PT Symmetry in Optics, Exceptional Points and Bound States in the Continuum

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Outline of the talk

- PT Symmetric Systems: Properties and Examples
- Exceptional Points and Bound States in the Continuum
- Scattering by PT Symmetric Systems
  - Equation of continuity and Scattering matrix
  - New symmetries and boundary conditions
  - Speculated scalar product
- PT Symmetric Solutions in KdV and Modified KdV Equations
- Conclusion(s)

PT Symmetric Systems

- The operator P represents parity reflection and the operator T represents time reversal.

- These operators are defined by their action on the position and momentum operators x and p:

$$P : x \to -x, \quad p \to -p,$$
$$T : x \to x, \quad p \to -p, \quad i \to -i.$$

  When the operators x and p are real, the canonical commutation relation $[x, p] = i$ PT-invariant.

- The Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2}x^2 + i\alpha x$$

  is PT-invariant with real eigenvalues.

- Non-Hermitian Hamiltonians yielding real eigenvalues: pseudo-Hermitian.

$$H^\dagger \neq H. \tag{1}$$

  But for,

$$\begin{aligned} E_n &= E_n^*, \\ < \psi_n | H | \psi_n > &= < \psi_n | H^\dagger | \psi_n > \end{aligned}$$

- The norm has to be redefined (Das *et al.*, Bender, Mostafazadeh):

$$< \phi|\psi >_\eta := < \phi|\eta|\psi > . \qquad (2)$$

- But only for certain parameter range, otherwise complex-conjugate pairs of energies.

- Antilinearity of time reversal (T) operation:

$$[H, T] = 0 \qquad (3)$$

  does not ensure common eigenfunction always: Spontaneous breaking of PT-symmetry.

- Explicit example of PT-symmetric potential is

$$V(x) = V_{even}(x) + iV_{odd}(x)$$

  in the Schrödinger equation.

PT Symmetric Potential: Example

- Spectral Bifurcation for certain complex potentials:

$$V(x;\alpha) = -V_1 \mathrm{sech}^2(\alpha x) - iV_2 \mathrm{sech}(\alpha x)\tanh(\alpha x). \quad (4)$$

We start with the superpotential:

$$W_P^{\pm}T = (A \pm iC^{PT})\tanh(\alpha x) + (\pm C^{PT} + iB)\mathrm{sech}(\alpha x), \quad (5)$$

yielding,

$$V_-^{\pm}(x) = -\left[(A \pm iC^{PT})(A \pm iC^{PT} + \alpha) - (\pm C^{PT} + iB)^2\right]\mathrm{sech}^2(\alpha x)$$
$$-i(\pm iC^{PT} - B)\left[2(A \pm iC^{PT}) + \alpha\right]\mathrm{sech}(\alpha x)\tanh(\alpha x),$$

which is PT-symmetric only if,

$$C^{PT}\,[2(A - B) + \alpha] = 0. \quad (6)$$

PT Symmetric Potential: Example

- For $C^{PT} = 0$:

$$W_{PT}(x) \equiv W_{real}(x) = A\tanh(\alpha x) + iB\mathrm{sech}(\alpha x), \tag{7}$$

yielding,

$$V_-^{\pm}(x) \equiv V_-(x) = -\left(A(A+\alpha) + B^2\right)\mathrm{sech}^2(\alpha x) + i\,(B(2A+\alpha))\,\mathrm{sech}(\alpha x)\tanh(\alpha x). \tag{8}$$

- Corresponding eigenvalues and eigenfunctions are:

$$\begin{aligned} E_n &= -(n\alpha - A)^2, \\ \psi_n(x) &\propto (\mathrm{sech}(\alpha x))^{\frac{A}{\alpha}} \exp[-i\frac{B}{\alpha}\tan^{-1}(\sinh(\alpha x))]P_n^{-\frac{A}{\alpha}-\frac{B}{\alpha}-\frac{1}{2}, -\frac{A}{\alpha}+\frac{B}{\alpha}-\frac{1}{2}}[i\sinh(\alpha x)]\,. \end{aligned}$$

PT Symmetric Potential: Example

- For $C^{PT} \neq 0$ and thus, $A = B - \frac{\alpha}{2}$:

$$W^{\pm}_{PT}(x) \equiv W^{\pm}_{c}(x) = \left(A \pm iC^{PT}\right) \tanh(\alpha x) + \left[\pm C^{PT} + i\left(A + \frac{\alpha}{2}\right)\right] \operatorname{sech}(\alpha x), \qquad (9)$$

yielding,

$$V^{\pm}_{-}(x) \equiv V^{c}_{-}(x) = -\left[2A(A+\alpha) - 2(C^{PT})^2 + \frac{\alpha^2}{4}\right] \operatorname{sech}^2(\alpha x)$$
$$+ i\left[2A(A+\alpha) + 2(C^{PT})^2 + \frac{\alpha^2}{2}\right] \operatorname{sech}(\alpha x) \tanh(\alpha x).$$

- Corresponding eigenvalues and eigenfunctions are:

$$E^{\pm}_n = 2n\left(A \pm iC^{PT}\right)\alpha + (n\alpha)^2,$$
$$\psi^{\pm}_n(x) \propto (\operatorname{sech}(\alpha x))^{\frac{1}{\alpha}(A \pm iC^{PT})} \exp\left[\left(-\frac{i}{\alpha}\left(A + \frac{\alpha}{2}\right) \mp \frac{C^{PT}}{\alpha}\right) \tan^{-1}(\sinh(\alpha x))\right] P_n^{\mp i2\frac{C^{PT}}{\alpha}, 2\frac{A}{\alpha}+\frac{1}{2}} \left[i \sinh(\alpha x)\right].$$

SUSY and PT symmetric Phases

Potential is invariant under the transformation $A \leftrightarrow B - \frac{\alpha}{2}$:

$$V_-(x) = -\left(A(A+\alpha) + B^2\right) \text{sech}^2(\alpha x) + i\,(B(2A+\alpha))\,\text{sech}(\alpha x)\,\tanh(\alpha x) \quad (10)$$

implying two superpotentials: W_1 and W_2.

$$W_1(x) = A\ tanh(\alpha x) + iB\ sech(\alpha x). \quad (11)$$

and

$$W_2(x) = (B - \frac{\alpha}{2}) tanh(\alpha x) + i(A + \frac{\alpha}{2}) sech(\alpha x). \quad (12)$$

The two superpotentials lead to two sets of real energy eigenvalues,

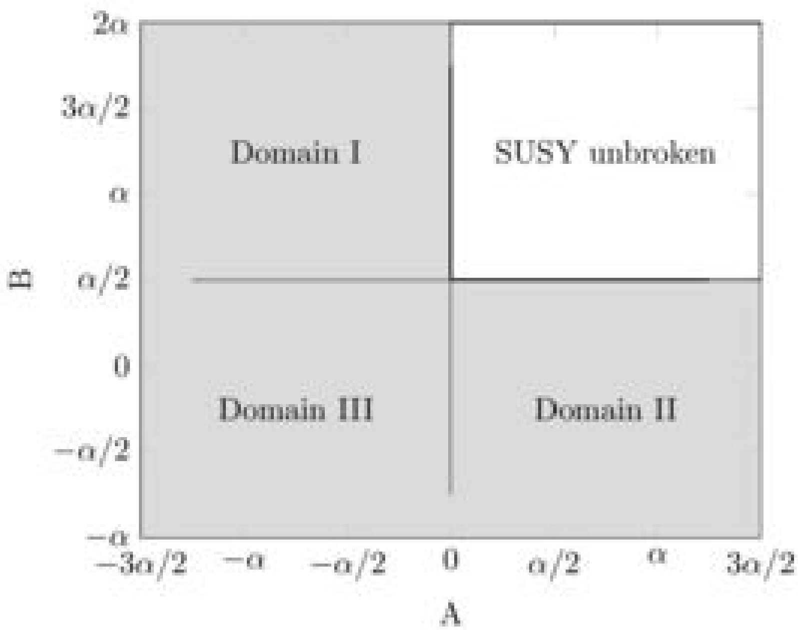
$$E_n^1 = -(A - n\alpha)^2, \quad (13)$$

and

$$E_n^2 = -(B - \frac{\alpha}{2} - n\alpha)^2. \tag{14}$$

with $A > 0$ and $B > \frac{\alpha}{2}$.

Broken phases of SUSY occur for different parameter domains with real eigenvalues.



The shaded regions I, II and III are areas of broken SUSY.

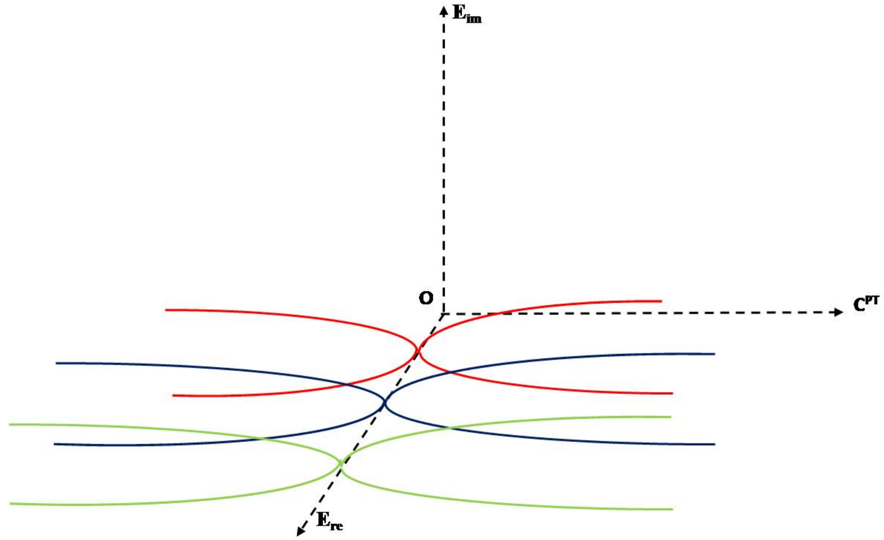
Exceptional Point, Bound States in Continuum

At exceptional point the PT-symmetry unbroken phase undergoes a transition to the PT-symmetry broken phase through the set of conditions in the parameter space defined as,

$$C^{PT} = 0, \quad A = B + \frac{\alpha}{2}, \tag{15}$$

where the two real energy eigenvalue sets and the two complex energy eigenvalue sets from both sides merge to form a single set of real energy eigenvalues.

PT Symmetric Potential: Example



Splitting of real energies into CC pairs on spontaneous PTSY breaking.

Bound States in Continuum

In the PT-broken phase, energy can be still real provided

$$A = n\alpha, \tag{16}$$

for which the energy is given as,

$$E_R = (C^{PT})^2. \tag{17}$$

They can be obtained through poles of transmission and reflection coefficients. These are spectral singularities with zero width resonance. Note that the exceptional point also has spectral singularities, where $E_R = 0$.

For the PT-broken phase, where $A = B - \frac{\alpha}{2}$ and $C^{PT} \neq 0$,

$$V_C(x) = -\Big[2A(A+\alpha) - 2(C^{PT})^2 + \frac{\alpha^2}{4}\Big] sech^2(\alpha x) + i\Big[2A(A+\alpha) + 2(C^{PT})^2 + \frac{\alpha^2}{2}\Big] sech(\alpha x) tanh(\alpha x) \quad (18)$$

with the pair of superpotentials,

$$W_C^{\pm} = (A \pm iC^{PT}) tanh(\alpha x) + \Big[\pm C^{PT} + i(A + \frac{\alpha}{2})\Big] sech(\alpha x). \quad (19)$$

with the complex energy eigenvalues,

$$E_n^{\pm} = -(n\alpha - A \pm iC^{PT})^2. \quad (20)$$

Scattering by PT Symmetric Systems

- The study of S-matrix of a PT-symmetric system helps in understanding the uniqueness of analytic structure of such systems.
- Altered boundary conditions and properties of S-matrix sheds light on 'flux' conservation, and hence on the question of scalar product, which is yet to be settled for generic systems corresponding to open symmetry groups.
- The Schrödinger equation for a PT-symmetric system has unchanged Hamiltonian upon PT-transformation, not complex-conjugation.
- Subsequently, one obtains a 'continuity equation':

$$\frac{\hbar}{i2\pi}\frac{\partial}{\partial x}\left(\psi(x,t)\frac{\partial}{\partial x}\psi^*(-x,t)-\psi^*(-x,t)\frac{\partial}{\partial x}\psi(x,t)\right)=i\frac{\partial}{\partial t}\left(\psi^*(-x,t)\psi(x,t)\right). \quad (21)$$

- The conserved 'density' is now:

$$\psi^*(-x,t)\psi(x,t), \quad (22)$$

which is *not* positive definite.

Scattering by PT Symmetric Systems

- Conservation of 'flux' (incoming vs outgoing) relates the asymptotic coefficients as:

$$AD^* - DA^* = CB^* - BC^*. \tag{23}$$

A, B and C, D are the coefficients of scattering states when $x \to \infty$ and $x \to -\infty$, respectively.

- The S-matrix obeys,

$$S^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{24}$$

- This shows that the S-matrix is *not* unitary!

- This is a general property of the pseudo-Hermitian S-matrix:

$$\eta S^{-1} = S^\dagger \eta. \tag{25}$$

enabling the identification: $\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the 'norm operator'.

- Caution: may not be integrable for all systems.

Scattering by PT Symmetric Systems

- Further, from the definition of S-matrix and complex-conjugation of the PT-symmetric potential $V(x) = V_{even}(x) + iV_{odd}(x)$in the Schrödinger equation, asymptotically,

$$S^{\dagger}\left(V_{odd}\right) S\left(-V_{odd}\right) = I, \tag{26}$$

- This is the *Duality* condition already well-appreciated in the optical analogues of PT -symmetric systems.

- As an working example, we consider the PT-symmetric potential:

$$V(x) = A^2 + (B^2 - A^2 - A\alpha)\frac{1}{\cosh^2(\alpha x)} + B(2A+\alpha)\frac{\tanh(\alpha x)}{\cosh(\alpha x)}. \tag{27}$$

Scattering by PT Symmetric Systems

- Under the usual Dirac definition of scalar product, the reflection and transmission coefficients add up to:

$$|\Re|^2+|\Im|^2 = 1+\left[\frac{2\cos^2(\frac{\pi A}{\alpha})\sin^2(\frac{\pi B}{\alpha})\sinh^2(\frac{\pi k}{\alpha})+\sin(2\frac{\pi A}{\alpha})\sin(2\frac{\pi B}{\alpha})\sinh(\frac{2\pi k}{\alpha})}{\left(\sinh^2(\frac{\pi k}{\alpha})+\sin^2(\frac{\pi A}{\alpha})\cos^2(\frac{\pi B}{\alpha})\right)\cosh^2(\frac{\pi k}{\alpha})-\cos^2(\frac{\pi A}{\alpha})\sin^2(\frac{\pi B}{\alpha})\sinh^2(\frac{\pi k}{\alpha})}\right], \tag{28}$$

  $\rightarrow$ The flux is not conserved.

- One can calculate the asymptotic coefficients corresponding to the given potential, and apply Eq.25 to have the expected result: $|\Re|^2+|\Im|^2=1$.

- Thus, a new definition of norm, which eventually belongs to a rigged Hilbert space.

Scattering by PT Symmetric Systems

- Boundary conditions for scattering states get modified in case of PT symmetric potentials.

- Spontaneously broken PT-symmetry yields complex conjugate eigenvalues in pairs, whereas unbroken sector has real eigenvalues.

- Such 'phase transition' is not seen in Hermitian systems as the reality of the spectrum is ensured by the construction of the Hilbert space.

- Thus, it is interesting to observe the effect on the 'conserved flux'.

- If PT-symmetry is not spontaneously broken, then,

$$\psi^*(-x,t) = \psi(x,t), \tag{29}$$

making the 'flux' in Eq.21 is identically zero and hence the 'density'

$$\psi^*(-x,t)\psi(x,t) \equiv |\psi(x,t)|^2, \tag{30}$$

is stationary.

Scattering by PT Symmetric Systems

- Similar to Hermitian systems, this still allows unitary time evolution, necessary for well-defined energy.
- This restricts the definition of PT-symmetry modulo the unitary 'energy exponential' on physical grounds.
- As scattering necessitates 'flux' to be detected asymptotically, PTSY is expected to be necessarily broken for the scattering states.
- This restricts the form of asymptotic states further than that for Hermitian systems. Particularly, a single plane wave will result into vanishing flux.
- Thus we need a non-PT-symmetric superposition: Incident waves from both directions! $\rightarrow$ The condition for PT-CPA.
- Bound states, with real discrete energies without gain or loss, should correspond to PT-symmetric wave-functions, leading to no asymptotic currents. They arise as poles of S-matrix elements, in terms of proper asymptotic coefficients.

Scattering by PT Symmetric Systems

- There are speculations regarding whether PT-symmetric systems are pseudo-Hermitian or not.

- For PT-symmetric systems,

$$PTHPT = H. \tag{31}$$

- This, from the definition of pseudo-Hermiticity and also from Eq.22, hints towards a norm:

$$\int_{-\infty}^{\infty} \phi(x,t) PT\psi(x,t) dx \equiv \int_{-\infty}^{\infty} \psi^*(x,t) P\phi(x,t) dx. \tag{32}$$

- This prompts interpreting the operator Tas a product of complex conjugation followed by interchange of ‘in’ and ‘out’ states, which is *not* exactly transposition.

- Eventually, the anti-unitarity of T makes it difficult to visualize a scalar-product as a generalization of the Dirac-von-Neumann norm.

Light Stops at Exceptional Point

The group velocity of light has been shown to vanish at exceptional point. For the PT-symmetric case,

$$v_g = (d\beta/d\omega)^{-1} = \frac{2c^2\beta \int \psi^2 dx}{\int [\partial(n^2\omega^2)/\partial\omega]\psi^2 dx}. \tag{33}$$

Noting that $\psi^*(-x)$ is also a solution, one obtains in general,

$$v_g = (d\beta/d\omega)^{-1} = \frac{2c^2\beta \int \psi^*(-x)\psi(x) dx}{\int [\partial(n^2\omega^2)/\partial\omega]\psi^*(-x)\psi(x) dx}. \tag{34}$$

Here $\psi^*(-x)\psi(x)$ is the conserved correlation.

Isospectral Deformation and Connection with KdV

The earlier mentioned two superpotentials W_1 and W_2 (Eq.11,12) are related by $W_2 = W_1 + \phi$ with $\phi(x) = (B - \frac{\alpha}{2} - A) tanh(\alpha x) + i(B - \frac{\alpha}{2} - A) sech(\alpha x)$. Isospectrality requires ϕ to satisfy the Bernoulli's equation:

$$\frac{d\phi(x)}{dx} - W(x)\phi(x) - \phi^2(x) = 0. \tag{35}$$

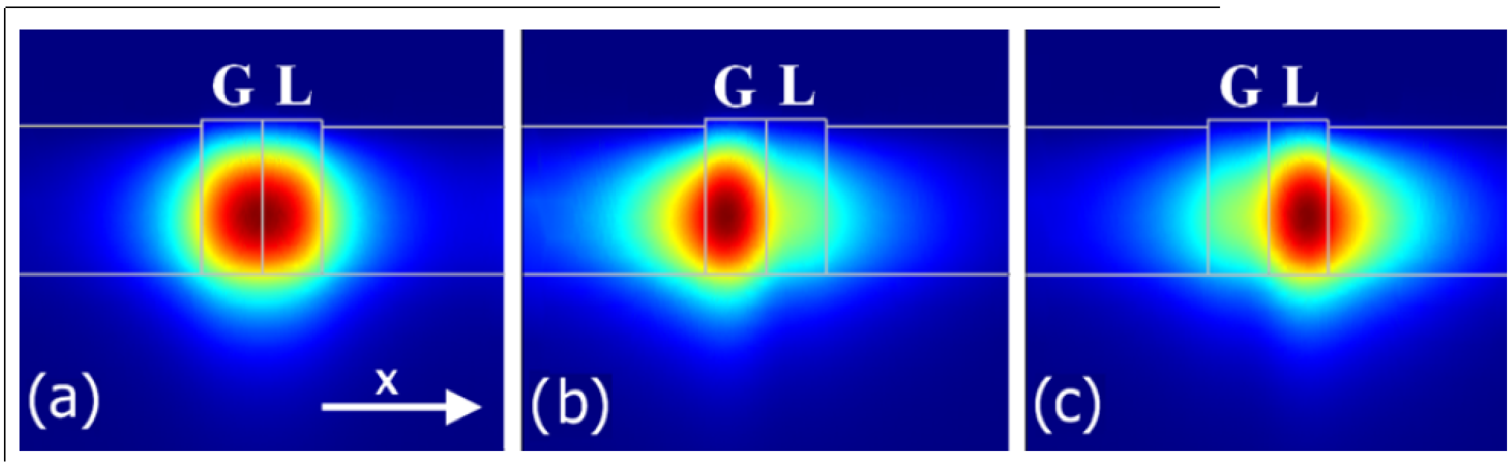
possible for $B - \frac{\alpha}{2} = -A$. ϕ satisfies the mKdV equation:

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{36}$$

and the corresponding potential satisfies the KdV equation:

$$u_t - 6u u_x + u_{xxx} = 0. \tag{37}$$

Experimental observation of PTSY-QM



A. Guo *et al.*, PRL **103**, 093902 (2009)

- 1st experimental proof of spontaneous breaking of PT-symmetry.
- For certain values of critical optical loss coefficient, PT-symmetry is preserved: (a).
- For other values, the fundamental mode breaks up into the gain (b) and loss (c) modes.

Coherent Perfect Absorber (CPA)

[Wan *et al.*, *Science* **331**, 889 (2011); Chong *et al.*, PRL **105**, 053901 (2010)]

- Helmholtz equation:

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \omega^2 n^2(x) \frac{\vec{E}}{c_0^2} = 0, \quad n = n_1 + i n_2, \tag{38}$$

  with solution (in 1-D): $E(x) = A\,\exp(iknx) + B\,\exp(-iknx)$.

- Time reversal symmetry at Classical level: $E(x) \rightarrow E^*(x), \quad n(x) \rightarrow n^*(x)$.

- This interchanges the notions of incoming/outgoing and gain/loss.

- The S-matrix definition changes from $\mathbf{S}[n(\vec{r}k)].a = b$ to $\mathbf{S}[n^*(\vec{r}k)].b^* = a^*$.

- If we had a lasing media prior to time reversal, after that we obtain a perfect laser absorber!

- The relative phase of the incoming signals become crucial $\rightarrow$ 'Destructive interference'.

Coherent Perfect Absorber (CPA)

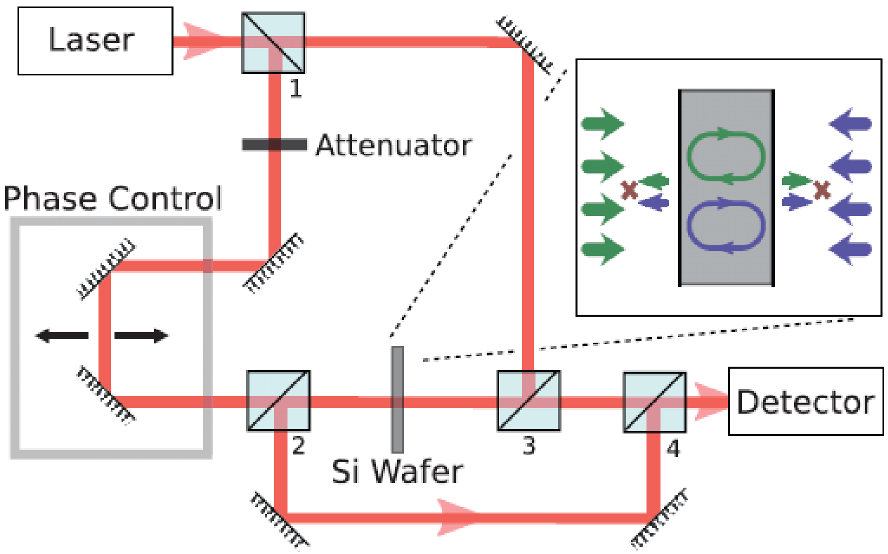
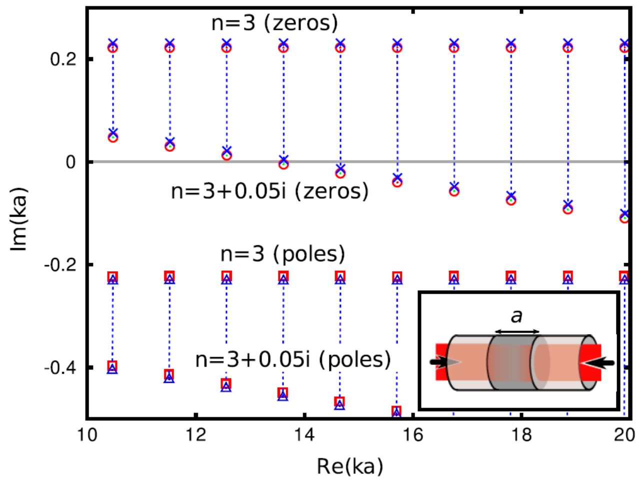


Fig. 1. A laser beam from a tunable (800 to 1000 nm) continuous-wave Ti:sapphire source enters a beam splitter (designated 1). The two split beams are directed normally onto opposite sides of a silicon wafer of thickness ~110 μm, using a Mach-Zehnder geometry. A phase delay in one of the beam paths controls the relative phase of the two beams. An additional attenuator ensures that the input beams have equal intensities, compensating for imbalances in the beam splitters and other imperfections. The output beams are rerouted, via beam splitters (designated 2, 3, and 4), into a spectrometer. The inset is a schematic of the CPA mechanism. The incident beams from left and right multiply scatter within the wafer with just the right amplitude and phase so that the total transmitted and reflected beams destructively interfere on both sides, leading to perfect absorption.

Wan *et al.*, *Science* **331**, 889 (2011)

Coherent Perfect Absorber (CPA)

- For initial input $a$being small, after time reversal, output a^*will be small $\rightarrow$ Zero of the S-matrix.
- The poles and zeros have symmetric distribution, owing to the time reversed pair $n_1 + in_2$and $n_1 - in_2$.
- For some value of n_2, the frequency crosses the real axis, resulting perfect absorption.
- Does not rely on certain specific modes.
- High Q value can lead to perfect absorption even for small single pass absorption.



Chong *et al.*, PRL **105**, 053901 (2010)

Coherent Perfect Absorber (CPA)

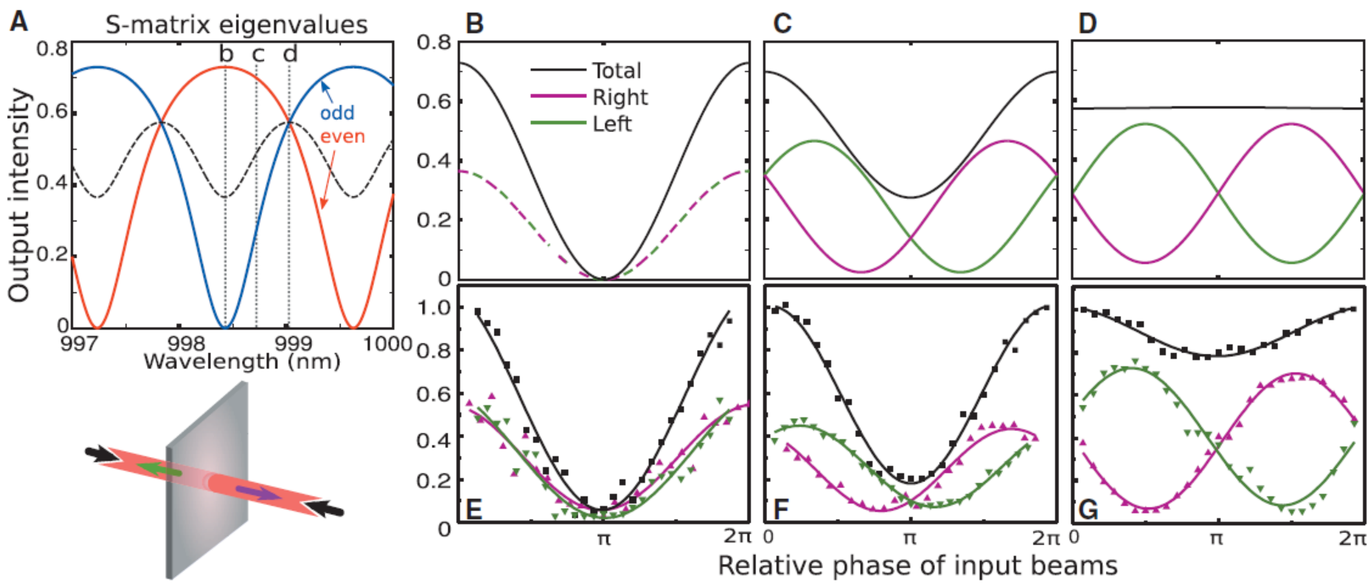


Fig. 2. Phase modulation of beam absorption. (**A**) Theoretical plot of normalized total output intensities as a function of wavelength λ for parity-odd (blue) and parity-even (red) scattering eigenmodes. The dashed black line is the result for incoherent input beams. (**B** to **D**) Theoretical output intensities at three representative values of λ as the relative phase of the input beams is varied, showing intensities emitted to the right (magenta) and left (green) sides of the slab, and the total intensity (black). Values of λ corresponding to (B) to (D) are marked by vertical lines in (A); (B) is the CPA resonance. (**E** to **G**) Experimental results at values of λ approximately corresponding to (B) to (D). Solid lines are fits to the data, not theory curves; results are normalized to max(I_{out}) of the fit.

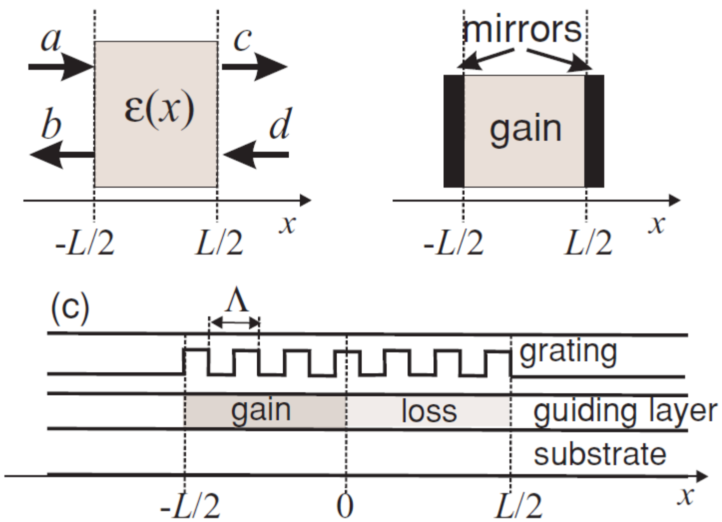
Wan *et al.*, *Science* **331**, 889 (2011)

PT-CPA Laser

[S. Longhi, *Physics* **3**, 61 (2010); PRA **82**, 031801(R) (2010)]

- A lasing setup with the complex dielectric constant satisfying $\epsilon(-\vec{r}) = \epsilon^*(\vec{r})$.
- Utilizes both Parity and Time Reversal symmetries of the Helmholtz equation.
- The solutions are related as: $E(x) = E^*(-x)$in 1-D. The complete solution:

$$E(x) = a\,\text{exp}(ikn_0x) + b\,\text{exp}(-ikn_0x), \quad x \leq -L/2,$$
$$E(x) = c\,\text{exp}(ikn_0x) + d\,\text{exp}(-ikn_0x), \quad x \geq L/2.$$



S. Longhi, PRA **82**, 031801(R) (2010)

PT-CPA Laser

- The transfer matrix:

$$\begin{pmatrix} c \\ d \end{pmatrix} = M(\omega) \begin{pmatrix} a \\ b \end{pmatrix}, \quad det(M) = 1. \tag{39}$$

- The laser threshold is achieved when the most unstable pole becomes real, with *no* inputs.

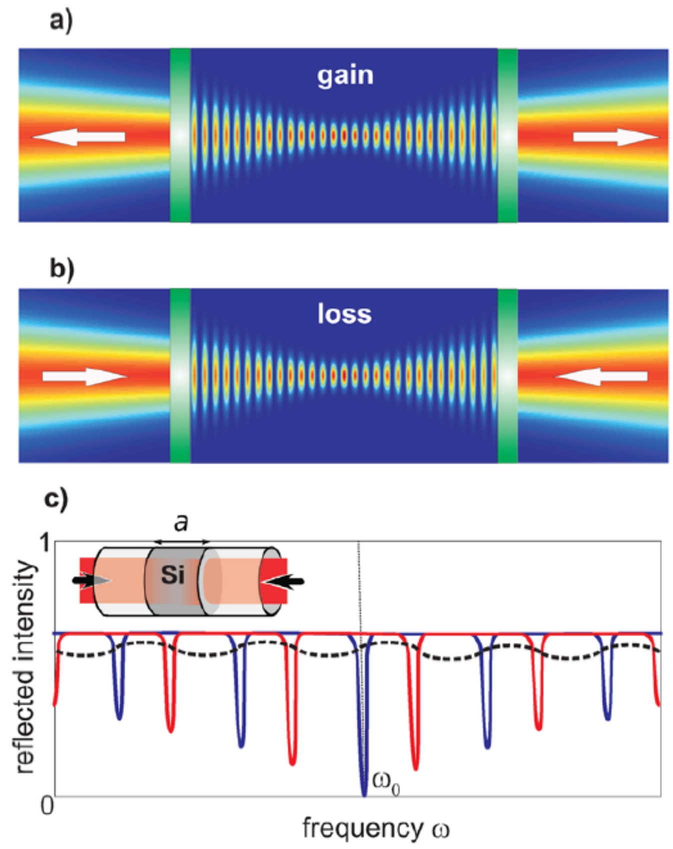
- Perfect absorption, with *no* outputs, cannot be satisfied by any frequency at laser threshold.

- For PT-symmetric laser, *both* the conditions are satisfied by certain real frequency.

- At proper resonance value, non-linear cancellation of intensities, with proper relative phase, results in perfect absorption.

- But CPA can be constructed in meta-materials, with linear superposition causing destructive interference, transferring the energy non-radiatively, even as plasmon excitations.

Coherent Perfect Absorber (CPA)



S. Longhi, *Physics* **3**, 61 (2010)

PT Symmetric Solutions in Modified KdV Equation

- The celebrated Korteweg-de Vries (KdV) equation is a well-studied non-linear dynamical system, first evoked for the description of solitary waves in shallow water

- It has both localized and periodic cnoidal wave solutions, which appear in the Lax equation as potentials, giving rise to bound states and band structure, respectively

- The KdV solutions are connected by the Miura transformation to the modified KdV (mKdV) equation, which also has found diverse physical applications

Modified KdV Equations

- There are two mKdV equations

$$v_{1,t} - 6v_1^2 v_{1,x} + v_{1,xxx} = 0$$
$$v_{2,t} + 6v_2^2 v_{2,x} + v_{2,xxx} = 0,$$

  whose solutions are related through $v \to iv$.

- The Miura transformation for the solution of KdV, $u = v^2 \pm v_x$, then implies that, $u = -v^2 \pm iv_x$, is also a solution.

- We find complex $\mathcal{PT}$-odd solutions for the mKdV equation, where the sum and the differences of the pair are also solutions.

Complex superposed solutions

- Let us start with KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0$$

where $u = u(x,t), u_t = \frac{\partial u}{\partial_t}, u_x = \frac{\partial u}{\partial x}, u_{xxx} = \frac{\partial^3 u}{\partial x^3}$

$cn^2 \pm isndn$ type solution

- It can be checked that the following pair of complex periodic solutions satisfy the KdV equation:

$$u(x,t) = Acn^2(\zeta, m) + iBsn(\zeta, m)dn(\zeta, m),$$

where $\zeta = \alpha(x - c\alpha^2 t)$, provided $A = -m\alpha^2, B = \pm\sqrt{m}\alpha^2$ and $c = (2m - 1)$, m being the modulus parameter.

- Here, velocity c exhibits two disjoint domains: for $\frac{1}{2} < m \leq 1$, the solution is right moving, while for the remaining half, $0 < m < \frac{1}{2}$, it is left moving.

$cn^2 \pm isncn$ type solutions

- The following factorizable, superposed solution also satisfies KdV equation:

$$u(x,t) = Acn^2(\zeta,m) + iBsn(\zeta,m)cn(\zeta,m) + \beta\alpha^2$$

provided $A = -m\alpha^2, B = \pm A$ and $c = (5m-4) - 6\beta$.

- In this case also, oppositely propagating modes occupy two different domains of m.

Some remarks

- For $m = 1$, superposed pairs reduce to PT-symmetric complex form: $u(x,t) = sech^2\zeta \pm isech\zeta tanh\zeta$. Using of the Cole-Hopf transformation: $v = \frac{\psi_x}{\psi}$, in one of the Miura route: $u = v^2 + v_x$,gives $u = \frac{\psi_{xx}}{\psi}$.

- The Galilean invariance of the KdV equation allows a constant shift in u, leading to $\psi_{xx} + [\lambda + u(x,t)]\psi = 0$, the one dimensional Schrödinger equation with a Scarf-type PT-symmetric potential.

- Soliton solutions can be generated through isospectral deformation of the potential, wherein both the wave function and the potential can change, leaving the spectrum invariant.

- Depending on the number of bound states, this iso-spectral flow introduces parameters, suitably interpretable as time variables.

- For the mKdV equation,

$$v_t - 6v^2 v_x + v_{xxx} = 0,$$

there exist solutions in the form of complex superposition.

$sn \pm icn$ type solutions

- The following parity odd superposition solution solves mKdV equation:

$$v = A\alpha sn(\zeta, m) + iB\alpha cn(\zeta, m),$$

provided $A = B = \pm\frac{\sqrt{m}}{2}$ and $c = \frac{m}{2} - 1$.

- Unlike KdV, the sum of the pair, as well as their difference, simultaneously satisfy the mKdV dynamics.

- In case of the sum, $v = 2A\alpha sn(\zeta, m)$ is an exact solution, when $c = 5(m-5)$ and $A = \pm\frac{\sqrt{m}}{2}$, while for the difference, $v = 2iB\alpha cn(\zeta, m)$ is also an exact solution, provided $c = (2m-1)$ and $B = \pm\frac{\sqrt{m}}{2}$.

$sn \pm idn$ type solutions

- Another form of complex periodic pair will satisfy the mKdV equation:

$$v = A\alpha sn(\zeta, m) + iB\alpha dn(\zeta, m),$$

  provided $A = \pm\frac{\sqrt{m}}{2}, B = \pm\frac{1}{2}$ and $c = \frac{1}{2} - m$.

- In case of $m = 1$, one obtains $v(x,t) = \frac{1}{2}(tanh\zeta \pm isech\zeta)$, which is odd under $\mathcal{PT}$ operation.

- It is interesting to note that, the velocity of the pair, $sn \pm icn$ is $-1/2$ times the velocity of dn solution and the velocity of the pair, $sn \pm idn$ is $-1/2$ times the velocity of cn solution.

Conclusion(s)

- PT-symmetric systems are experimentally accessible, with interesting applications regarding CPA and PT-CPA.

- Reveals exceptional points and bound states in the continuum.

- Study of S-matrix for such systems yields interesting boundary conditions, with accord to the experiments. Interestingly, incoming/outgoing 'fluxes' are needed from both directions, and insights regarding the norm are obtained.

Conclusion(s)

- Trivially PT-symmetric KdV equation is shown to possess PT-symmetric complex solutions, with asymptotically vanishing intensity for the solitons.

- For the mKdV equation, PT-odd solutions have been found to be exact solutions, which generate general PT-symmetric potentials for the KdV equation, through Miura transformation.

Thank you for your patience and attention