UNITARY QUANTUM EVOLUTION
IN
NON-HERMITIAN INTERACTION PICTURE

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MZ, "Non-Hermitian interaction representation and its use in relativistic QM."
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thanks to the organizers!

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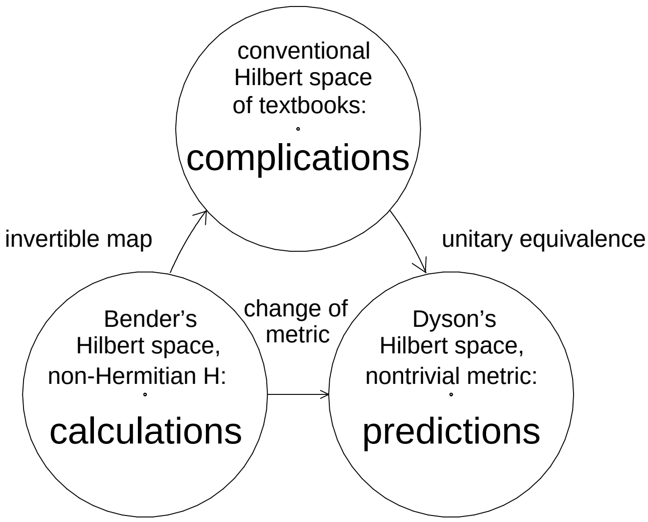
- ♢ **the zoo of formulations**
- ♠ **unitarily evolving systems**
- ♡ **non-Hermitian operators**
- ♣ **representations using triplets of Hilbert spaces**


conventional
Hilbert space
of textbooks:
complications
invertible map
unitary equivalence
Bender's
Hilbert space,
non-Hermitian H:
calculations
change of
metric
Dyson's
Hilbert space,
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predictions

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II. main message: NIP

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non-stationary non-Hermitian "representation" of the unitarily evolving systems

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interaction picture can be made non-Hermitian

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Hermiticity of the observables guaranteed indirectly

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how? via an *ad hoc* amendment of Hilbert space

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$\Longrightarrow$ Stone theorem not challenged

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2. QM in stationary dynamical regime:

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non-Hermitian Schrödinger picture

I. a bit of history

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Dyson (1956, ferromagnetism)

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Scholtz et al (1992, heavy nuclei)

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Bender et al (idea made popular between 1998 and 2004)

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Mostafazadeh's review (2010)

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scope extended to non-unitary systems

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scope extended beyond QM

II. the current state of art

♠ scepticism of mathematicians:

1960: the first (Dieudonné's) critique of the concept of quasi-Hermiticity

.

$\Longrightarrow$ 2005: pseudo-spectra: Trefethen and Embree

.

$\Longrightarrow$ 2015: quantum instabilities: Krejcirik et al

♣ Imperium strikes back:

1992: Scholtz et al: unbounded observables excluded

.

$\Longrightarrow$ alternative rescue attempts (Bagarello & MZ 2012, Mostafazadeh 2013, etc)

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$\Longrightarrow$ the safe territory of finite matrices

III. math of NSP

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(i) NSP must be equivalent to conventional SP

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for states $|\psi^{(SP)}(t)\succ \in \mathcal{H}^{(textbook)}$ in SP,

.

one predicts

$$\prec\psi^{(SP)}(t_f)|\mathfrak{q}_{(SP)}(t_f)|\psi^{(SP)}(t_f)\succ \tag{1}$$

one just solves

$$\mathrm{i}\frac{\partial}{\partial t}\,|\psi(t)\succ = \mathfrak{h}_{(SP)}\,|\psi(t)\succ\ , \qquad |\psi(t)\succ \in \mathcal{H}^{(textbook)} \tag{2}$$

$\boxed{\text{unitarity}}$ is guaranteed: $\mathfrak{h}_{(SP)} = \mathfrak{h}^{\dagger}_{(SP)}$ in $\mathcal{H}^{(textbook)}$

(ii) the Dyson's "non-Hermitian" 3HS paradigm:

$\boxed{\text{unusual}}$ stationary preconditioning

$$|\psi_n^{(correlated)}\succ = \Omega_{(Dyson)}|\psi_n^{(simplified)}\rangle \qquad (3)$$

$$\Omega_{(Dyson)} \neq \Omega_{(Dyson)}(t),\,, \qquad \Omega^\dagger_{(Dyson)}\Omega_{(Dyson)} = \Theta_{(stationary)} \neq I \qquad (4)$$

chosen $n-$independent,

$$|\psi^{(SP)}(t)\succ = \Omega_{(Dyson)}|\psi^{(Dyson)}(t)\rangle\,, \qquad |\psi^{(Dyson)}(t)\rangle \in \mathcal{H}^{(friendlier)}\,. \qquad (5)$$

.

$\boxed{\text{success,}}$ e.g., in IBM $\boxed{\text{or}}$ in CCM ($\Omega_{(Dyson)} = \exp S$):

$$\mathrm{i}\frac{\partial}{\partial t}\,|\psi^{(Dyson)}(t)\rangle = H_{(Dyson)}\,|\psi^{(Dyson)}(t)\rangle\,, \qquad |\psi^{(Dyson)}(t)\rangle \in \mathcal{H}^{(friendlier)} \qquad (6)$$

with $\boxed{\textit{simplified}}$ $H_{(Dyson)} = \Omega^{(-1)}_{(Dyson)}\mathfrak{h}_{(SP)}\Omega_{(Dyson)}$

(iii) a return to textbooks

tool:

$$\langle\psi| \;\to\; \langle\psi|\Theta_{(stationary)} \;\equiv\; \langle\psi_\Theta| \tag{7}$$

i.e.,

$$\mathcal{H}^{(friendlier)} \to \mathcal{H}^{(standard)} \tag{8}$$

.

the unphysical inner product $\langle\psi|\chi\rangle$ replaced by the physical one

$$\langle\psi|\chi\rangle \to \langle\psi|\Theta|\chi\rangle \;\equiv\; \langle\psi_\Theta|\chi\rangle \tag{9}$$

self-adjointness reinstalled,

$$H^\ddagger := \Theta^{-1}H^\dagger\Theta = H$$

(iv) practical Hermitization recipe

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metric $\Theta = \Theta_{(stationary)} = \Omega^\dagger \Omega$ usually $\boxed{\text{reconstructed}}$ by solving equation

$$H^\dagger \Theta = \Theta H \tag{10}$$

.

self-adjoint $\Omega_{(special)} = \sqrt{\Theta}$ often used

.

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further observables may be added,

$$Q^\dagger_{(Dyson)} \Theta = \Theta Q_{(Dyson)} \tag{11}$$

(math more complicated: MZ et al, 2017)

(v) example: stationary Klein-Gordon equation

in units $\hbar = c = 1$ (see also the talk by Semoradova),

$$\left(\frac{\partial^2}{\partial t^2} + D\right)\,\psi^{(KG)}(\vec{x},t) = 0\,, \qquad D = -\triangle + m^2\,, \quad m^2 = m^2(\vec{x}) \tag{12}$$

.

(a)===== Feshbach and Villars (FV) changed the variables

$$\psi^{(KG)}(\vec{x},t) \quad \rightarrow \quad \langle \vec{x}|\psi^{(FV)}(t)\rangle = \left(\begin{array}{c} \mathrm{i}\partial_t \psi^{(KG)}(\vec{x},t) \\ \psi^{(KG)}(\vec{x},t) \end{array} \right) \tag{13}$$

and replaced the hyperbolic partial differential Eq. (12) by its parabolic equivalent

$$\mathrm{i}\frac{\partial}{\partial t}\,|\psi^{(FV)}(t)\rangle = H_{(FV)}\,|\psi^{(FV)}(t)\rangle \tag{14}$$

(b)===== with stationary generator

$$H_{(FV)} = \begin{pmatrix} 0 & D \\ I & 0 \end{pmatrix} \neq H_{(FV)}(t)\,. \qquad (15)$$

Pauli and Weisskopf choose $\mathcal{H}^{(FV)} = \mathcal{L}^2(\mathbb{R}^3) \bigoplus \mathcal{L}^2(\mathbb{R}^3)$:

$$\langle \psi_1 | \psi_2 \rangle \quad \to \quad (\psi_1, \psi_2)_{(Krein)} = \langle \psi_1 | \mathcal{P}_{(FV)} | \psi_2 \rangle\,. \qquad (16)$$

.

(c)===== in the ultimate Mostafazadeh's presentation

$$(\psi_1, \psi_2)_{(Krein)} \quad \to \quad (\psi_1, \psi_2)_{(Mostafazadeh)} = \langle \psi_1 | \Theta_{(stationary)} | \psi_2 \rangle\,. \qquad (17)$$

the use of Hilbert-space metric

$$\Theta_{(stationary)} = \begin{pmatrix} 1/\sqrt{D} & 0 \\ 0 & \sqrt{D} \end{pmatrix}\,. \qquad (18)$$

yielded finally the first-quantized interpretation of the Klein-Gordon field.

IV. formalism *in nuce*

(i) predictions of measurements

physical observables $\quad Q_{(Dyson)} = \Omega^{(-1)}_{(Dyson)} \mathfrak{q}_{(SP)} \Omega_{(Dyson)} \neq Q^{\dagger}_{(Dyson)} \quad (19)$

equivalence

$$\prec \psi^{(SP)}(t_f)|\mathfrak{q}_{(SP)}(t_f)|\psi^{(SP)}(t_f)\succ = \langle \psi_{\Theta}^{(Dyson)}(t_f)|Q_{(Dyson)}(t_f)|\psi^{(Dyson)}(t_f)\rangle \,. \quad (20)$$

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(ii) special case: $\mathcal{PT}-$symmetric quantum mechanics

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flowchart inverted: tractable *input* $\quad H_{(Dyson)} \neq H^{\dagger}_{(Dyson)}$

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postulate added: let $H_{(Dyson)}$ be Krein-space self-adjoint:

$$H_{(Dyson)}\mathcal{PT} = \mathcal{PT} H_{(Dyson)} \quad (21)$$

3. non-Hermitian interaction picture

I. introductory notes

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Hermitian IP:

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- "no formulation produces a royal road to quantum mechanics"
- *amazing anomaly*: interaction picture not always mentioned
- question: is the Haag's theorem the reason?
- answer: not only

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II. the birth of NIP

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- pioneers and time-dependence explorers:
  - Faria and Fring (2006, 2007): linear time-dependence found NSP tractable
  - Mostafazadeh (2007): quasi-stationarity found NSP tractable; restrictions
- the end of the NSP era;
  - restrictions found merely terminological (MZ, quant-ph 0711.0535)
  - non-stationary formulation of QM in 3HS: MZ 2008 and 2009
  - the story to be retold today (KG/WDW and Liouvilleans added)

.

- applications:
  - KG(t) and WDW equations
  - 2 by 2 models (Bíla 2009, etc),
  - solvable models

    MZ 2013, Fring and Moussa 2015 and 2016, Miao et al 2016, Luiz et al 2016, Fring and Frith 2017 and 2018
  - the breakdown of adiabaticity

    Milburn et al (poster in Palermo), Doppler et al 2017, H.-L. Wang et al 2018

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- plus a few would-be theories beyond unitarity:

  Gong and Q.-H. Wang 2010 and 2013, Maamache et al 2015 and 2016, etc

III. notation conventions: Rosetta stone

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	concept	AM 2010	AF & al 2015	FS & al 1992	here
	observable Hamiltonian	H	$\widetilde{H}$	H	H
	Hilbert space metric	η_+	ρ	$\widetilde{T}$	Θ
	Dyson's map, see Eq. (22)	ρ	η	S	Ω
	state vector, Eq. (23)	$\|\psi\rangle$	Ψ	$\|\Psi\rangle$	$\|\psi\rangle$
	the generator of kets, Eq. (24)	–	H	–	G
	textbook Hamiltonian, Eq. (25)	h	h	–	$\mathfrak{h}$
	Coriolis Hamiltonian, Eq. (26)	–	unabbr.	–	Σ
	dual state vector, Eq. (27)	$\|\phi\rangle$	$\rho\Psi$	$\widetilde{T}\|\Psi\rangle$	$\|\psi_\Theta\rangle$

.

IV. the zoo of evolution equations in NIP

(unitarity preserved, the observability of the "Schrödinger Hamiltonian" $G(t)$ lost)

(a) Schrödinger equation for ket vectors

non-stationary ansatz

$$|\psi(t)\succ = \Omega(t)|\psi(t)\rangle \in \mathcal{H}^{(T)}\,, \qquad |\psi(t)\rangle \in \mathcal{H}^{(F)}\,. \tag{22}$$

Schrödinger equation

$$\boxed{\boxed{\mathrm{i}\frac{\partial}{\partial t}\,|\psi(t)\rangle = G(t)\,|\psi(t)\rangle}} \tag{23}$$

(1) with the <u>ket-evolution generator</u> *alias* "the first Schrödinger Hamiltonian"

$$G(t) = H(t) - \Sigma(t) \tag{24}$$

(2) with the <u>energy</u> operator *alias* "the quasi-Hermitian Hamiltonian"

$$H(t) = \Omega^{(-1)}(t)\mathfrak{h}_{(SP)}(t)\Omega(t) \tag{25}$$

(3) with the <u>Coriolis force</u> *alias* "the first Heisenberg Hamiltonian"

$$\Sigma(t) = \mathrm{i}\Omega^{-1}(t)\dot{\Omega}(t)\,, \qquad \dot{\Omega}(t) = \frac{d}{dt}\,\Omega(t)\,. \tag{26}$$

.

(b) Schrödinger equation for bra vectors

from the second non-stationary ansatz

$$|\psi(t)\succ = \left[\Omega^\dagger(t)\right]^{-1} |\psi_\Theta(t)\rangle \in \mathcal{H}^{(T)}\,, \qquad |\psi_\Theta(t)\rangle \,\equiv\, \Theta(t)|\psi(t)\rangle \in \mathcal{H}^{(F)}\,. \qquad (27)$$

one deduces the second non-stationary update of Schrödinger equation

$$\boxed{\boxed{\mathrm{i}\frac{\partial}{\partial t}\,|\psi_\Theta(t)\rangle = G^\dagger(t)\,|\psi_\Theta(t)\rangle}} \qquad (28)$$

which contains

.

(4) the bra-evolution generator a.k.a. "the second Schrödinger Hamiltonian"

$$G^\dagger(t) = H^\dagger(t) - \Sigma^\dagger(t) \qquad (29)$$

(5) containing the adjoint of the energy operator and
(6) the adjoint of the Coriolis force a.k.a. "the second Heisenberg Hamiltonian"

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(c) Liouvilleans

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in dyadic notation the pure states are elementary projectors

$$\pi_{\psi,\Theta}(t) = |\psi(t)\rangle \, \frac{1}{\langle \psi_\Theta(t)|\psi(t)\rangle} \, \langle \psi_\Theta(t)| \tag{30}$$

.

in statistical QM one works with the non-Hermitian density matrix

$$\widehat{\varrho}(t) = \sum_k |\psi^{(k)}(t)\rangle \, \frac{p_k}{\langle \psi^{(k)}_\Theta(t)|\psi^{(k)}(t)\rangle} \, \langle \psi^{(k)}_\Theta(t)| \,, \qquad \sum_k p_k = 1 \tag{31}$$

.

under assumption $p_k \neq p_k(t)$ one gets "the non-Hermitian Liouville evolution equation"

$$\mathrm{i}\,\partial_t \, \widehat{\varrho}(t) = G(t)\widehat{\varrho}(t) - \widehat{\varrho}(t)\, G(t) \,. \tag{32}$$

(d) last but not least: Heisenberg equation

any observable

$$Q^\dagger(t)\Theta(t) = \Theta(t)Q(t) \tag{33}$$

must obey the Heisenberg's evolution law

$$\mathrm{i}\,\frac{\partial}{\partial t}\,Q(t) = Q(t)\Sigma(t) - \Sigma(t)Q(t) + K(t)\,, \qquad K(t) = \Omega^{(-1)}(t)\mathrm{i}\,\dot{\mathfrak{q}}_{(SP)}(t)\Omega(t)\,. \tag{34}$$

practical recommendation

let us have the partial derivatives vanishing, $\dot{\mathfrak{q}}_{(SP)}(t) = 0 = K(t)$.

because

the general case (with $K(t)$ prescribed) is not user friendly.

4. a few notes on physics

I: why and when the 3HS NIP?

(i) the variability of assumptions

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physical framework: the ultimate experimental prediction

$$\boxed{\boxed{\langle\psi_\Theta(t_f)|Q(t_f)|\psi(t_f)\rangle}} \tag{35}$$

mathematical framework: identity

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\Theta(t) = \Theta(t)\Sigma(t) - \Sigma^\dagger(t)\Theta(t)\,. \tag{36}$$

alias

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\Theta(t) = G^\dagger(t)\Theta(t) - \Theta(t)G(t)\,. \tag{37}$$

.

(ii) the special stationary choice of $G \neq G(t)$

.

(a) $G = 0 \Longrightarrow$ non-Hermitian Heisenberg picture

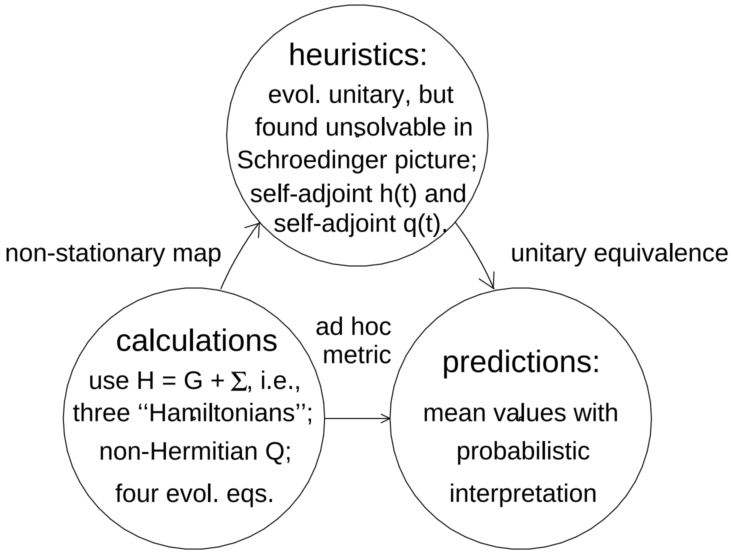
note: the HP metric operator *cannot* be non-stationary

$$\Sigma_{(HP)}(t) = H(t)$$

(b) $G \neq 0 \Longrightarrow$ extended NHH picture

time-independent non-vanishing Gs still user-friendly

(iii) the stationary–non-stationary parallels


heuristics:
evol. unitary, but
found unsolvable in
Schroedinger picture;
self-adjoint h(t) and
self-adjoint q(t).
non-stationary map
unitary equivalence
calculations
use H = G + Σ, i.e.,
three ‘‘Hamiltonians’’;
non-Hermitian Q;
four evol. eqs.
ad hoc
metric
predictions:
mean values with
probabilistic
interpretation

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II. strategy: NIP split $H(t) = G(t) + \Sigma(t)$

(see also the yesterday's talk by Mostafazadeh)

kinematical input: $G(t)$ **and/or** $\Sigma(t)$

the first step of the construction: basis

$t = 0$

.

initial $N-$plets

$$|\psi_1(0)\rangle\,, |\psi_2(0)\rangle\,, \ldots\,, |\psi_N(0)\rangle \tag{38}$$

and

$$|\psi_{1,\Theta}(0)\rangle\,, |\psi_{2,\Theta}(0)\rangle\,, \ldots\,, |\psi_{N,\Theta}(0)\rangle \tag{39}$$

$G(t)$ will generate the "future" $N-$plets

$$|\psi_1(t)\rangle\,, |\psi_2(t)\rangle\,, \ldots\,, |\psi_N(t)\rangle$$

and

$$|\psi_{1,\Theta}(t)\rangle\,, |\psi_{2,\Theta}(t)\rangle\,, \ldots\,, |\psi_{N,\Theta}(t)\rangle$$

theorem 1: the $t > 0$ friendliness

the $t = 0$ bi-orthonormality

$$\langle \psi_{m,\Theta}(0) | \psi_n(0) \rangle = \delta_{m,n} \,, \quad m, n = 1, 2, \ldots, N \tag{40}$$

and the $t = 0$ completeness

$$\sum_{n=1}^{N} |\psi_n(0)\rangle \langle \psi_{n,\Theta}(0)| = I \,. \tag{41}$$

survive in time,

$$\sum_{n=1}^{N} |\psi_n(t)\rangle \langle \psi_{n,\Theta}(t)| = I \,, \qquad \langle \psi_{1,\Theta}(t) | \psi_n(t) \rangle = \delta_{m,n} \,, \quad m, n = 1, 2, \ldots, N \,. \tag{42}$$

the second step: the metric

spectral representation

$$H(t) = \sum_{n=1}^{N} |\psi_n(t)\rangle E_n(t) \langle \psi_{n,\Theta}(t)| \,. \tag{43}$$

"tractable" scenario

$$E_n(t) = E_n(0) = E_n$$

theorem 2

For a given generator $G(t)$ and for the two initial vector sets (38) and (39) with properties (40) and (41), the metric operator $\Theta(t)$ has the following formal representation in $\mathcal{H}^{(F)}$,

$$\Theta(t) = \sum_{n=1}^{N} |\psi_{n,\Theta}(t)\rangle \, \langle \psi_{n,\Theta}(t)| \,. \tag{44}$$

the third step: the Dyson map $\Omega(t)$

.

take, in practice, $N < \infty$, and diagonalize,

$$\Theta(t) = \mathcal{U}^\dagger(t)\theta^2(t)\mathcal{U}(t)\,. \tag{45}$$

up to a new unitary-matrix ambiguity $\mathcal{V}(t)$ we have

$$\Omega(t) = \mathcal{V}^\dagger(t)\theta(t)\mathcal{U}(t)\,; \tag{46}$$

at this moment we construct the Coriolis force $\Sigma(t)$.

.

ultimately, we have $H(t) = \widetilde{H}(t) = G(t) + \Sigma(t)$;

.

the construction proves consistent and complete.

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note added: constant-ket limit: non-Hermitian Heisenberg picture (MZ 2015)

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III. a few final notes on applications

dynamical input: $H(t)$ **etc**

on merits

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the initial choice of $\Theta(0)$ enables us to generate the set of initial bras (39) from a pre-selected set (38) of initial kets.

.

vice versa, at $t_i = 0$ as well as at $t_i > 0$ the choice of the biorthonormal basis determines the unique metric $\Theta(t_i)$ as its byproduct.

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in principle, the operator evolution-equation might define $\Theta(t)$ but its solution would be far from economical. The solution of the two Schrödinger equations will decisively be a better strategy.

on the use in relativistic quantum mechanics

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sample: non-stationary KG scenario:

$m^2 = m^2(\vec{x}, t)$ would be highly desirable

.

non-stationary KG

$$\left(\frac{\partial^2}{\partial t^2} + D(t)\right)\,\psi^{(KG)}(\vec{x},t) = 0\,, \qquad D(t) = -\triangle + m^2(\vec{x},t)\,. \tag{47}$$

and non-stationary

$$\langle \vec{x}|\psi^{(NIP)}(t)\rangle = \left(\begin{array}{c} \mathrm{i}\partial_t\psi^{(KG)}(\vec{x},t) \\ \psi^{(KG)}(\vec{x},t) \end{array}\right) \tag{48}$$

imply

$$\mathrm{i}\frac{\partial}{\partial t}\,|\psi^{(NIP)}(t)\rangle = \left(\begin{array}{cc} 0 & D(t) \\ I & 0 \end{array}\right)\,|\psi^{(NIP)}(t)\rangle \tag{49}$$

$$\mathrm{i}\frac{\partial}{\partial t}\,|\psi^{(NIP)}_\Theta(t)\rangle = \left(\begin{array}{cc} 0 & I \\ D^*(t) & 0 \end{array}\right)\,|\psi^{(NIP)}_\Theta(t)\rangle\,, \qquad m^2(\vec{x},t) \in \mathbb{C}\,. \tag{50}$$

.

the second step of the recipe: solve

$$\mathrm{i}\frac{\partial}{\partial t}\,\Omega^{(NIP)}(t)\rangle = \Omega^{(NIP)}(t)\,\Sigma^{(NIP)}(t)\,, \tag{51}$$

.

the third step of the recipe: solve the Heisenberg equations

$$\mathrm{i}\,\frac{\partial}{\partial t}\,H^{(NIP)}(t) = H^{(NIP)}(t)\Sigma^{(NIP)}(t) - \Sigma^{(NIP)}(t)H^{(NIP)}(t) + K^{(NIP)}(t) \tag{52}$$

.

in the final step, notice that

$$\mathrm{i}\,\frac{\partial}{\partial t}\,H^{(NIP)}(t) = G^{(NIP)}(t)H^{(NIP)}(t) - H^{(NIP)}(t)G^{(NIP)}(t) + K^{(NIP)}(t) \tag{53}$$

remember and take home

Unitary evolution of a non-relativistic quantum system $\mathcal{S}$ is most often described in Schroedinger picture (SP). The Hilbert space $\mathcal{H}^{(physical)}$ is chosen in advance (frequently, $\mathcal{H}^{(physical)} = L^2(\mathbb{R}^3)$). The generic operators of observables $q_{(SP)}(t)$ are required, due to Stone theorem, self-adjoint in $\mathcal{H}^{(physical)}$. The evolution of wave functions is generated by a dedicated observable $g_{(SP)}(t)$ called Hamiltonian.

According to the "non-Hermitian Schroedinger picture" (NSP) idea initiated by Dyson and made widely popular by Bender et al, a double change of the representation space $\mathcal{H}^{(physical)} \to \mathcal{H}^{(unphysical)}_{(auxiliary)} \to \mathcal{H}^{(physical)}_{(amended)}$ can be motivated by a simplifying upgrade $g_{(SP)} \to G_{(NSP)}$ of the generator in Schroedinger equation.

In 2008 we opposed certain widespread no-go beliefs and we showed that the three-Hilbert-space NSP formalism admits a straightforward generalization. As a non-Hermitian extension of the Dirac's interaction picture (abbreviated as NIP) it offered an innovative formulation of quantum theory characterized, first of all, by the emergence of the non-Hermitian Coriolis forces $\Sigma_{(NIP)}(t)$. This causes that the non-Hermitian and non-stationary generator $G_{(NIP)}(t)$ in Schroedinger equation ceases to be observable.

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thanks for your patient curiosity!