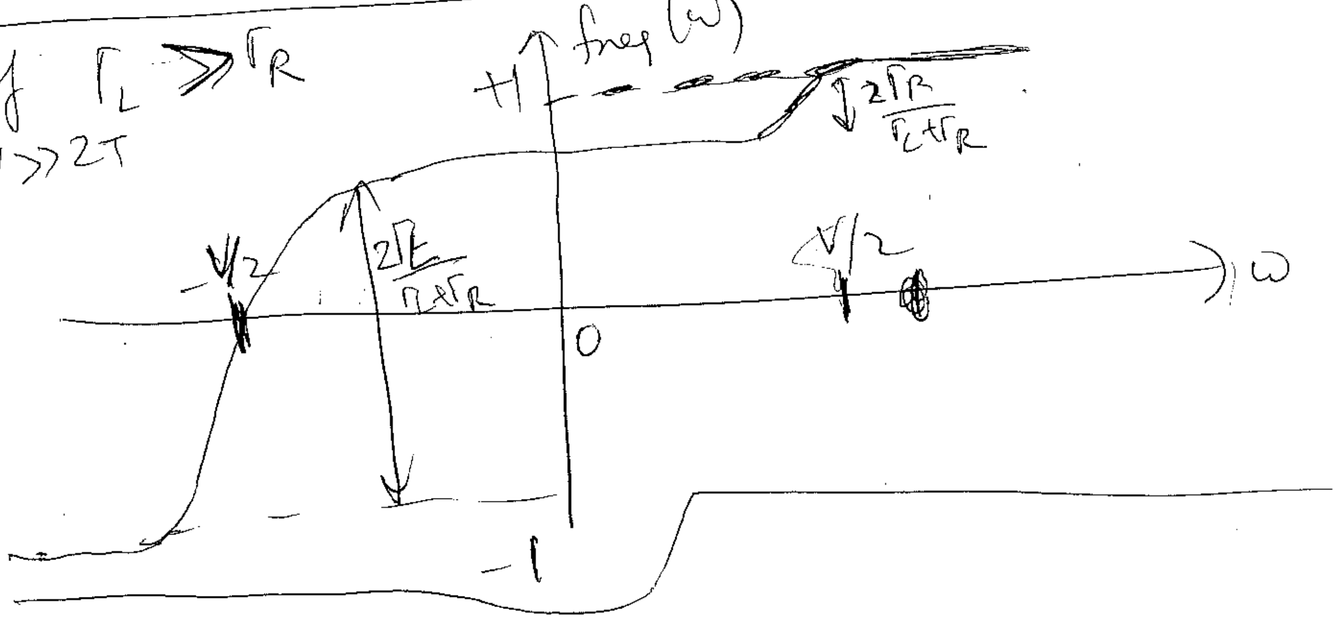
Recall



If $\mu_L = -\frac{V}{2}$ & $\mu_R = \frac{V}{2}$
and $\Gamma_L = \Gamma_R$ & $V >> 2T$



If $\Gamma_L >> \Gamma_R$
$V >> 2T$



Voltage gives a broadening. Thus it behaves as an effective temperature.

4. Spatially extended system coupled to bath.

$$H = \sum_{k\sigma} \varepsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

↓ Hubbard model.

Goal is to understand how mean-field theory + fluctuations about mean-field + Renormalization group can be generalized to systems in a nonequilibrium steady state.

Before this, let recap. equilibrium approach.

Write $U n_{i\uparrow} n_{i\downarrow} = \frac{U N_i}{2} - \frac{U}{2} (n_{i\uparrow} - n_{i\downarrow})^2$

because $n^2_{i\uparrow,\downarrow} = n_{i\uparrow,\downarrow}$.

This is a convenient representation when one is interested in magnetism in the Hubbard model.

We outline how one may study magnetic phase transitions in equilibrium.

$$Z = \mathrm{Tr}\, e^{-\beta H}$$

$$H = \sum_{k\sigma} c^{\dagger}_{k\sigma} c_{k\sigma} + \frac{UN}{2} - \frac{U}{2} \sum_i (n_{i\uparrow} - n_{i\downarrow})$$

N: Total particle number,

Path integral

$$Z = \int \mathcal{D}[\bar{c}, c]\, e^{-\left\{\int_0^\beta d\tau \left[\bar{c}\, g^{-1} c - \frac{U}{2}(n_{i\uparrow} - n_{i\downarrow})^2\right]\right\}}$$

Introduce Hubbard - Stratonovich field

$$\int dm\, e^{-\frac{Um^2}{2} + Um(n_\uparrow - n_\downarrow)} = e^{\frac{U}{2}(n_\uparrow - n_\downarrow)^2}$$

$$\therefore Z = \int \mathcal{D}[\bar{c}, c, m]\, e^{-\int_0^\beta d\tau \left\{\bar{c}\, g^{-1} c + \frac{Um^2}{2} - Um(n_\uparrow - n_\downarrow)\right\}}$$

Now integrate out fermions:

$$Z = \int \mathcal{D}[m]\, \cancel{e^{-\frac{Um^2}{2}}}\, e^{-\frac{U}{2}\int_0^\beta d\tau\, m^2 + \mathrm{Tr}\ln\left[g^{-1} + Um\sigma_z\right]}$$

Now suppose $\langle m \rangle = 0$

$$\mathrm{Tr}\ln\left[g^{-1} + Um\sigma_z\right] = \mathrm{Tr}\ln g^{-1}\left(1 + gUm\sigma_z\right)$$

If $\langle m^2 \rangle$, $\langle m^4 \rangle \dots$ small, then may Taylor expand

Taylor expansion leads to a quantum Ginzburg Landau theory (suppressing spin labels).

$$Z = \int \mathcal{D}[m]\, e^{-\int_0^\beta d\tau \int_0^\beta d\tau' \int dx \int dx' \left[m_{x\tau}\, g_{x\tau,x'\tau'}\, m_{x'\tau'}\, g_{x'\tau' x\tau} + \text{quartic terms} \right]}$$

$$\int_{x\tau}\int_{x'\tau'} m(x\tau)\, m(x'\tau') \underbrace{\left[g_{x\tau,x'\tau'}\, g_{x'\tau',x\tau} \right]}$$

Determines how quantum the system is.

Let us suppose there are no conservation laws (we will consider next an explicit example where the 2d Hubbard model will be coupled to leads).

Then

$$g_{x\tau,x'\tau'}\, g_{x'\tau',x\tau} = \begin{cases} \delta_{x-x'}\, \delta_{\tau-\tau'} & \text{if classical} \\ \delta_{x-x'}\, \dfrac{1}{\tau-\tau'} & \text{if quantum} \end{cases}$$

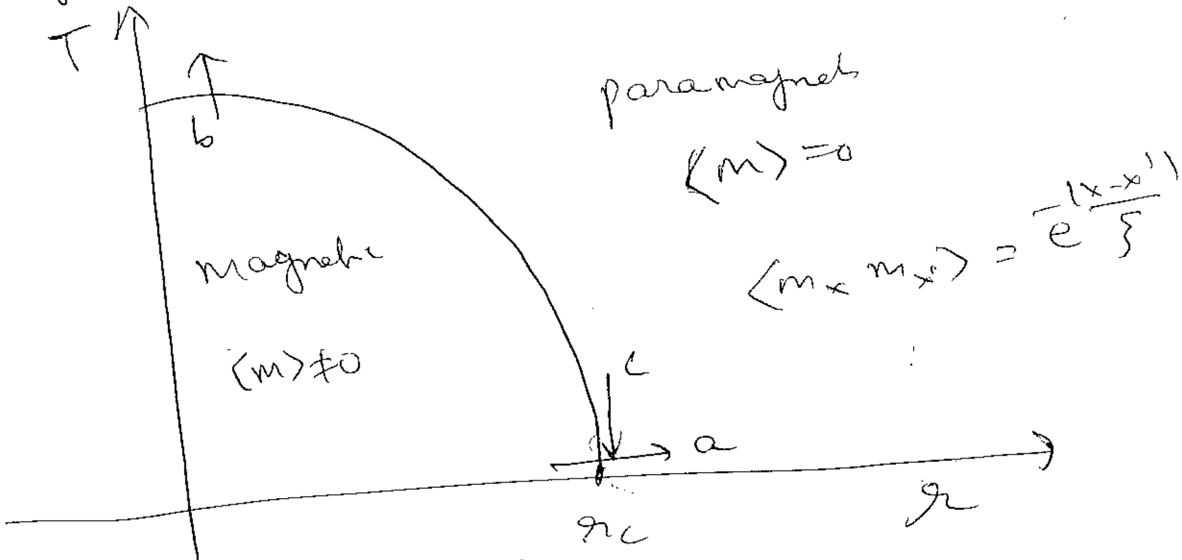
Fourier transform of $\dfrac{1}{\tau-\tau'} = |\omega|$.

So, for an open system

$$Z_k = \int \mathcal{D}[m]\, e^{-\int_0^\beta d\tau \int dx\, F[m]}$$

$$F[m] = \frac{|\omega|}{\Gamma} m^2 + c(\nabla m)^2 + r m^2 + u m^4 .$$

This is then analyzed using perturbation theory and RG to obtain phase diagram



(a) $\rightarrow$ $\xi = |r - r_c|^{-\nu_q}$

(b) $\rightarrow$ $\xi = |T - T_c|^{-\nu_{cl}}$

(c) $\xi = T^{-1/z}$ $\longrightarrow$ thermal decoherence cuts off diverging length correlation length.

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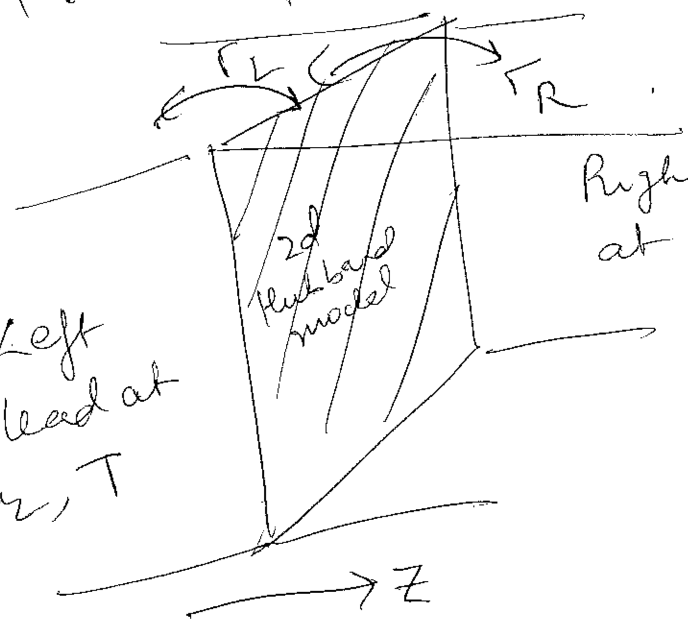
Now, we turn to open nonequilibrium system

$$H = H_{hubbard} + H_{leads} + H_{tunneling}$$



$$\Gamma_{L,R} = + \pi \nu (t^2_{L,R})$$

$$H_{hubbard} = \sum_{k\sigma} \varepsilon_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma} + \underbrace{U \sum n_{i\uparrow} n_{i\downarrow}}_{\frac{Un}{2} - \frac{U}{2}\sum_i (n_{i\uparrow} - n_{i\downarrow})^2}$$

$$H_{leads} = \sum_{k\sigma\alpha} \varepsilon_{k\sigma\alpha} d^\dagger_{k\sigma\alpha} d_{k\sigma\alpha}$$

$$H_{tunneling} = \sum_{i\sigma k_z} \left(t_{k_z,\alpha} c^\dagger_{i\sigma} d_{ik_z\sigma\alpha} + h.c. \right)$$

i = site index | $c^\dagger_{i\sigma}$ layer electron, $d_{ik_z\sigma\alpha}$ lead electrons.

$\cancel{d^\dagger_{k_z\sigma\alpha}}$

Tunneling conserves in plane momentum. k_z is transverse momentum. We assume as before $t_{k_z\alpha} = t_\alpha$

Nonequilibrium because $\mu_L \neq \mu_R$.

Suppose $g_{R,K}$ ~~are~~ is green's funct for layer electrons including the coupling to leads. Then, generalizing the results of the ~~eq~~ two-level system:-

$$\boxed{\Gamma = \Gamma_L + \Gamma_R}$$

$$g_R(\omega, \vec{k}) = \frac{1}{\omega - \varepsilon_k + i\Gamma}$$

$$g_K(\omega, k) = \frac{-2\pi i \Gamma}{(\omega-\varepsilon_k)^2 + \Gamma^2}\left[\frac{\Gamma_L}{\Gamma}\tanh\left(\frac{\omega-\mu_L}{2T}\right) + \frac{\Gamma_R}{\Gamma}\tanh\left(\frac{\omega-\mu_R}{2T}\right)\right]$$

Now we write Keldysh action for this problem:

$$Z_K = \int \mathcal{D}(\bar{c}, c)\, e^{iS_K}$$

$$S_K = (\bar{c}_q \;\; \bar{c}_{cl})\, g^{-1} \begin{pmatrix} c_{cl} \\ c_q \end{pmatrix} + \frac{U}{2}(n_{i\uparrow -} - n_{i\downarrow -})^2 - \frac{U}{2}(n_{i\uparrow +} - n_{i\downarrow +})^2$$

Warning: In previous two-level example I denoted "c" for reservoir and "d" for two-level system. Here I have reversed Notation

Now introduce Hubbard-Stratonovich decoupling

$$e^{\frac{iU}{2}(n_{i\uparrow -}-n_{i\downarrow -})^2 - \frac{iU}{2}(n_{i\uparrow +}-n_{i\downarrow +})^2}$$

$$= \int dm_- \, dm_+ \left[e^{-\frac{iU}{2}(m_{i-}^2 - m_{i+}^2) - iU(\bar{c}_{\sigma -} \; \bar{c}_{\sigma +})\sigma \begin{pmatrix} m_- & 0 \\ 0 & -m_+ \end{pmatrix} \begin{pmatrix} c_{\sigma -} \\ c_{\sigma +} \end{pmatrix}} \right.$$

Perform rotation to q, cl fields:-

$$S_K = \int d^dx \, dt \left[2U m_q m_{cl} + (\bar{c}_{\sigma q} \; \bar{c}_{\sigma cl}) g^{-1} \begin{pmatrix} c_{\sigma cl} \\ c_{\sigma q} \end{pmatrix} \right.$$
$$\left. + U (\bar{c}_{\sigma q} \; \bar{c}_{\sigma cl}) \sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} \begin{pmatrix} c_{\sigma cl} \\ c_{\sigma q} \end{pmatrix} \right]$$

Note the reservoir has already been integrated out and enters in form of "g" as outlined in previous pages.

Now integrate out fermions to obtain

$$S_K = \int d^d x\, dt \left[2U\, m_q m_{cl} \right] + \mathrm{Tr}\ln\left[g^{-1} + U\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} \right]$$

$$g = \begin{pmatrix} g_R & g_K \\ 0 & g_A \end{pmatrix}$$ and given in previous page.

Now assume $\langle m_q \rangle = 0$, $\langle m_{cl} \rangle = 0$ and expand in powers of m_q, m_{cl} $\longrightarrow$

$$\mathrm{Tr}\ln\left[g^{-1} + U\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} \right]$$

$$= \mathrm{Tr}\ln g^{-1} + \mathrm{Tr}\ln\left[1 + gU\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} \right]$$

$$= \mathrm{Tr}\ln g^{-1} + \mathrm{Tr}\left\{ gU\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} gU\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} \right\}$$

$$- \frac{1}{2}\underbrace{\mathrm{Tr}\left\{ gU\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} gU\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} gU\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} gU\sigma \begin{pmatrix} m_{cl} & m_q \\ m_q & m_{cl} \end{pmatrix} \right\}}_{\text{quartic terms}}$$

$$= \text{Const} + \underbrace{S_K^{(2)}}_{\text{quadratic}} + \underbrace{S_K^{(4)}}_{\text{quartic}}$$

$$S_K = S_K^{(2)} + S_K^{(4)}$$

$$S_K^{(2)} = \sum_{q,\omega} (m_{cl}^{*} \;\; m_q^{*})(q,\omega) \begin{pmatrix} 0 & \Pi^A(q,\omega) \\ \Pi^R(q,\omega) & \Pi^K(q,\omega) \end{pmatrix} \begin{pmatrix} m_{cl} \\ m_q \end{pmatrix} (q,\omega)$$

$$\Pi^R(q,\omega) = i\omega - (Dq^2 + \delta)$$

Detuning from critical point.

$$\Pi^A(q,\omega) = -i\omega - (Dq^2 + \delta)$$

$$\Pi^K(q,\omega) = -2i \sum_{\alpha\beta = L,R} \frac{\Gamma_\alpha \Gamma_\beta}{\Gamma^2} \coth\left(\frac{\omega + \mu_\alpha - \mu_\beta}{2T}\right)(\omega + \mu_\alpha - \mu_\beta)$$

Case (1):- $\mu_\alpha - \mu_\beta = 0$, since $\sum_{\alpha\beta} \frac{\Gamma_\alpha \Gamma_\beta}{\Gamma^2} = 1$, expression simplifies

$\Rightarrow$ FDT obeyed. $\Pi^K(q,\omega) = 2\left[\mathrm{Im}\, \Pi^R(q,\omega)\right] \coth\left(\frac{\omega}{2T}\right)$

$$= -2i\omega \coth\left(\frac{\omega}{2T}\right) \rightarrow \text{quantum noise.}$$

$T \ll \omega \quad \Pi^K \longrightarrow -2i|\omega|$

$T \gg \omega \quad \Pi^K \longrightarrow -2i(2T) \rightarrow$ classical noise

Now consider $\mu_L - \mu_R \neq 0$

Then

$$\Pi^K(\omega=0) = (-2i)\left(\frac{2\Gamma_L\Gamma_R}{\Gamma^2}\right)\coth\left(\frac{\mu_L-\mu_R}{2T}\right)(\mu_L-\mu_R)$$

If $|\mu_L-\mu_R| \gg T$

$$\Pi^K(\omega=0) \to (-2i)\,\frac{2\Gamma_L\Gamma_R}{\Gamma^2}\,|\mu_L-\mu_R|$$

$$\therefore\ T_{eff} = \frac{\Gamma_L\Gamma_R}{\Gamma^2}|\mu_L-\mu_R|$$

This denotes shot noise due to current flowing across the 2d layer.
In general shot noise is the low frequency limit of a more complicated colored noise source as seen in the general expression for $\Pi^K(\omega\neq 0, \mu_L-\mu_R\neq 0)$ when $\omega \gg |\mu_L-\mu_R| \to$ noise again becomes like an equilibrium quantum noise.

In general terms that come proportional to M_q^2 are noise terms. This is apparent from the pure temporal decay of the reduced density matrix

$$\rho \equiv e^{\pm i\{(M_q^2 \Pi^K(q,\omega))\}}$$

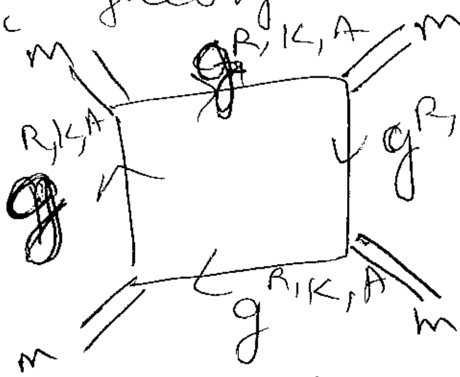
where $\Pi^K(q,\omega)$ is purely imaginary.

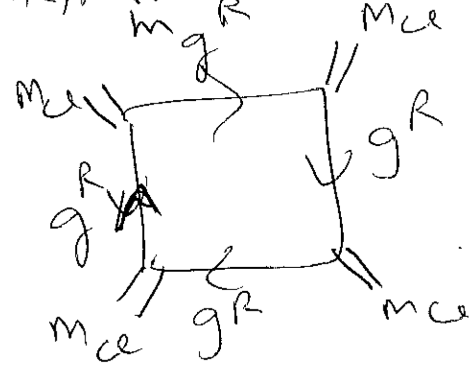
We will also show this more formally by mapping the problem to a Langevin equation.

Let us consider quartic terms.

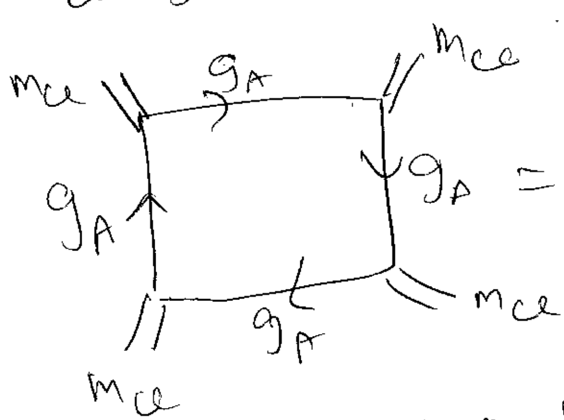
$$S_K^{(4)} = \int d^dx\, dt \left[u_1 M_q M_{cl}^3 + u_2 M_q^2 M_{cl}^2 + u_3 M_q^3 M_{cl} + u_4 M_q^4 \right]$$

$u_{1,2,3,4}$ all determined from underlying microscopic theory where

$u_i \equiv$ 

Terms such as 

$= 0$



$= 0$

Thus causality imposes that purely M_{cl} terms (M_{cl}^2) & (M_{cl}^4) ... terms are absent.

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Some general comments about $u_{i=1,2,3,4}$

When $T=0=V=0$

then $u_2 = u_4 = 0$ and $u_1 = u_3$

This means $S_4^{(4)} \equiv \int_{x,t} \left[m_q M_{cl}^3 + M_q^3 M_{cl} \right]$

Thus underlying dynamics of scalar field m is unitary in that it appears to come from a Hamiltonian which is

$$H = \frac{\gamma}{2} m^2 + \frac{u}{4} m^4 + m_t \frac{1}{(t-t')} m_{t'}$$

zero temperature fermi reservoirs.

In Keldysh this translates to

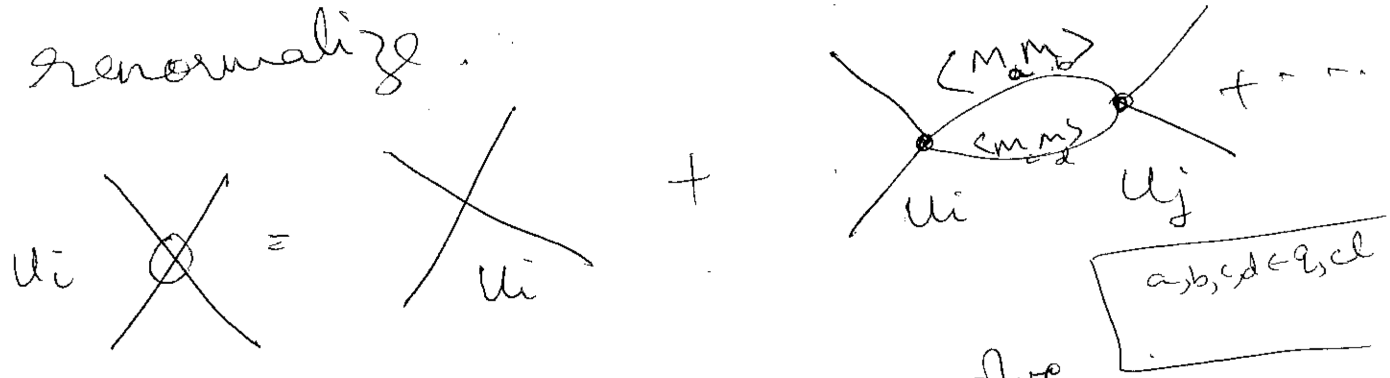
$$\frac{\gamma}{2}(m_-^2 - m_+^2) + \frac{u}{4}(M_-^4 - M_+^4)$$

$$2\frac{\gamma}{2} m_q m_{cl} + \frac{u}{4} m_q M_{cl}(M_q^2 + M_{cl}^2)$$

~~Of course there is still~~

When $T \neq 0$ and ~~or~~ $V \neq 0$, then $u_{1,2,3,4} \neq 0$

~~then~~ One may study how $u_{1,2,3,4}$ renormalize.



Then the internal loops involve $\langle M_q M_{cl} \rangle$ averages or $\langle M_{cl} M_{cl} \rangle$ averages.

At $T \neq 0$ or $V \neq 0$ and larger $\langle M_{cl} M_{cl} \rangle \propto O\left(\frac{V}{\Gamma}, \frac{T}{\Gamma}\right)$ that $\langle M_q M_{cl} \rangle$.

Thus vertices with large number of classical fields dominate due to large boson occupation number.

Thus in high T or high V limit we may retain only $u_1 m_q m_{cl}^3$.

Thus

$$S_K = i\int_{x,t} \left[-m_q \left(\partial_t m_{cl} - D\nabla^2 m_{cl} + u_1 m_{cl}^3\right) + 2i\gamma T_{eff}\, m_q^2 \right]$$

where $T_{eff} = T$ or $\dfrac{\Gamma_L \Gamma_R |V|}{\Gamma^2}$.

Now we show the mapping to Langevin equation. The key is to keep terms $O(m_q)$ and $O(m_q^2)$, while there is no restriction on the number of powers of the classical field m_{cl}.

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Introduce auxiliary field ξ

$$e^{-2\gamma T_{eff}\int dt\, m_q^2} = \int \mathcal{D}(\xi)\, e^{-\int_{-\infty}^{+\infty} dt \left[\frac{\xi^2}{2\gamma T_{eff}} - 2i\xi(t)\, m_q(t)\right]}$$

Then,

$$Z = \int \mathcal{D}(\xi)\, e^{-\int_{-\infty}^{+\infty} dt \frac{\xi^2}{2\gamma T_{eff}}} \int \mathcal{D}[m_q, m_{cl}]\, e^{-i\int dt\, m_q\left[\partial_t m_{cl} - D\nabla^2 m_{cl} + u_1 m_{cl}^3 - \xi\right]}$$

Integrating out m_q gives

$$\delta\left[\partial_t m_{cl} - D\nabla^2 m_{cl} + u_1 m_{cl}^3 - \xi\right) = 0$$

Langevin Eq.

$$\Rightarrow \boxed{\begin{aligned} &\partial_t m_{cl} - D\nabla^2 m_{cl} + u_1 m_{cl}^3 = \xi \\ &\text{where } \langle \xi\xi\rangle = 2\gamma T_{eff}\, \delta(t-t') \end{aligned}}$$

Thus nonzero voltage difference or non-zero current flow, maps the dynamics to an Langevin equation with noise proportional to $T_{eff} \simeq \frac{\Gamma_L \Gamma_R}{\Gamma^2} \underbrace{|\mu_L - \mu_R|}_{V}$

Thus ~~voltage~~ current can drive a magnetic-paramagnetic phase transition with the same universal behavior as the thermal transition