A class of exactly solvable Calogero type extended potentials associated with exceptional orthogonal polynomials

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Outline

- Introduction
- Rationally extended three-body potentials
    - Extended real case
    - Extended complex potential case
- Rational extension of many-body Calogero's model
- Summary

Introduction

1. **Three-body problem:**

- Calogero[1] in 1969 solved a problem of three particles interacting pairwise by inverse square potential in addition to the harmonic potential ($\hbar = 2m = 1$)

$$H = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + V_C, \tag{1}$$

with

$$V_C = V_H + V_I, \tag{2}$$

$$V_H = \frac{\omega^2}{8} \sum_{i<j} (x_i - x_j)^2; \quad V_I = g \sum_{i<j} (x_i - x_j)^{-2}. \tag{3}$$

[1] F. Calogero, *J. Math. Phys.* **10** (1969) 2191.

Three-body problem

- Solved by defining

$$\rho^2 = \frac{1}{3}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2], \qquad (4)$$

and using the Jacobi co-ordinates

$$R = \frac{1}{3}(x_1 + x_2 + x_3),$$

with

$$x = \frac{(x_1 - x_2)}{\sqrt{2}}, \qquad y = \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}}. \qquad (5)$$

- In polar co-ordinates

$$x = \rho \sin\phi, \qquad y = \rho\cos\phi; \qquad 0 \le \rho \le \infty, \quad 0 \le \phi \le 2\pi, \qquad (6)$$

Three-body problem

- We can easily show

$$(x_1 - x_2) = \sqrt{2}\rho \sin \phi,$$
$$(x_2 - x_3) = \sqrt{2}\rho \sin(\phi + 2\pi/3),$$
$$(x_3 - x_1) = \sqrt{2}\rho \sin(\phi + 4\pi/3). \quad (7)$$

- Thus, the Schrödinger equation $H\Psi = E\Psi$ is now solvable by separation of variables as

$$\Psi(\rho, \phi) = R(\rho)\Phi(\phi) \quad (8)$$

with
$R(\rho) \longrightarrow f(\rho)L_n^{(\alpha)}(g(\rho))$ and $\Phi(\phi) \longrightarrow f(\phi)F(g(\phi))$.

Many-body problem

2. Many-body problem[2]:

- The Hamiltonian ($\hbar = 2m = 1$)

$$H = -\sum_{i<j}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\{\frac{1}{8}\omega^2(x_i - x_j)^2 + g(x_i - x_j)^{-2}\}. \quad (9)$$

- The solution of this system is given as

$$\Psi(\mathbf{x}) = z^{a+\frac{1}{2}} R(\rho) P_k(\mathbf{x}), \quad (10)$$

where

$$z = \prod_{i=2}^{N}\prod_{j=1}^{i-1}(x_i - x_j)$$
$$\rho^2 = \frac{1}{N}\sum_{i=2}^{N}\sum_{j=1}^{i-1}(x_i - x_j)^2$$
$$a = (g + \tfrac{1}{2})^{\frac{1}{2}}, \quad (11)$$

[2] F. Calogero, *J. Math. Phys.* **12** (1971) 419.

Many-body problem

- $P_k(\mathbf{x})$ is homogeneous polynomial of degree $k(=0,1,2,...)$ and satisfies the generalized Laplace equation

$$\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+2(a+\frac{1}{2})\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{1}{(x_i-x_j)}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\right]P_k(\mathbf{x})=0. \tag{12}$$

- The function $R(\rho)$ satisfies the equation

$$R''(\rho)+\left(N+2k-2+N(N-1)(a+\frac{1}{2})\right)\frac{1}{\rho}R'(\rho)+(E-V(\rho))R(\rho)=0. \tag{13}$$

where $V(\rho)=\frac{1}{8}N\omega^2\rho^2$. The solution of the above Eq.

$$R(\rho)=\exp\left(-\sqrt{N/2}(\omega\rho^2)/4\right)L_n^{(b)}(\sqrt{N/2}(\omega\rho^2/2))$$

$$E_{n,k}=\sqrt{\frac{N}{2}}\omega(2n+b+1);\, b=[\frac{(N-3)}{2}+k+\frac{1}{2}N(N-1)(a+\frac{1}{2})].$$

The Calogero-Wolfes type 3-body problems (contd...)

- In 1974 Wolfes[3] showed that a three-body problem

$$V_W(g) = g[(x_1+x_2-2x_3)^{-2}+(x_2+x_3-2x_1)^{-2}+(x_3+x_1-2x_2)^{-2}], \quad (14)$$

  is also solvable when it is added to V_C with or without the inverse square potential V_I.
- Later on, Khare and Bhaduri[4] defined different Wolfes type of interaction terms V_{int} and obtained the exact solutions of a number of three body potentials in one dimension.
- Thus, we have the solution of $V = V_C + V_W + V_{int}$.

[3] J. Wolfes *J. Math. Phys.* **15** (1974) 1420.
[4] A. Khare and R. K. Bhaduri, *J. Phys. A* **27** (1994) 2213.

The Calogero-Wolfes type 3-body problems (contd...)

- Examples of V_{int}:

$$(a)\quad \frac{3f_1}{2\sqrt{2}\rho}\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)^2}+\text{c.p}\right].$$

$$(b)\quad \frac{-f_1}{(\sqrt{6})\rho}\left[\frac{(x_1+x_2-2x_3)}{(x_1-x_2)^2}+c.p\right].$$

$$(c)\quad \frac{\sqrt{3}}{2\rho^2}f_1\left[\frac{(x_1+x_2-2x_3)}{(x_1-x_2)}+\text{c.p}\right];\quad (f_1\rightarrow if_1)$$

$$(d)\quad \frac{-3\sqrt{3}}{2\rho^2}f_1\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)}+c.p\right];\quad (f_1\rightarrow if_1)$$

Note: The solutions of $V = V_C + V_W + V_{int}$ are obtained in the form of product of classical Laguerre and Jacobi orthogonal polynomials.

Rationally extended 3-body problems

- We extended the potential of the form[5]

$$V = V_C + V_W(g) + V_{int} + V_{rat}^{(1)}(\mathbf{x}) + V_{rat}^{(2)}(\mathbf{x}), \qquad (15)$$

- The exact forms of $V_{rat}^{(1)}(\mathbf{x})$ and $V_{rat}^{(2)}(\mathbf{x})$ are

$$V_{rat}^{(1)}(\mathbf{x}) = \frac{a\sum_{i<j}(x_i - x_j)^2 + c_1}{\left(b\sum_{i<j}(x_i - x_j)^2 + c_2\right)^2}, \quad \text{and} \qquad (16)$$

$$V_{rat}^{(2)}(\mathbf{x}) = \frac{\delta}{\rho^2}\left[\frac{k_1}{(k_2 + k_3\xi(\mathbf{x}))} - \frac{k_4}{(k_2 + k_3\xi(\mathbf{x}))^2}\right]. \qquad (17)$$

[5] N. Kumari, R.K.Yadav, A. Khare, B.P.Mandal, *Annals of Physics, 385 (2017) 57.*

RE 3-body potentials (contd...)

- Different forms[6] of ξ:

$$\begin{aligned}
&(1.a) \quad \frac{\sqrt{2}}{\rho^2}(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \\
&(1.b) \quad -\frac{3}{2\sqrt{2}\rho}\left(\sum_{i<j}(x_i - x_j)^{-1}\right)^{-1} \\
&(2.a) \quad \frac{3}{2\rho^2}\left[(x_1 + x_2 - 2x_3)^{-1} + c.p\right]^{-2} \\
&(2.b) \quad (\frac{2}{3\sqrt{6}})^2 \frac{1}{\rho^6}[(x_1 + x_2 - 2x_3)(x_2 + x_3 - 2x_1)(x_1 + x_3 - 2x_3)]^2
\end{aligned} \tag{18}$$

[6] N. Kumari, R.K.Yadav, A. Khare, B.P.Mandal, *Annals of Physics, 385 (2017) 57.*

RE 3-body potentials (contd...)

$$(3.a) \quad \frac{3}{\sqrt{6}}\frac{1}{\rho}\left[(x_1+x_2-2x_3)^{-1}+c.p\right]^{-1}.$$

$$(3.b) \quad -\frac{4}{6\sqrt{6}\rho^3}(x_1+x_2-2x_3)(x_2+x_3-2x_1)(x_1+x_3-2x_2).$$

$$(4) \quad \frac{1}{\sqrt{3}}\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)}+c.p\right] \tag{19}$$

RE 3-body real potential

- Now we consider a potential of the form

$$\begin{aligned} V &= V_H + V_W(g) + V_{int} + V_{rat}^{(1)} + V_{rat}^{(2)}, \\ &= \frac{\omega^2}{8}\sum_{i<j}(x_i - x_j)^2 + 3g[(x_1 + x_2 - 2x_3)^{-2} + \text{c.p}] \\ &+ \frac{3f_1}{2\sqrt{2}\rho}\left[\frac{(x_1 - x_2)}{(x_1 + x_2 - 2x_3)^2} + \text{c.p}\right] + V_{rat}^{(1)} + V_{rat}^{(2)}. \end{aligned}$$

$$\begin{aligned} \text{with} \quad \xi &= \frac{\sqrt{2}}{\rho^3}(x_1 - x_2)(x_2 - x_3)(x_3 - x_1); \\ \text{or} \quad \xi &= -\frac{3}{\sqrt{2}\rho}\left(\sum_{i<j}(x_i - x_j)^{-1}\right)^{-1}. \end{aligned} \tag{20}$$

- It leads to two different 3-body problems with same $\xi = -\sin(3\phi)$.

RE 3-body real potentials(contd...)

- In polar co-ordinates (ρ, ϕ), the Schrödinger Eq. for V is

$$\left[-\frac{d^2}{d\rho^2}-\frac{1}{\rho}\frac{d}{d\rho}-\frac{1}{\rho^2}\frac{d^2}{d\phi^2}\right]\psi_{n\ell}(\rho,\phi)+V(\rho,\phi)\psi_{n\ell}(\rho,\phi)=E_{n\ell}\psi_{n\ell}(\rho,\phi)\,. \tag{21}$$

- Using

$$\psi_{n\ell}(\rho,\phi)=\frac{R_{n\ell}(\rho)}{\rho^{1/2}}\Phi_\ell(\phi)\,, \tag{22}$$

- we get

$$\left[-\frac{d^2}{d\rho^2}+V_{ext}(\rho)\right]R_{n\ell}(\rho)=E_{n\ell}R_{n\ell}(\rho), \tag{23}$$

$$\left[-\frac{d^2}{d\phi^2}+V_{ext}(\phi)\right]\Phi_\ell(\phi)=\lambda_\ell^2\Phi_\ell(\phi), \tag{24}$$

RE 3-body real potential (contd...)

- The extended potential[7]

$$V_{ext}(\rho) = V_{Con}(\rho) + V_{rat}^{(1)}(\rho), \tag{25}$$

with

$$\begin{aligned} V_{Con}(\rho) &= \frac{3}{8}\omega^2\rho^2 + \frac{(\lambda_\ell^2 - 1/4)}{\rho^2}, \\ V_{rat}^{(1)}(\rho) &= \frac{(3a\rho^2 + c_1)}{(3b\rho^2 + c_2)^2}. \end{aligned} \tag{26}$$

- If, we set the parameters a, b, c_1 and c_2 as $a = 2\omega^2$; $b = (\sqrt{1/6})\omega$; $c_1 = -4(\sqrt{3/2})\omega c_2$; and $c_2 = 2\lambda_\ell$,

[7] N. Kumari, R.K.Yadav, A. Khare, B.P.Mandal, *Annals of Physics, 385 (2017) 57.*

RE 3-body real potential (contd...)

- The reduced potential matches exactly with the rationally extended radial oscillator with the solutions[8]

$$R_{n\ell}(\rho) \propto \frac{\rho^{\lambda_\ell+1/2} \exp\left(-\frac{1}{4}(\sqrt{3/2})\omega\rho^2\right)}{\left[(\sqrt{3/2})\omega\rho^2 + 2\lambda_\ell\right]} \hat{L}^{(\lambda_\ell)}_{n+1}\left((\sqrt{3/8})\omega\rho^2\right), \tag{27}$$

$$E_{n\ell} = (\sqrt{3/2})\omega(2n + \lambda_\ell + 1), \tag{28}$$

with $n = 0, 1, 2, ...; \qquad \ell = 0, 1, 2..., \qquad \lambda_\ell > 0.$

- $\hat{L}^{(\alpha)}_{n+1}(z) = -(z + \alpha + 1)L^{(\alpha)}_n(z) + L^{(\alpha)}_{n-1}(z).$

[8] C. Quesne, *J.Phys.A* **41** (2008) 392001.
Ullate et al, *J. Math. Phys.* **43** (2010) 434016.

RE 3-body real potential (contd...)

- The potential $V_{ext}(\phi)$ takes the form

$$V_{ext}(\phi) = V_{Con}(\phi) + V_{rat}^{(2)}(\phi)\,, \tag{29}$$

$$V_{Con}(\phi) = \frac{9g}{2}\sec^2(3\phi) - \frac{9f_1}{2}\sec(3\phi)\tan(3\phi), \tag{30}$$

$$V_{rat}^{(2)}(\phi) = \delta\left[\frac{k_1}{(k_2 - k_3\sin(3\phi))} - \frac{k_4}{(k_2 - k_3\sin(3\phi))^2}\right]. \tag{31}$$

defined over the range $-\frac{\pi}{6} < \phi < \frac{\pi}{6}$ and the parametric restriction $0 < k_3 < k_2 - 3$.

- If we set the parameters:
$\frac{9g}{2} = A(A-3) + B^2$; $\frac{9f_1}{2} = B(2A-3), \delta = 9$; $k_1 = 2k_2$; $k_2 = 2A - 3$; $k_3 = 2B$; and $k_4 = 2(k_2^2 - k_3^2)$.

RE 3-body real potential (contd...)

- The extended potential is equivalent to the RE trigonometric Scarf potential[9] with the solutions

$$\Phi_\ell(\phi) \propto \frac{(1-\sin(3\phi))^{\frac{1}{6}(A-B)}(1+\sin(3\phi))^{\frac{1}{6}(A+B)}}{(2A-3-2B\sin(3\phi))}\hat{P}^{(\alpha,\beta)}_{\ell+1}(\sin(3\phi)), \quad (32)$$

$\alpha = (A/3 - B/3 - 1/2); \quad \beta = (A/3 + B/3 - 1/2),$
and the energy eigenvalues

$$\lambda_\ell^2 = (A+3\ell)^2; \quad \ell = 0, 1, 2, ..., . \quad (33)$$

- $\hat{P}^{(\alpha,\beta)}_{n+1}(z) = -\frac{1}{2}(z-b)P^{(\alpha,\beta)}_n(z) + \frac{bP^{(\alpha,\beta)}_n(z) - P^{(\alpha,\beta)}_{n-2}(z)}{(\alpha+\beta+2n)},$

[9] C. Quesne, *J.Phys.A* **41** (2008) 392001.

RE 3-body complex potential

- Let us consider a complex potential V of the form[10]

$$\begin{aligned} V &= V_H + V_I + V_{int} + V_{rat}^{(1)} + V_{rat}^{(2)}, \\ &= \frac{\omega^2}{8}\sum_{i<j}(x_i - x_j)^2 + g\sum_{i<j}(x_i - x_j)^{-2} \\ &+ \frac{\sqrt{3}}{2\rho^2} i f_1 \left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)} + \text{c.p}\right] + V_{rat}^{(1)} + V_{rat}^{(2)}. \end{aligned} \quad (34)$$

- Keeping $V_{rat}^{(1)}$ same but different $V_{rat}^{(2)}$ by defining new form of function ξ i.e.

$$\xi = \frac{1}{3\sqrt{3}}\left(\frac{x_1 + x_2 - 2x_3}{x_1 - x_2} + \text{c.p}\right), \quad (35)$$

[10] N. Kumari, R.K.Yadav, A. Khare, B.P.Mandal, *Annals of Physics, 385 (2017) 57.*

RE 3-body complex potential (contd....)

- We get, the ϕ-dependent potential

$$V_{ext}(\phi) = V_{Con}(\phi) + V^{(2)}_{rat}(\phi), \tag{36}$$

with the equivalent conventional PT symmetric trigonometric Eckart potential

$$V_{Con}(\phi) = \frac{9}{2} g cosec^2(3\phi) + \frac{9}{2} i f_1 \cot(3\phi), \tag{37}$$

and the rational term

$$V^{(2)}_{rat}(\phi) = \delta\left[\frac{k_1}{(k_2 + k_3\cot(3\phi))} + \frac{k_4}{(k_2 + k_3\cot(3\phi))^2}\right]. \tag{38}$$

RE 3-body complex potential (contd....)

- The potential $V(\phi)$ is equivalent to the rationally extended PT symmetric[11] complex trigonometric Eckart potential [12]

$$V(\phi) = A(A-3)cosec^2(3\phi) + 2iB\cot(3\phi) + \frac{9}{A^2(A-3)^2}\left[\frac{-4iB[A^2(A-3)^2 - B^2]}{(iB + A(A-3)\cot(3\phi))} + \frac{2[A^2(A-3)^2 - B^2]^2}{(iB + A(A-3)\cot(3\phi))^2}\right], \quad (39)$$

[11] By P (i.e. parity) we mean here $\phi \to \pi - \phi$ while by T (i.e. time reversal) we mean $t \to -t$ and $i \to -i$.

[12] Which is easily obtained by complex co-ordinate transformation $x \to ix$ of the rationally extended hyperbolic Eckart potential [**C. Quesne,** *SIGMA* **8 (2012) 080.]**

RE 3-body complex potential (contd....)

- The associated wavefunction $\Phi_\ell(\phi)$ is given by

$$\Phi_\ell(\phi) \propto \frac{(z-1)^{\frac{\alpha_\ell}{2}}(z+1)^{\frac{\beta_\ell}{2}}}{(iB + A(A-3)\cot(3\phi))} y_\ell^{(A/3,B/3)}(z), \qquad (40)$$

  with $z = i\xi = i\cot(3\phi)$.

- with the polynomial function

$$\begin{aligned} y_\ell^{(A/3,B/3)}(z) &= \frac{2(\ell+\alpha_\ell)(\ell+\beta_\ell)}{(2\ell+\alpha_\ell+\beta_\ell)} q_1^{(A/3,B/3)}(z) P_{\ell-1}^{(\alpha_\ell,\beta_\ell)}(z) \\ &- \frac{2(1+\alpha_1)(1+\beta_1)}{(2+\alpha_1+\beta_1)} P_\ell^{(\alpha_\ell,\beta_\ell)}(z), \end{aligned} \qquad (41)$$

- where $q_1^{(A/3,B/3)}(z) = P_{p=1}^{(\alpha_p,\beta_p)}$.

RE 3-body complex potential (contd....)

- The parameters α_ℓ and β_ℓ in terms of A and B are given by

$$\alpha_\ell = -(A/3 - 1 + \ell) + \frac{B/9}{(A/3 - 1 + \ell)};$$
$$\beta_\ell = -(A/3 - 1 + \ell) - \frac{B/9}{(A/3 - 1 + \ell)}. \quad (42)$$

- The energy eigenvalues are same as given by Eq. (28), where λ_l is given by

$$\lambda_\ell^2 = 9(\ell + a - \frac{1}{2}) - \frac{9f_1^2}{16(\ell + a - \frac{1}{2})^2}; \quad \ell = 0, 1, 2, ... \quad (43)$$

- Note Φ_ℓ has to be such that $\lambda_\ell^2 > 0$.

Extension of Many-body problem

- The extended Many-body Hamiltonian H_{ext} is given by

$$H_{ext} = H + V_{new}(\rho) \tag{44}$$

where

$$V_{new}(\rho) = \frac{(\alpha_1 + \alpha_2\omega^2\rho^2)}{(\beta_1 + \beta_2\omega^2\rho^2)^2}; \quad \rho^2 = \frac{1}{N}\sum_{i=2}^{N}\sum_{j=1}^{i-1}(x_i - x_j)^2. \tag{45}$$

- The solution of this system is now given as

$$\Psi(\mathbf{x}) = z^{a+\frac{1}{2}} R_{ext}(\rho) P_k(\mathbf{x}), \tag{46}$$

where $R_{ext}(\rho)$ satisfies

$$R''_{ext}(\rho) + \left(N + 2k - 2 + N(N-1)(a + \tfrac{1}{2})\right)\frac{1}{\rho}R'_{ext}(\rho) + (E - V_{ext}(\rho))R_{ext}(\rho) = 0. \tag{47}$$

Extension of Many-body problem (contd...)

- The effective many-body potential reduces to the extended radial oscillator potential

$$V_{ext}(\rho) = V(\rho) + V_{new}(\rho); \text{where} \quad V(\rho) = \frac{1}{8}N\omega^2\rho^2. \quad (48)$$

with the well known solutions

$$R_{ext}(\rho) \simeq \frac{\exp\left(-\sqrt{N/2}(\omega\rho^2)/4\right)}{(\sqrt{N/2}\omega\rho^2 + 2b)} \hat{L}^{(b)}_{n+1}(\sqrt{N/2}(\omega\rho^2/2)). \quad (49)$$

Note: We can also generalize these problems corresponding to the X_m Laguerre or X_m Jacobi cases. In particular case for $m = 0$, the whole extended (three-body or many-body) problems can be reduces to the usual Calogero models

Summary

- New exactly solvable rationally extended Calogero-Wolfes type three body real and complex PT symmetric problems are constructed by adding new types of rational interaction terms.
- The solutions of these new 3-body systems are obtained as a product of the X_1 Laguerre times X_1 Jacobi EOPs.
- Many-body Calogero model is also considered and obtained the solutions by adding new rational type interaction term.

THANK YOU