Measuring non-exponential decay at the bound state in continuum

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- Japan Society for the Promotion of Science (JSPS) – Grant No. JP18K034666
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Bound states in continuum

Bound states in continuum (**BICs**) represent localized states with energy directly in the scattering continuum.

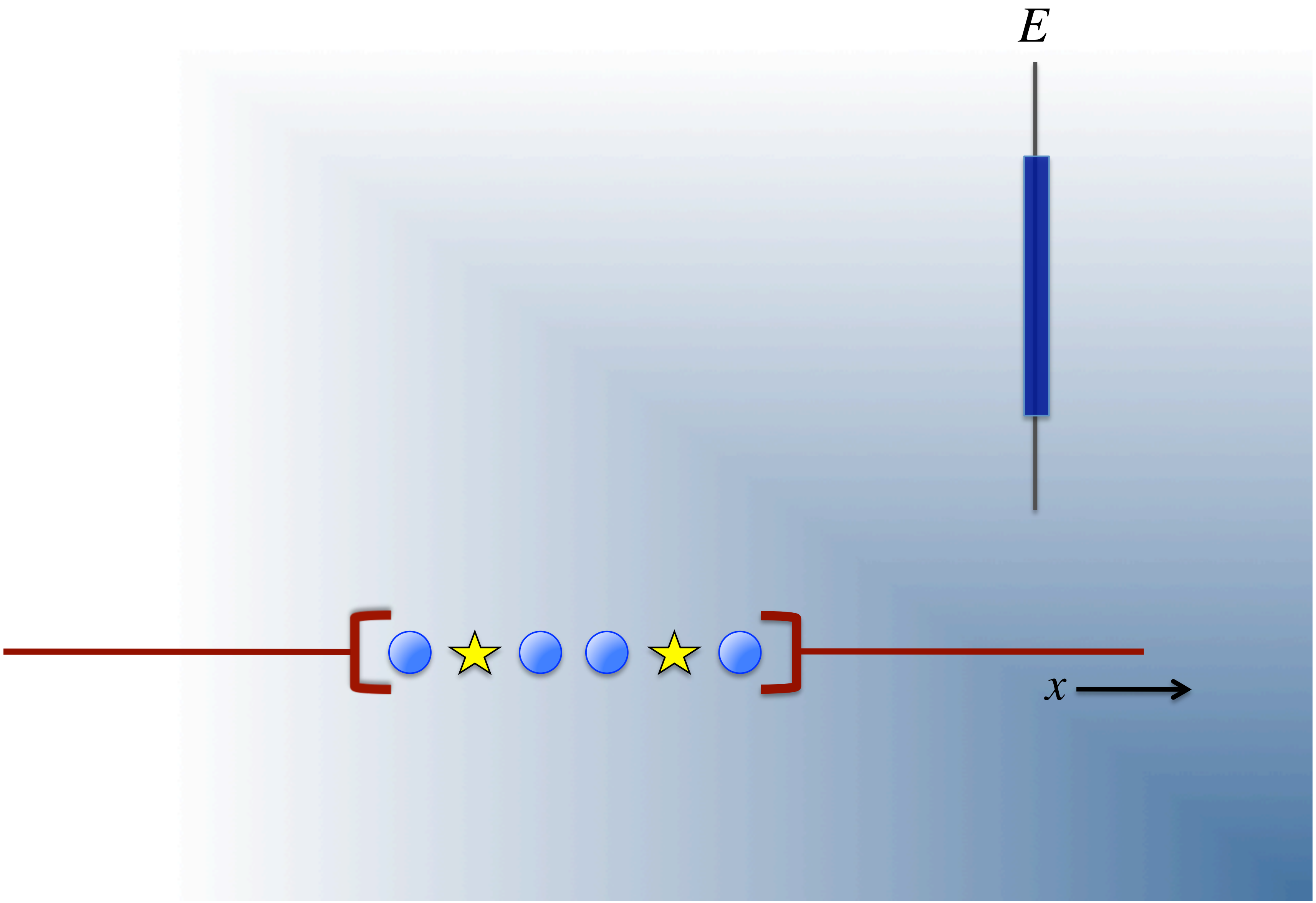
- BICs appear due to interference effects in quantum mechanics

- Predicted in 1929 by von Neumann and Wigner

  J. von Neumann and E Wigner, Phys. Z. **30**, 465 (1929).

- BICs can only appear in ***open* quantum systems** (discrete levels and energy continuum)

Open Quantum Systems – Discrete States



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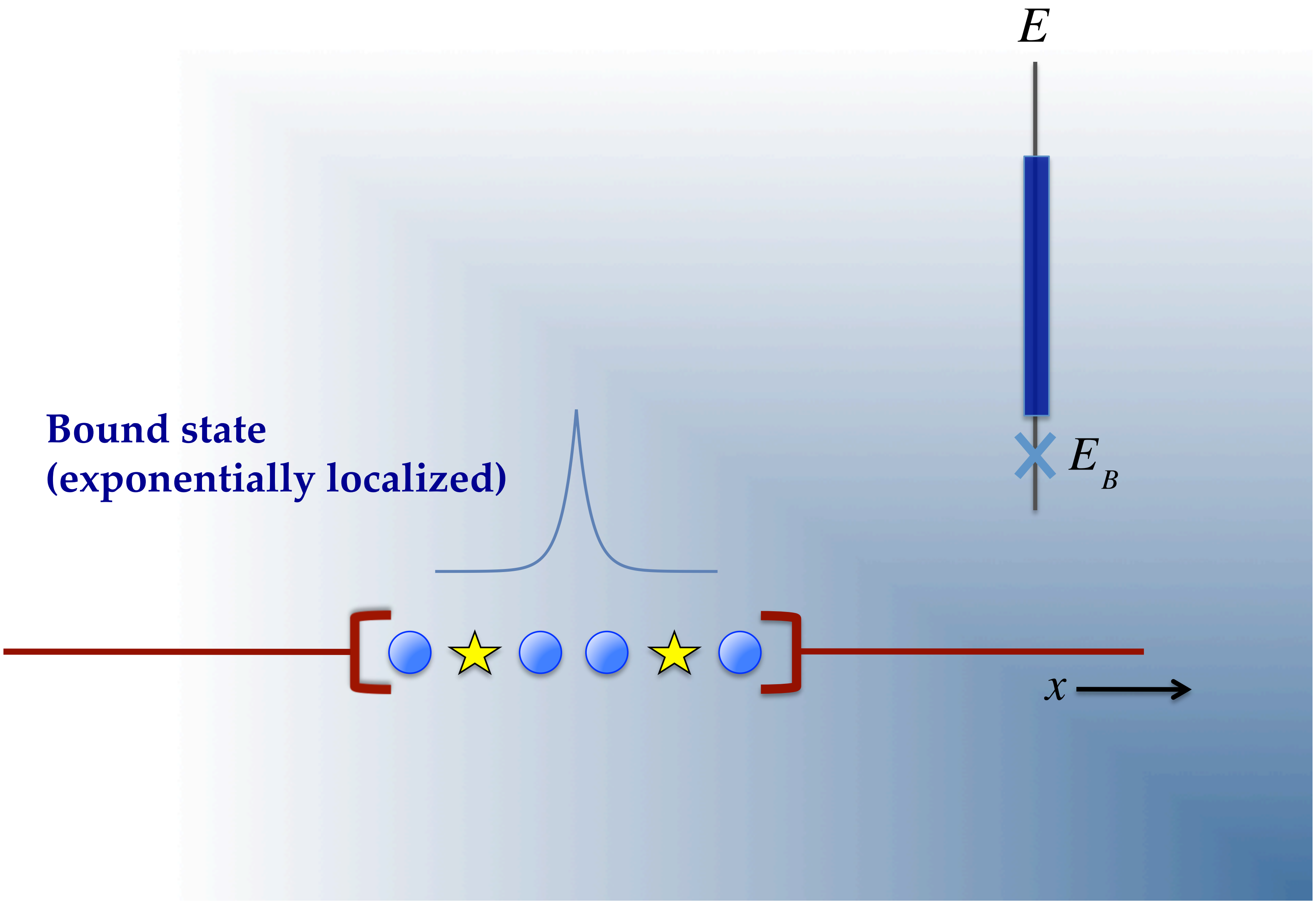

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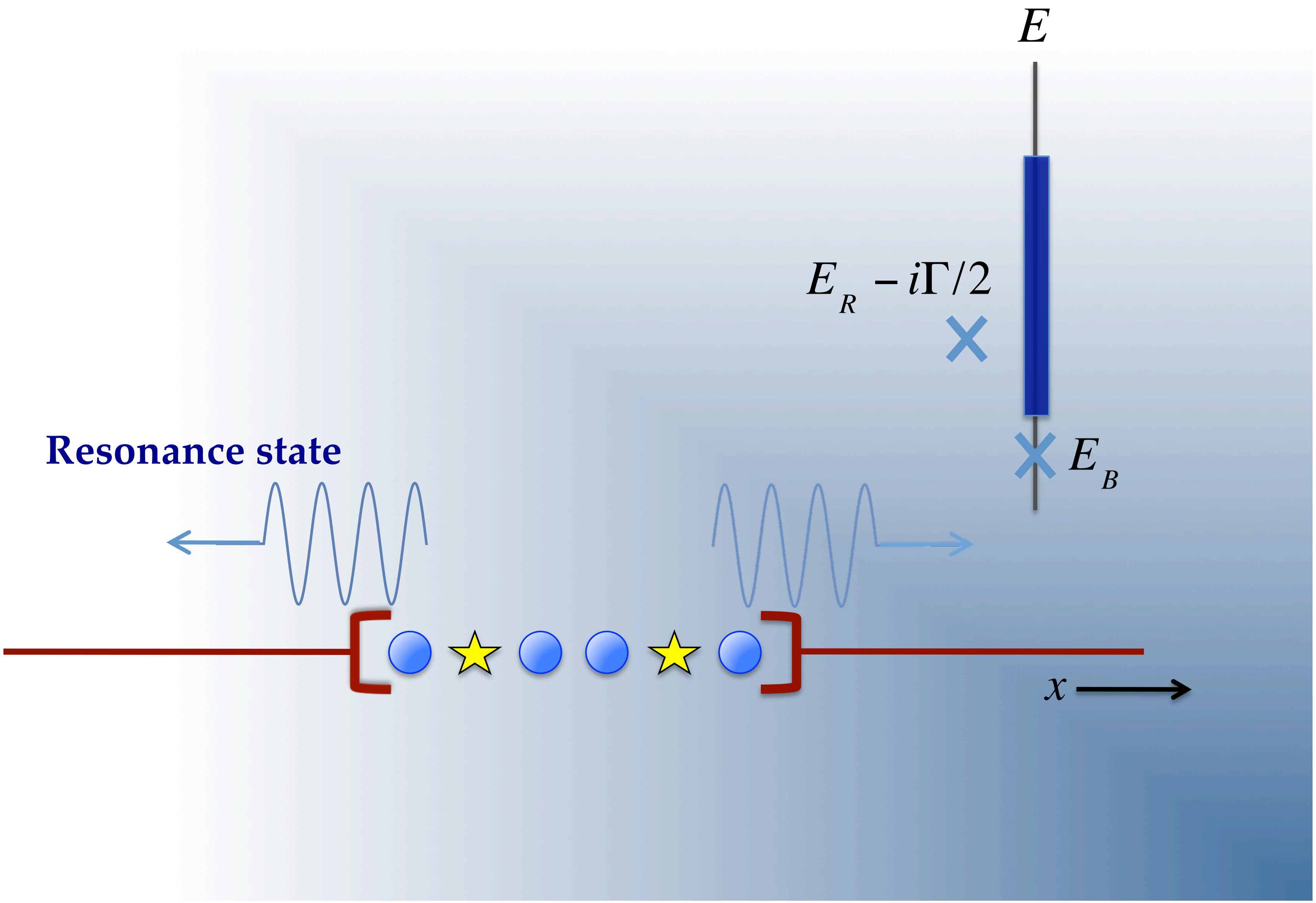

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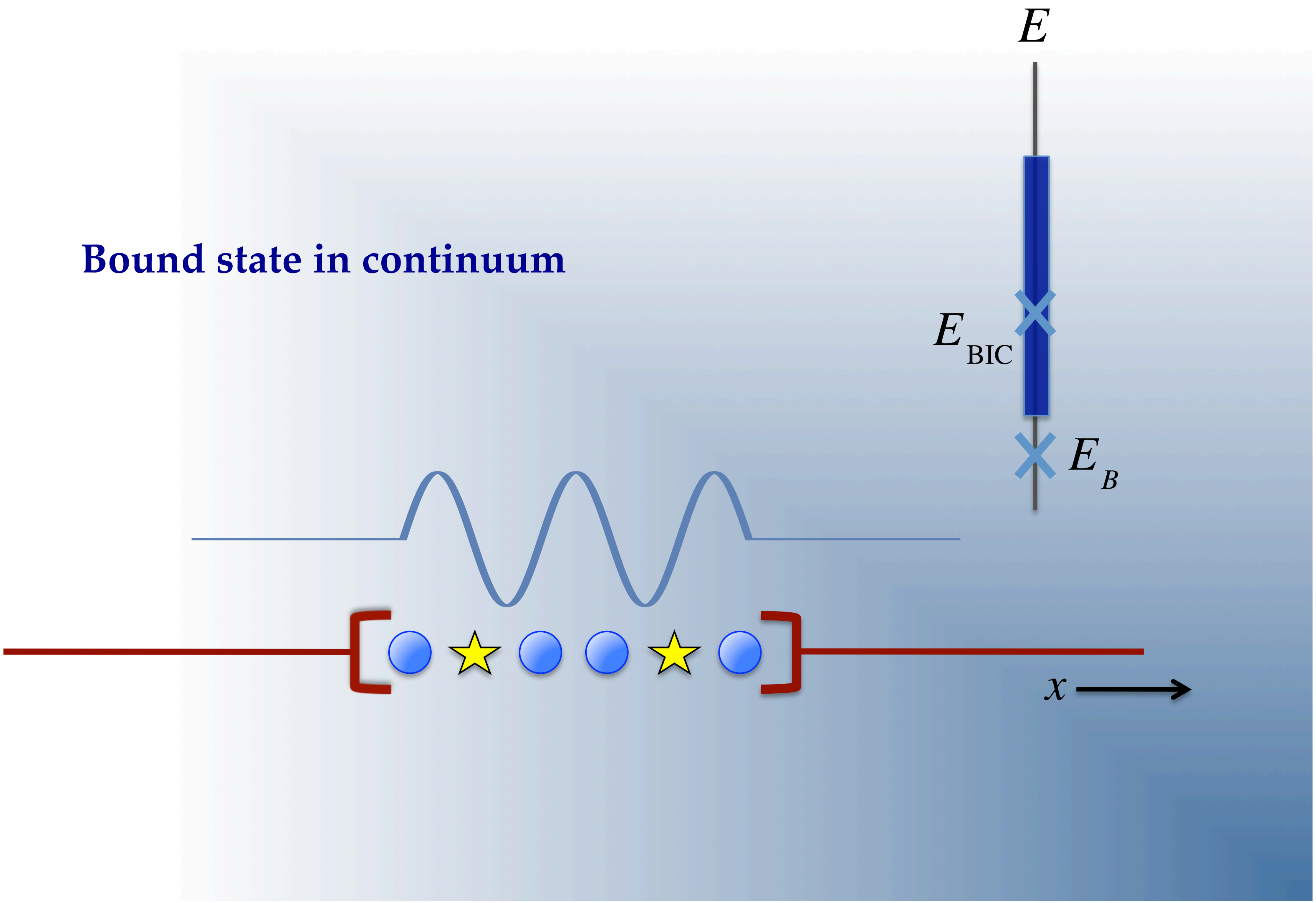


Bound states in continuum: Waveguide array

Although predicted in 1929, BICs are only very recently observed in optical waveguide array experiments.

BIC mode (spatially decoupled)

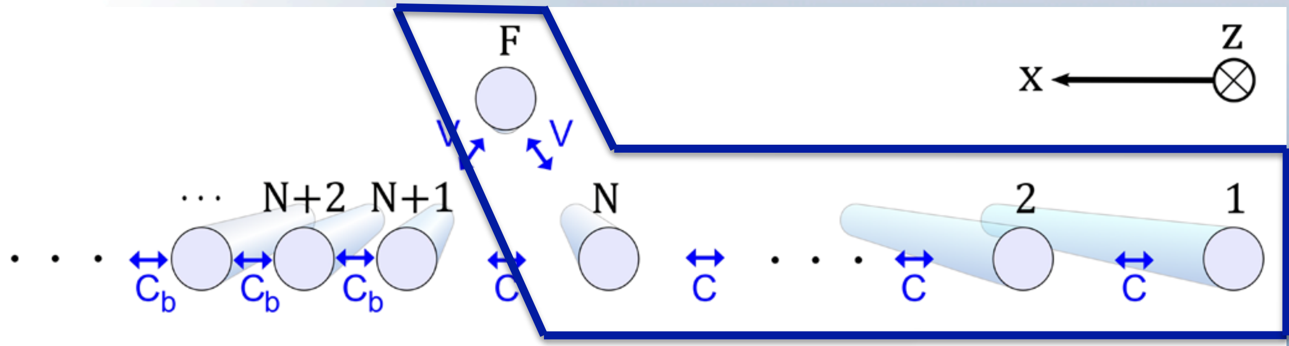
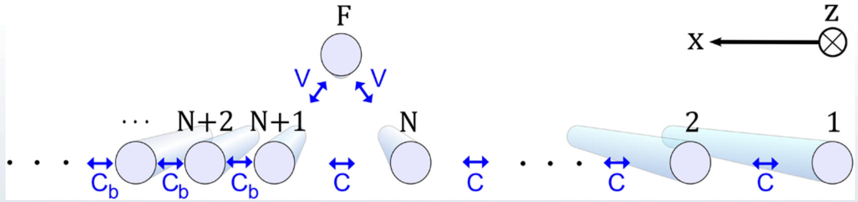


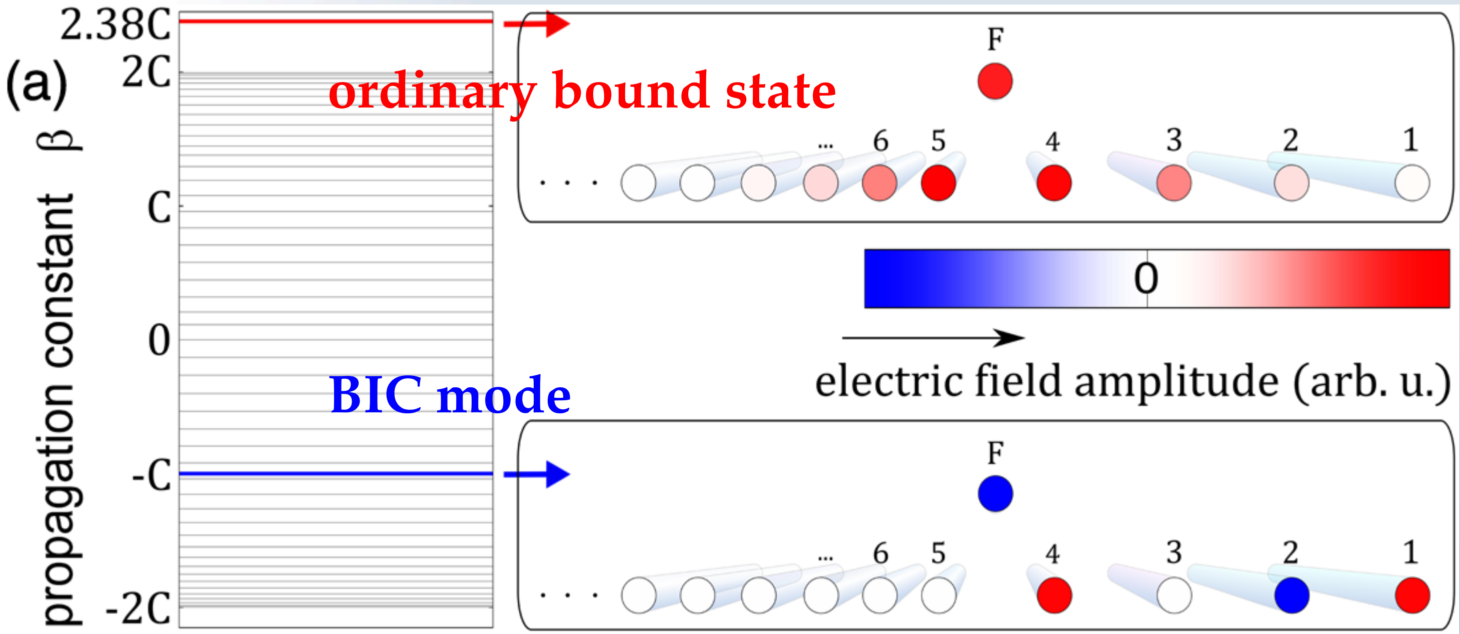
FIG. 1 (color online). Sketch of a semi-infinite waveguide array, where a side-coupled waveguide is introduced to provide the Fano resonance.

S. Weimann, *et al*, Phys. Rev. Lett. **111**, 240403 (2013).

Bound states in continuum: Waveguide array

Both bound states and BICs were observed:





case: $V = C$

$N = 4$

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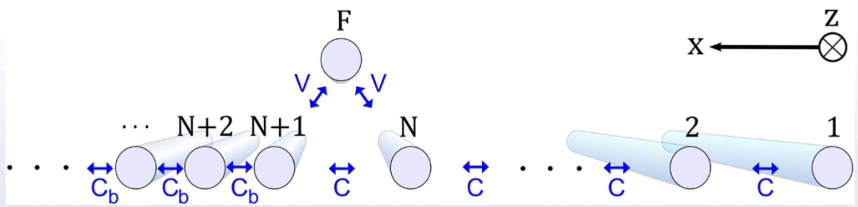

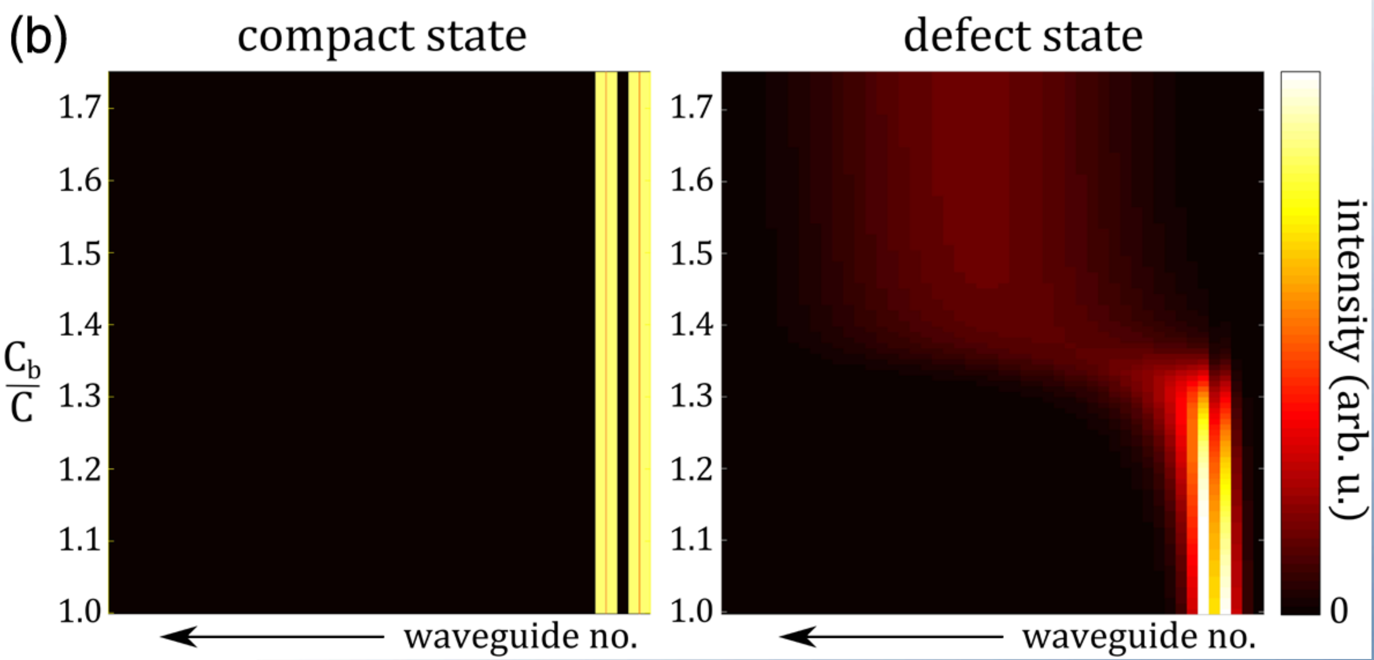

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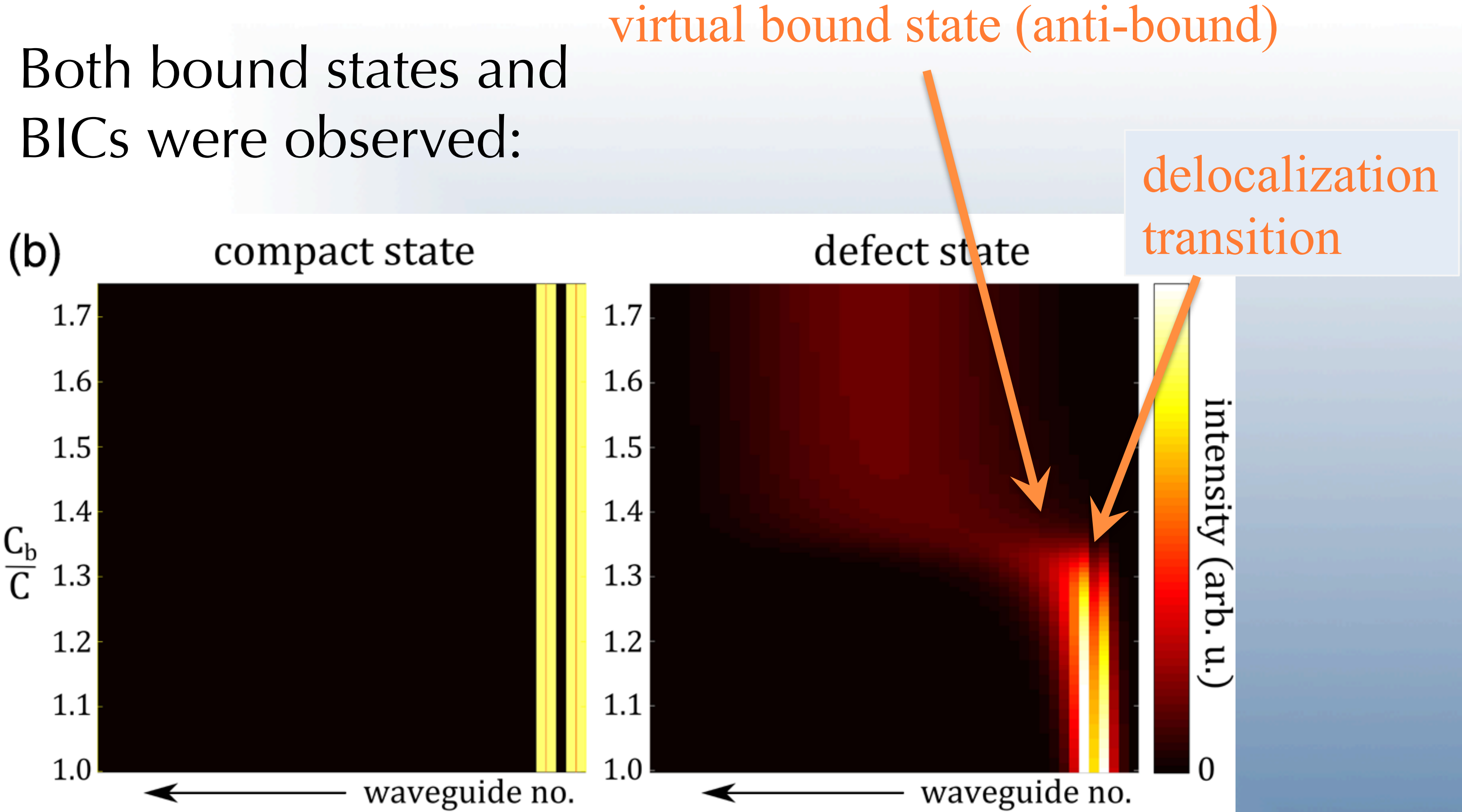


BIC mode

case: $V = C$

$N = 4$

S. Weimann, *et al*, Phys. Rev. Lett. **111**, 240403 (2013).

Motivation for the present study

We have seen BICs that occur due to interference and have been detected in optical systems.

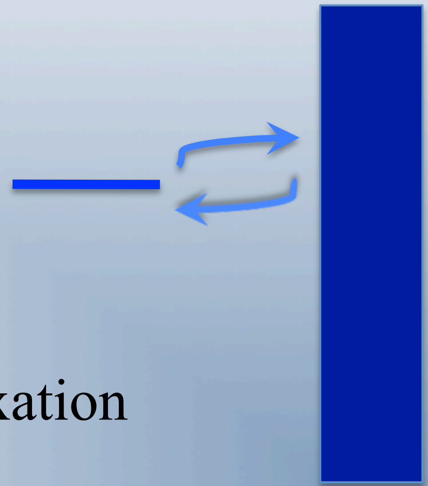
We propose to apply this recent progress on the BICs to make progress on another subtle effect: **deviations from exponential decay** in quantum and optical systems.

Resonance condition and exponential decay

Usually when the so-called resonance condition is satisfied, exponential decay is the dominant process

resonance eigenvalue: $E_R - i\frac{\Gamma}{2}$

$$P(t) = P_0 e^{-\Gamma t}$$



Examples: nuclear decay, atomic relaxation

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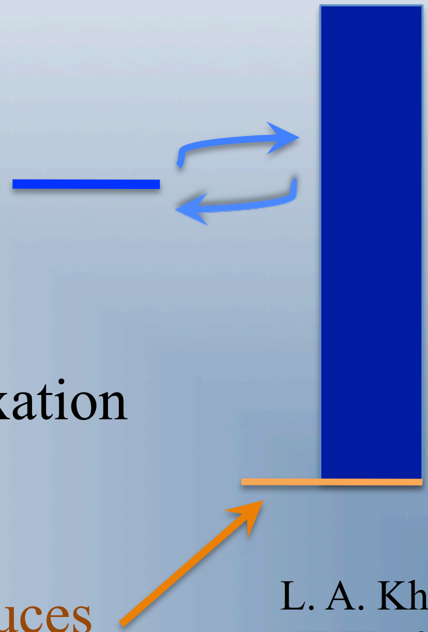
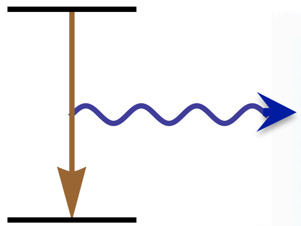


Examples: nuclear decay, atomic relaxation

However:
<u>continuum threshold</u> introduces deviations from exponential decay

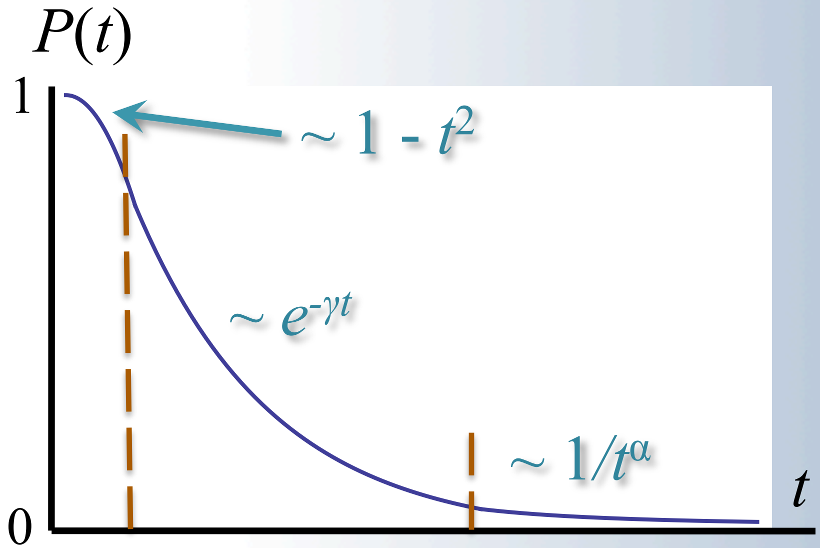
L. A. Khalfin, Sov. Phys.-JETP, **6**, 1053 (1958).

Deviations from exponential decay (1/2)



More detailed picture: deviations from exp. decay always exist in quantum mechanics

- Deviations from exponential decay exist at least on **extremely short** and **extremely long** time scales

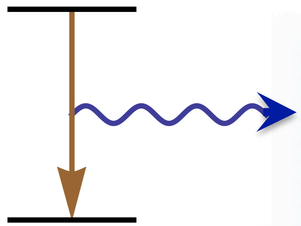


Survival Probability:

$$P(t) = \left|\langle \psi_0 | e^{-iHt} | \psi_0 \rangle\right|^2$$

C. B. Chiu, B. Misra, and E. C. G. Sudarshan, Phys. Rev. D, **16**, 520 (1977).

Deviations from exponential decay (1/2)



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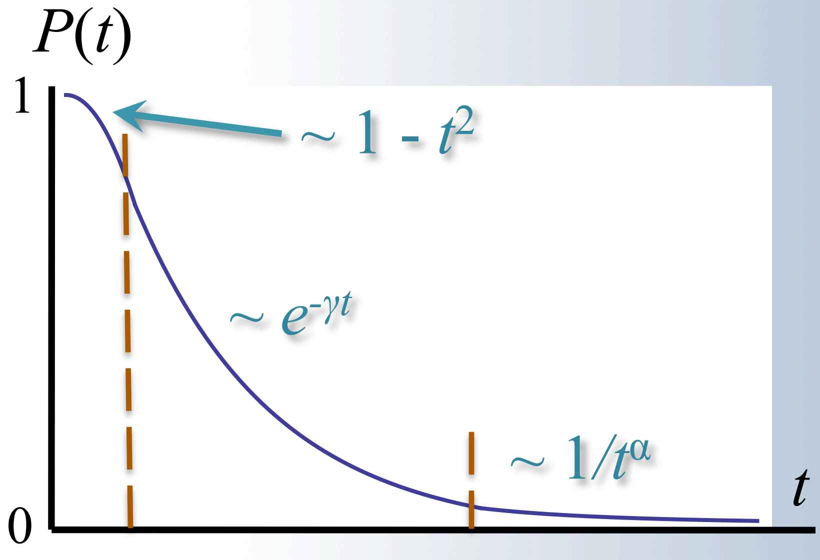
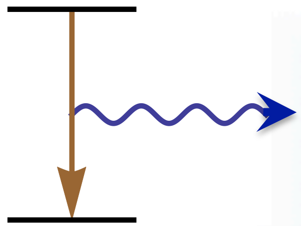


Typical long-time power law:

$$\alpha = 3, \qquad P(t) \sim \frac{1}{t^3}$$

C. B. Chiu, B. Misra, and E. C. G. Sudarshan, Phys. Rev. D, **16**, 520 (1977).

Deviations from exponential decay (2/2)



Deviations from exponential decay in quantum mechanics

However, these deviations are typically extremely difficult to detect in experiment.

For example, the long time deviation usually does not appear until after many exponential lifetimes have passed.

One experimental detection:

C. Rothe, S. I. Hintschich, and A. P. Monkman, Phys. Rev. Lett. **96**, 163601 (2006).

Motivation for the present study

In this work, we propose a new approach to studying non-exponential decay in quantum and optical systems.

Initial observation: exponential decay is suppressed at BIC

$|\psi_{\text{BIC}}\rangle$ BIC state itself ⇨ Hamiltonian eigenstate

(stable evolution)

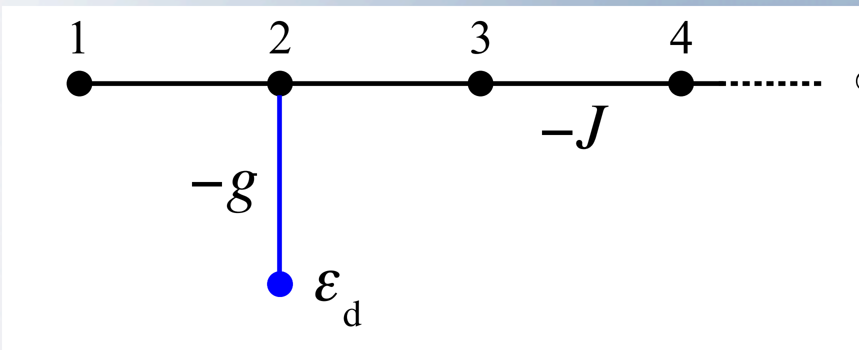
Our proposal: consider the time evolution of a state that lies orthogonal to the BIC

$$\langle \psi_{\perp} | \psi_{\text{BIC}} \rangle = 0$$

Model: semi-infinite optical waveguide array

We use a slightly simplified variation of the geometry from the optical waveguide experiment:

$$H = \varepsilon_{\mathrm{d}} |d\rangle\langle d| - J \sum_{n=1}^{\infty} \left[|n\rangle\langle n+1| + |n+1\rangle\langle n| \right] - g\left[|d\rangle\langle 2| + |2\rangle\langle d| \right]$$



Semi-infinite waveguide array: continuum

$$H = \varepsilon_{\mathrm{d}} |d\rangle\langle d| - J \sum_{n=-\infty}^{\infty} \left[|n\rangle\langle n+1| + |n+1\rangle\langle n| \right] - g\left[|d\rangle\langle 2| + |2\rangle\langle d| \right]$$

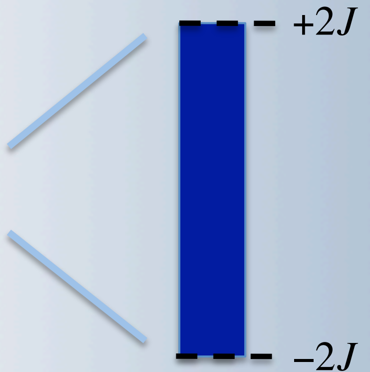
Half-range Fourier transform

$$|n\rangle = \sqrt{\frac{2}{\pi}} \int_0^{\pi} dk \sin nk |k\rangle$$

$$H = \varepsilon_{\mathrm{d}} |d\rangle\langle d| + \int_0^{\pi} E_k |k\rangle\langle k| dk + g\int_0^{\pi} V_k \left[|d\rangle\langle k| + |k\rangle\langle d| \right] dk$$

continuum:

$$E_k = -2J\cos k$$



$$V_k = \sqrt{\frac{2}{\pi}} \sin 2k$$

Discrete Spectrum: Green's function

We can obtain the discrete spectrum from the poles of the impurity site Green's function

$$\langle d|\frac{1}{z-H}|d\rangle = \frac{1}{z-\varepsilon_d - \Sigma(z)}$$

with the self-energy function

$$\Sigma(z) \;=\; g^2\int_0^{\pi} dk\,\frac{V_k^2}{z-E_k} \;=\; \frac{zg^2}{2}\left(z^2-2-z\sqrt{z^2-4}\right)$$

Discrete Spectrum: 4th order equation

We obtain the (4th order) discrete dispersion equation

$$z - \varepsilon_d = \frac{zg^2}{2}\left(z^2 - 2 - z\sqrt{z^2 - 4}\right)$$

In the range of typical physical values $\varepsilon_d \in (-2, 2); \;\; g \sim 1$
we find:

(2) solutions: bound states/virtual states

real eigenvalue

(2) solutions: resonance/anti-resonance pair

complex eigenvalues $z = E_R \pm i\Gamma$

Discrete Spectrum: 4th order equation

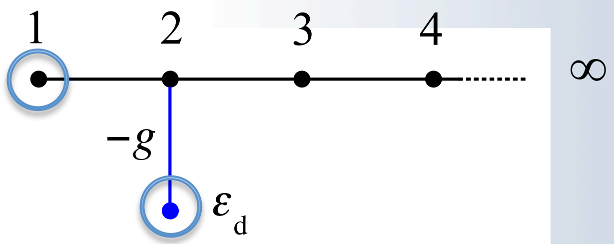
We obtain the (4th order) discrete dispersion equation

$$z - \varepsilon_d = \frac{zg^2}{2}\left(z^2 - 2 - z\sqrt{z^2 - 4}\right)$$

If we choose $\varepsilon_d = 0$ it is easy to see that a solution appears at $z = 0$, which is the BIC.

resonance solution: $E_R - i\Gamma \to E_{BIC} = 0$

as $\varepsilon_d \to 0$

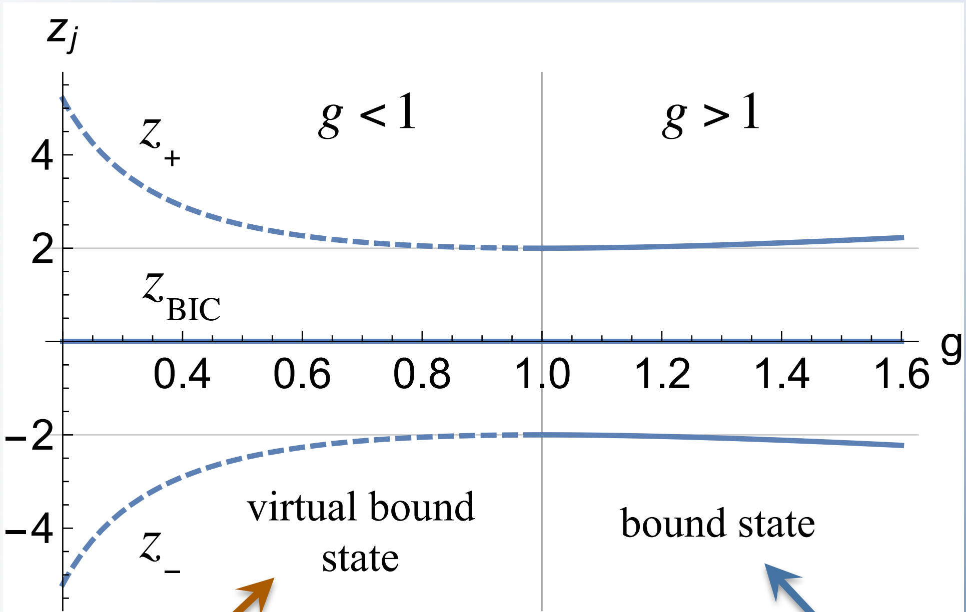


$$\left|\psi_{BIC}\right\rangle = \frac{1}{\sqrt{1+g^2}}\left(\left|d\right\rangle - g\left|1\right\rangle\right)$$

Discrete Spectrum for $\varepsilon_d = 0$

Let us focus on the spectrum in the case $\varepsilon_d = 0$ as the coupling g varies

$$z_\pm = \pm\left(g + \frac{1}{g}\right)$$



pure non-exponential decay

Will tend to suppress non-exponential dynamics

Survival probability for BIC-orthogonal state

The BIC eigenstate is given by

$$\left|\psi_{\mathrm{BIC}}\right\rangle = \frac{1}{\sqrt{1+g^2}}\left(\left|d\right\rangle - g\left|1\right\rangle\right)$$

We will study the time evolution of the simplest BIC-orthogonal state, given by

$$\left|\psi_{\perp}\right\rangle = \frac{1}{\sqrt{1+g^2}}\left(g\left|d\right\rangle + \left|1\right\rangle\right)$$

survival probability:

$$P_{\perp}(t) = \left|A_{\perp}(t)\right|^2$$

$$A_{\perp}(t) = \left\langle\psi_{\perp}\right|e^{-iHt}\left|\psi_{\perp}\right\rangle$$

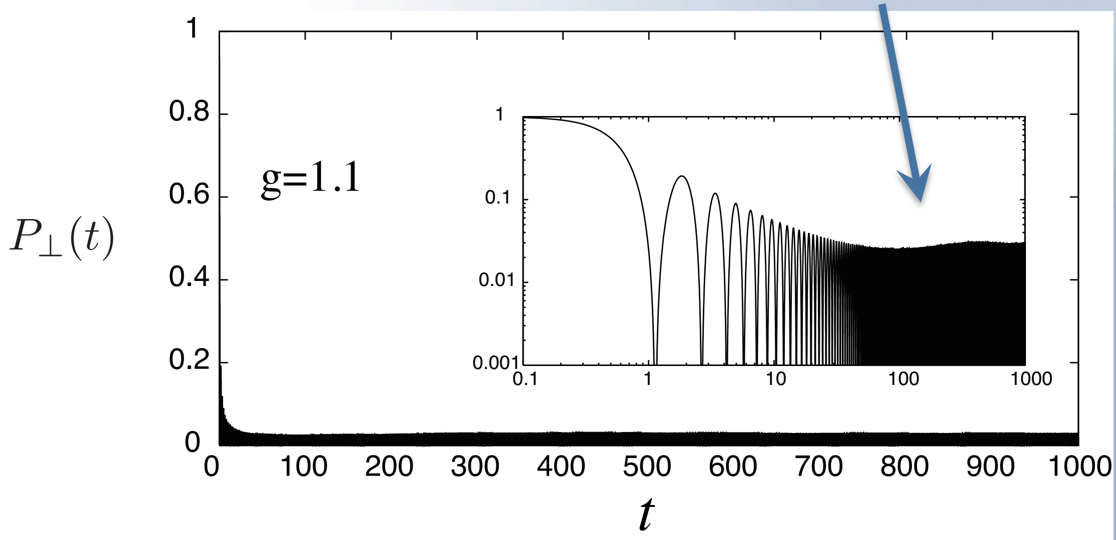
Time evolution due to bound states

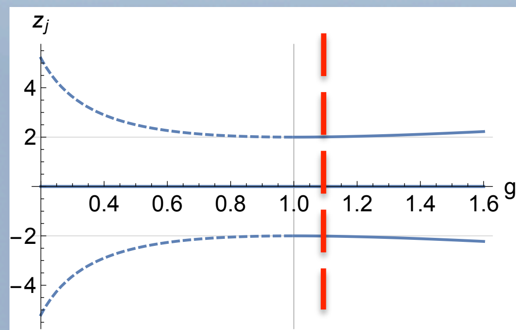
For $g > 1$ case we find

$$A_{\perp}(t) = \frac{g^2 - 1}{g^2} \cos\left(\left|z_{\pm}\right| t\right) + A_{\mathrm{br}}(t)$$

bound state dynamics

non-exponential dynamics

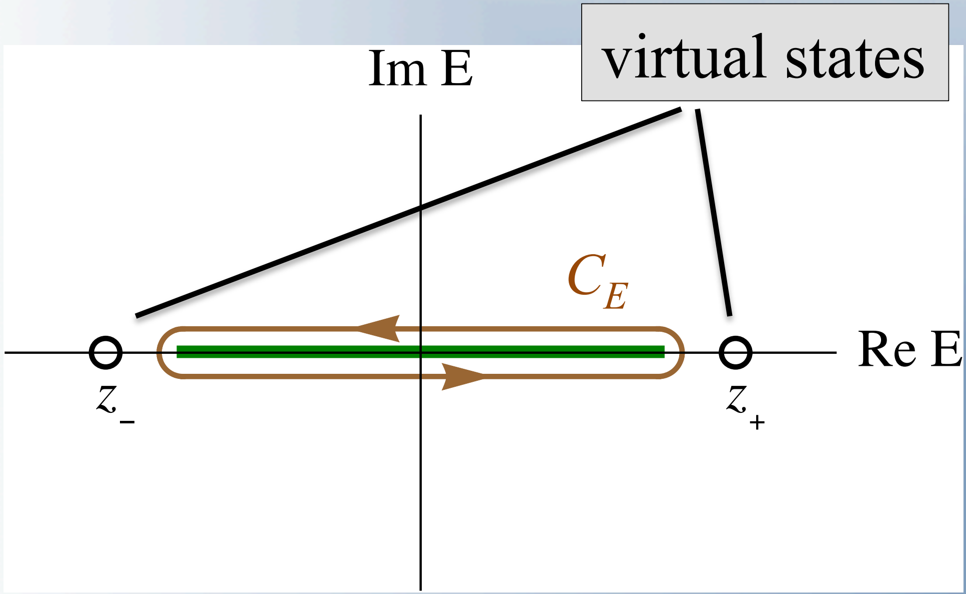




Time evolution: branch cut integral

For $g \leq 1$ the non-exponential dynamics from the branch cut will dominate the evolution.

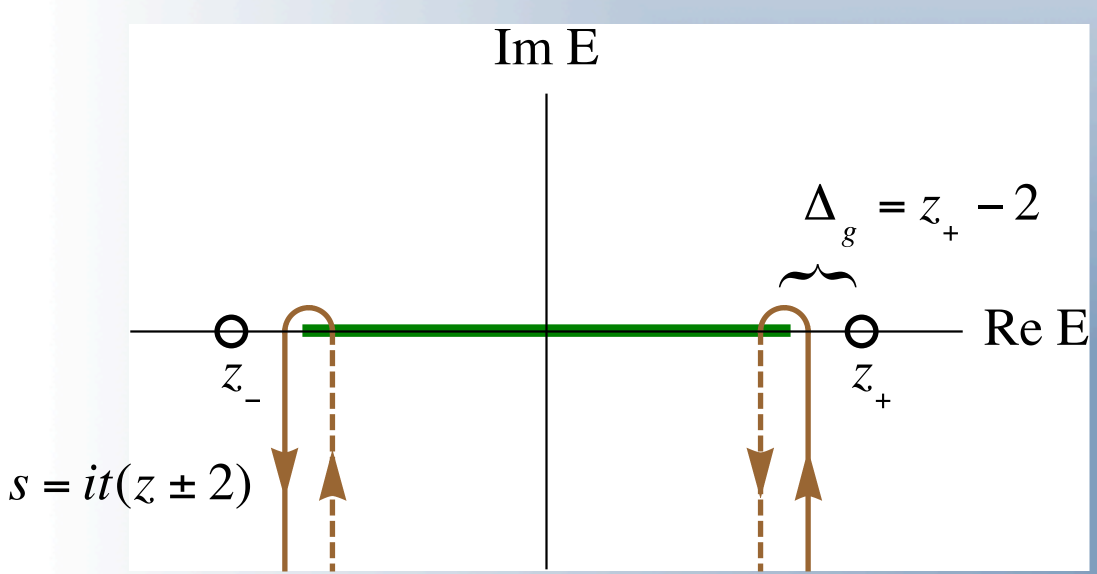
$$A_{\text{br}}(t) = \frac{1+g^2}{4\pi i g^2} \int_{C_E} e^{-izt} \frac{\sqrt{z^2 - 4}}{(z - z_+)(z - z_-)}$$



Time evolution: branch cut integral

We can deform the contour and apply an integration variable transform to obtain $A_{\mathrm{br}}(t) = A_+(t) + A_-(t)$

$$A_{\mp}(t) = \frac{i(1+g^2)e^{\pm 2it}}{2\pi g^2 t^2} \int_0^\infty ds e^{-s} \frac{\sqrt{s^2 \mp 4ist}}{\mp\Delta_g\left(2+|z_+|\right) + 4i\frac{s}{t} \mp \frac{s^2}{t^2}}$$



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Two key time zones:

inverse power law
far zone (FZ)

inverse power law
near zone (NZ)

$$P_{\mathrm{NZ}}(t) \approx \frac{(1+g^2)^2\cos^2(2t-\pi/4)}{4\pi g^4 t}$$

$$P_{\mathrm{FZ}}(t) \approx \frac{(1+g^2)^2\cos^2(2t-3\pi/4)}{\pi g^4(2+z_g)^2\Delta_g^2 t^3}$$

$T_Z << t << T_\Delta$

$T_\Delta \sim 1/\Delta_g$

$T_\Delta << t$

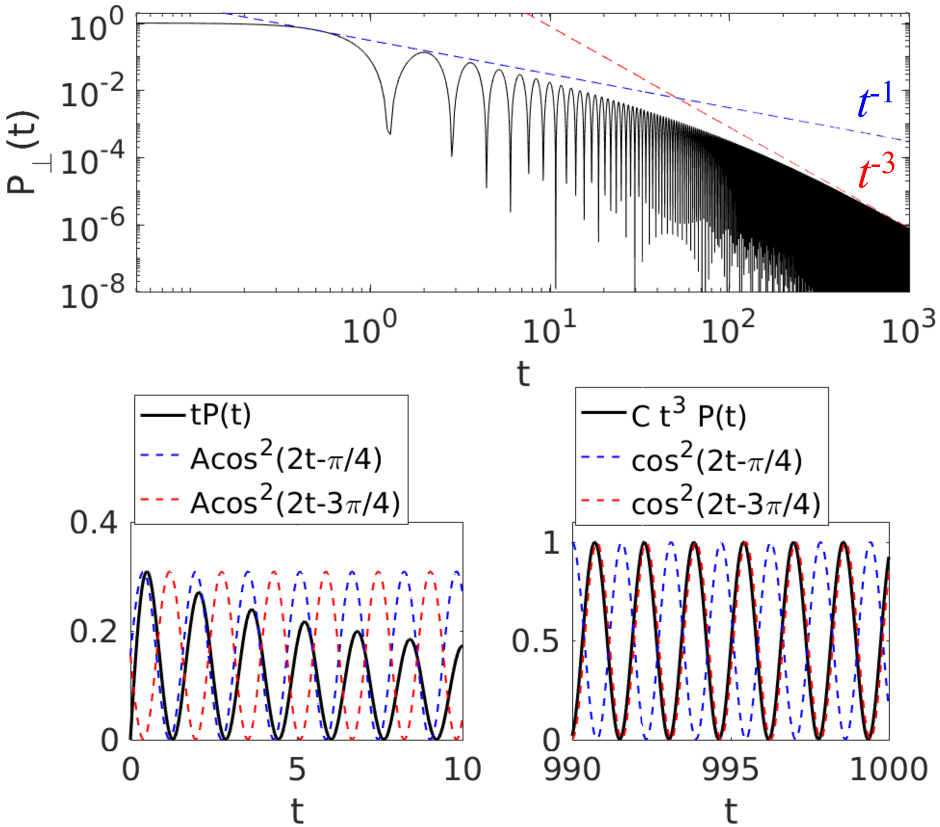
Time evolution: phase shift between time zones

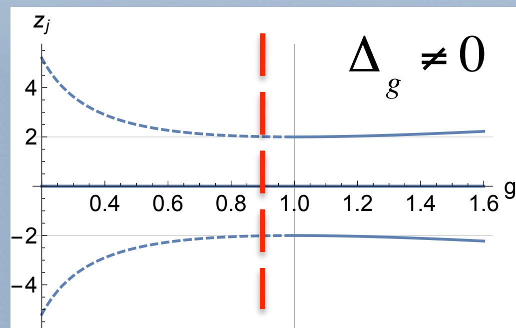
For $g = 0.9$, we see transition from near zone to far zone:

$$P_{\mathrm{NZ}}(t) \approx \frac{(1+g^2)^2\cos^2(2t-\pi/4)}{4\pi g^4 t}$$

$$T_\Delta \sim 1/\Delta_g$$

$$P_{\mathrm{FZ}}(t) \approx \frac{(1+g^2)^2\cos^2(2t-3\pi/4)}{\pi g^4 (2+z_g)^2 \Delta_g^2 t^3}$$



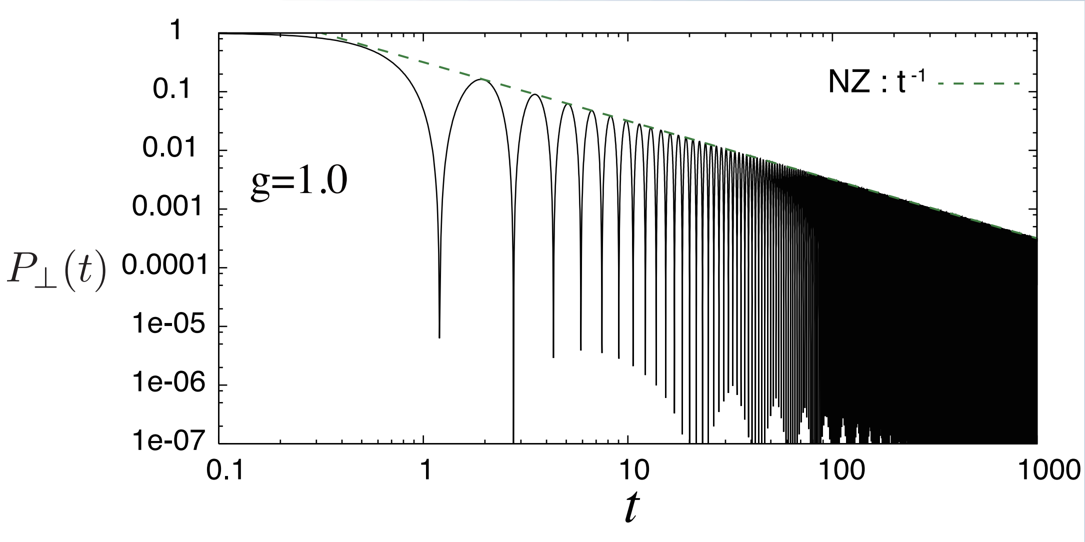


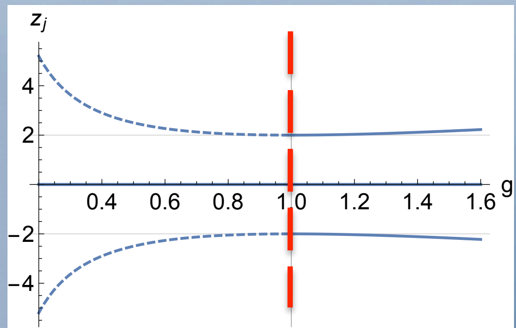
Time evolution at the delocalization transition

For g = 1, the near zone dynamics drive the evolution:

$$P_{\mathrm{NZ}}(t) \approx \frac{(1+g^2)^2 \cos^2(2t - \pi/4)}{4\pi g^4 t}$$

$$\Delta_g = 0 \quad \Rightarrow \quad T_\Delta \sim 1/\Delta_g \to \infty$$





virtual state influence on dynamics:

S. Garmon, T. Petrosky L. Simine, and D. Segal, Fortschr. Phys. **61**, 261 (2013).

Experimental picture: two considerations

To bring our analysis closer to real experiment, we consider the following issues:

(1) BIC detuning:

$$\text{Exact BIC} \xrightarrow{\varepsilon_d \neq 0} \text{resonance with narrow width}$$

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(1) BIC detuning:

Exact BIC $\xrightarrow{\varepsilon_d \neq 0}$ resonance with narrow width

(2) it may be difficult to measure output state $\langle \psi_\perp |$

We should consider the quantities: $\left| \langle d | e^{-iHt} | \psi_\perp \rangle \right|^2$

$$\left| \langle 1 | e^{-iHt} | \psi_\perp \rangle \right|^2$$

Experimental picture: two considerations

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Exact BIC $\xrightarrow{\varepsilon_d \neq 0}$ resonance with narrow width

(2) it may be difficult to measure output state $\langle \psi_\perp |$

In fact, we will find it convenient to study

$$P_{1d}(t) = \left| \langle d | e^{-iHt} | \psi_\perp \rangle \right|^2 + \left| \langle 1 | e^{-iHt} | \psi_\perp \rangle \right|^2$$

as $P_{1d}(t) = P_\perp(t)$ for $\varepsilon_d = 0$

Resonance imprint from BIC detuning

As we detune the system from the BIC, we expect the resonance will begin to impact the dynamics.

$$E_{\mathrm{BIC}} \Rightarrow E_{\mathrm{R}} - i\Gamma = \frac{\varepsilon_d}{1+g^2} - i\frac{g^2\varepsilon_d^2}{\left(1+g^2\right)^3}$$

We estimate this impact from the resonance pole:

$$P_{\perp,\mathrm{res}}(t) \approx \frac{g^4\varepsilon_d^4}{\left(1+g^2\right)^8} e^{-\Gamma t}$$

Lowest *and* next-lowest orders have canceled! Hence, our method is remarkably robust against detuning.

Measuring output quantity $P_{1d}(t)$

$$P_{1d}(t) = \left|\langle d|e^{-iHt}|\psi_\perp\rangle\right|^2 + \left|\langle 1|e^{-iHt}|\psi_\perp\rangle\right|^2$$

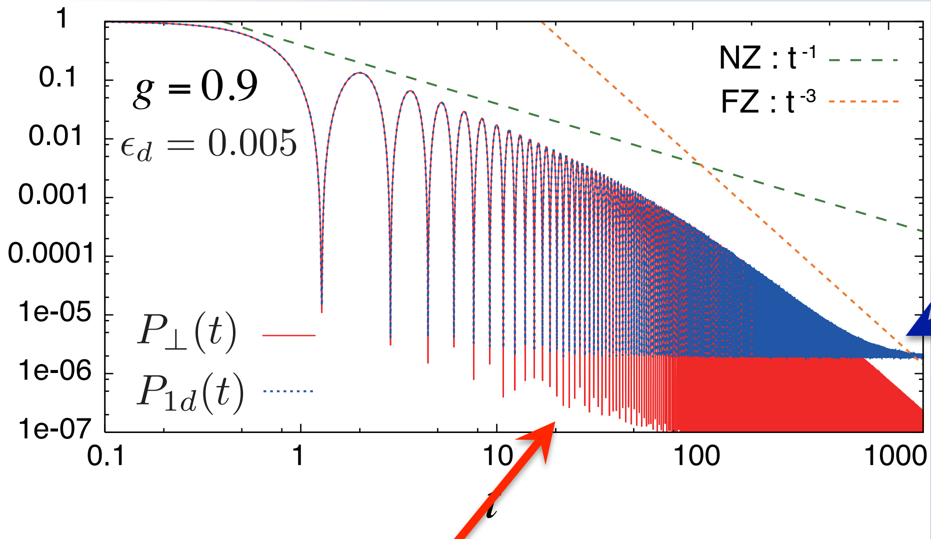
We find that $P_{1d}(t)$ is essentially the same as $P_\perp(t)$ for much of the evolution.

The one significant difference is the resonance effect is suppressed only to lowest order:

$$P_{1d,\mathrm{res}}(t) \approx \frac{g^2\varepsilon_d^2}{\left(1+g^2\right)^4} e^{-\Gamma t}$$

Impact of small detuning

For small detuning, the difference between the two quantities does not appear until late times:



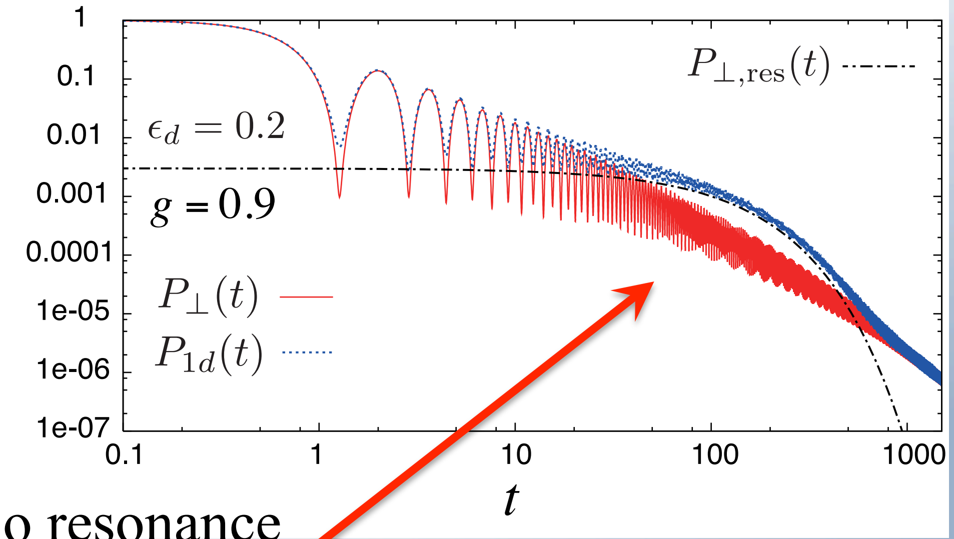
resonance effect eventually appears for $P_{1d}(t)$

$$\frac{g^2\varepsilon_d^2}{\left(1+g^2\right)^4} \sim 2\times 10^{-6}$$

the resonance has no noticeable impact on $P_\perp(t)$ $\approx \dfrac{g^4\varepsilon_d^4}{\left(1+g^2\right)^8} \sim 10^{-12}$

Impact of modest detuning

For larger detuning, the difference between the two appears only in the (short-lived) exponential regime:



still no resonance impact on $P_{\perp}(t)$

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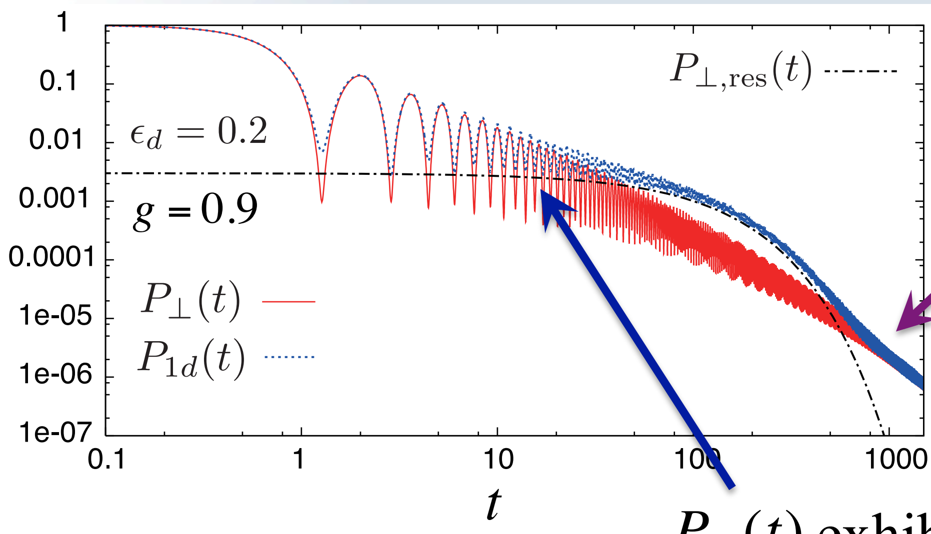

the two quantities agree again in the t^{-3} FZ regime

$P_{1d}(t)$ exhibits interesting pre-exponential inverse power law decay

Conclusion (1/2)

We have shown that populating a BIC-orthogonal state provides an interesting opportunity to study deviations from exponential decay.

In the case there are no other bound states or resonance states present, we obtain fully non-Markovian dynamics.

- ❖ Near zone ($1/t$) to far zone ($1/t^3$) transition associated with virtual bound state energy gap
- ❖ BIC in the middle of the band: coherent oscillations appear [$\pi/2$ phase shift between the two zones]
  - ❖ Including chain sites in initial state induces decoherence

Conclusion (2/2)

Our method is remarkably robust against BIC detuning:

$$P_{\perp,\mathrm{res}}(t) \approx \frac{g^4 \varepsilon_d^4}{\left(1+g^2\right)^8} e^{-\Gamma t}$$

We introduced a new measurement quantity for which the resonance influenced is somewhat less suppressed:

$$P_{1d}(t) = \left|\left\langle d \right| e^{-iHt} \left| \psi_\perp \right\rangle\right|^2 + \left|\left\langle 1 \right| e^{-iHt} \left| \psi_\perp \right\rangle\right|^2$$

$$P_{1d,\mathrm{res}}(t) \approx \frac{g^2 \varepsilon_d^2}{\left(1+g^2\right)^4} e^{-\Gamma t}$$

Recalling our initial motivation is to observe deviations from exponential decay, maybe $P_{1d}(t)$ is *more* interesting.