Symmetries of quantum potentials:

From $\mathcal{PT}$- to super-, broken to unbroken

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1. On the effect of symmetries on physical systems

Symmetry ⇒ special spectral patterns

What if there are **two** distinct symmetries?

2. Symmetries of quantum mechanical potentials

The basics of $\mathcal{PT}$ symmetry

The basics of supersymmetric quantum mechanics

How do they match: general considerations

3. Examples from exactly solvable potentials

The basics on potential types: shape-invariant, Natanzon, etc.

Combining unbroken/broken $\mathcal{PT}$ symmetry with unbroken/broken SUSY

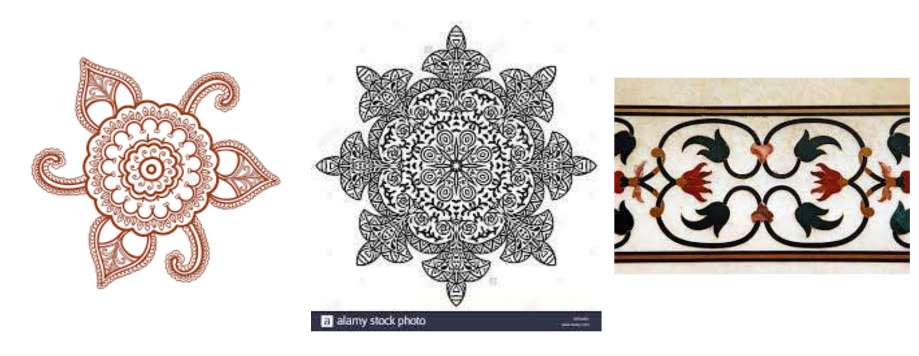
What do we learn from it?

1. On the effect of symmetries on physical systems

A simple example from classical physics

Symmetry No. 1: *Discrete rotations*

C_n: n-fold rotations



Symmetry No. 2: *Translations and rotations in 2D space*

E_2: Euclidean group in 2D

What about their joint effect?

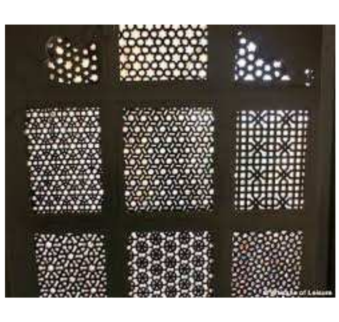
Tile the 2D plane with C_n-symmetric plates

They restrict each other:

Not an n remain allowed

Not all translations and rotations remain allowed

BUT: their combined effect is more spectacular...



(??? But why tile the praeri???)

2. Symmetries of quantum mechanical potentials

1D, $\hbar = 2m = 1$

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + V(x)\Psi(x) = E\Psi(x)$$

$x \in (-\infty, \infty)$

$x \in [-a, a]$, $a < \infty$ — x can be shifted to this domain

$x \in [0, \infty)$ — *Radial* problems; extended to $x \in (-\infty, \infty)$ in $\mathcal{PT}$QM

Possible symmetries:

Supersymmetric quantum mechanics (SUSYQM)

Connects different isospectral potentials

Unbroken or broken, depending on the SUSY transformation

$\mathcal{PT}$-symmetric quantum mechanics

Special type of complex potentials

Unbroken or broken, depending on the potential

Symmetries based on (Lie) algebras

Possible after defining further (differential) operators

We combine here SUSYQM and $\mathcal{PT}$QM

What are the consequences? What if they are broken?

The basics of $\mathcal{PT}$ symmetry

(only features relevant here)

$$\mathcal{PT}: \quad V(x) \equiv V_R(x) + \mathrm{i}V_I(x) \Rightarrow V^*(-x)$$
$$V_R(-x) = V(x), \qquad V_I(-x) = -V(x)$$

Energy eigenvalues:

Real with states $\qquad \mathcal{PT}\psi_n^{(q)}(x) = \psi_n^{(q)}(x)$

and/or

Complex conjugate with states $\qquad \mathcal{PT}\psi_n^{(q)}(x) = \psi_n^{(-q)}(x)$

Real energy eigenvalues can merge to complex conjugate pairs:

Breakdown of $\mathcal{PT}$ symmetry

Merging/re-emerging states carry the quasi-parity $q = \pm 1$ quantum number

Note: The breakdown of $\mathcal{PT}$ symmetry **does not** occur for every potenial

Potentials can be defined off the real x axis Reconsidered boundary conditions

A simple case:

Imaginary coordinate shift: $\delta = -\mathrm{i}c$ It has no effect on the spectrum

Avoids singularities e.g. at $x = 0$: $[0, \infty) \Rightarrow (-\infty, \infty)$

The basics of supersymmetric quantum mechanics

The superalgebra of $N = 2$ SUSYQM

$$\{Q, Q^\dagger\} = \mathcal{H} \qquad\qquad Q^2 = (Q^\dagger)^2 = 0$$

$$[Q, \mathcal{H}] = [Q^\dagger, \mathcal{H}] = 0$$

Realization of the charge operators and the SUSY Hamiltonian in terms of 2×2 matrices:

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \equiv \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

H_- "bosonic" and H_+ "fermionic" components

The SUSY of $\mathcal{H}$ implies isospectrality of H_- and H_+

$$H_-\psi^{(-)} = E^{(-)}\psi^{(-)} \qquad\qquad H_+\psi^{(+)} = E^{(+)}\psi^{(+)}$$

$$H_+(A\psi^{(-)}) = AA^\dagger(A\psi^{(-)}) = AH_-\psi^{(-)} = E^{(-)}A\psi^{(-)}$$

except when $A\psi^{(-)} = 0$ holds

$E = 0$

a b c

Unbroken SUSY Broken SUSY

Take the differential realization

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x), \qquad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x),$$

$$H_\pm \psi^{(\pm)}(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x)\right)\psi^{(\pm)}(x) = E^{(\pm)}\psi^{(\pm)}(x).$$

where

$$V_\pm(x) = W^2(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}W(x).$$

For unbroken SUSY the $W(x)$ superpotential is related to $\psi_0^{(-)}(x)$ as

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0^{(-)}(x)$$

For broken SUSY $\psi_0^{(-)}$ is replaced with other eigenfunctions: $H_-\chi = \epsilon\chi$

$\chi(x)$: unphysical, but nodeless

Case b): unbound at $x \to \pm\infty$ Case c): unbound at $x \to +/-\infty$

ϵ: factorization energy

If $\chi(x) = \psi_0^{(-)}(x)\xi(x)$, then

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\chi(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\psi_0^{(-)}(x) + \ln\xi(x)\right] \equiv W_0(x) + \widetilde{W}(x).$$

Separate the **even(e)**/**odd(o)** AND real(R)/imaginary(I) components:

$$V_{\pm}(x) = V_{\pm R}(x) + \mathrm{i}V_{\pm I}(x)$$

$$W(x) = W_{Re}(x) + W_{Ro}(x) + \mathrm{i}W_{Ie}(x) + \mathrm{i}W_{Io}(x)$$

Then

$$V_{-}(x) = \left(W_{Re}^2 + W_{Ro}^2 - W_{Ie}^2 - W_{Io}^2 - W'_{Ro} + \epsilon_R\right) + (2W_{Re}W_{Ro} - 2W_{Ie}W_{Io} - W'_{Re})$$

$$+\mathrm{i}\,(2W_{Re}W_{Ie} + 2W_{Ro}W_{Io} - W'_{Io} + \epsilon_I) + \mathrm{i}\,(2W_{Re}W_{Io} + 2W_{Ro}W_{Ie} - W'_{Ie})$$

$V_{-}(x)$ is $\mathcal{PT}$-symmetric, if

$$W'_{Re}(x) - W_{Ro}(x)W_{Re}(x) + 2W_{Ie}(x)W_{Io}(x) = 0$$

$$W'_{Io}(x) - 2W_{Ie}(x)W_{Re}(x) - 2W_{Ro}(x)W_{Io}(x) = \epsilon_I$$

Inhomogeneous system of first-order differential equation for $W_{Re}(x)$ and $W_{Io}(x)$

What about $V_+(x)$?

$$V_+(x) = V_-(x) + 2(W'_{Re}(x) + W'_{Ro}(x) + \mathrm{i}W'_{Ie}(x) + \mathrm{i}W'_{Io}(x))$$

$V_+(x)$ is $\mathcal{PT}$-symmetric if $W'_{Re} = 0$ and $W'_{Io} = 0$.

For $W(x) \neq const.$ this means $W_{Re} = 0$, $W_{Io} = 0$, implying $\epsilon_I = 0$

This implies

$$\mathcal{PT} : W(x) = W_{Ro}(x) + \mathrm{i}W_{Re}(x) \quad \Rightarrow \quad -W(x)$$

Therefore,

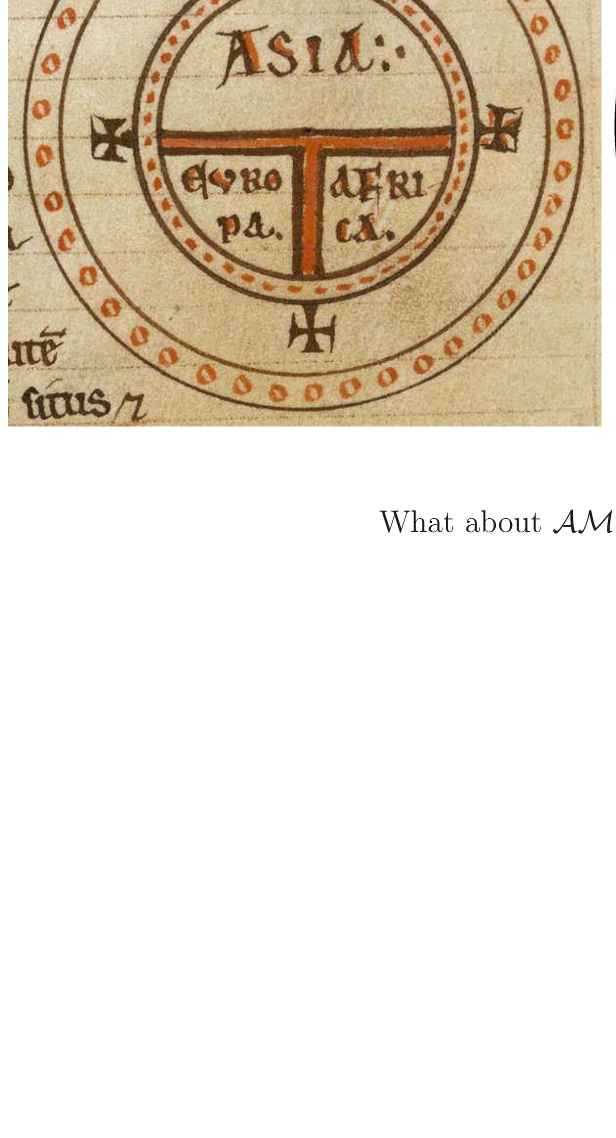
$$\chi(x) \sim \exp\left(-\int^x W(y)\mathrm{d}y\right)$$

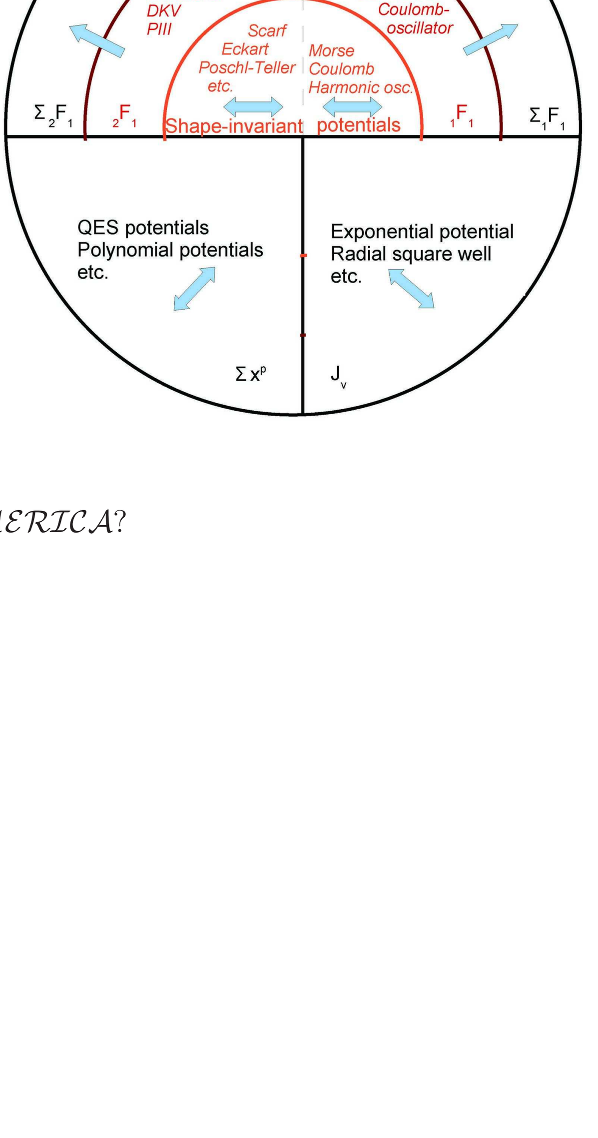
is the eigenfunction of $\mathcal{PT}$: $\mathcal{PT}\chi(x) \sim \chi(x)$

The $\mathcal{PT}$ symmetry of $V_+(x)$ requires real factorization energy ϵ

3. Examples from exactly solvable potentials

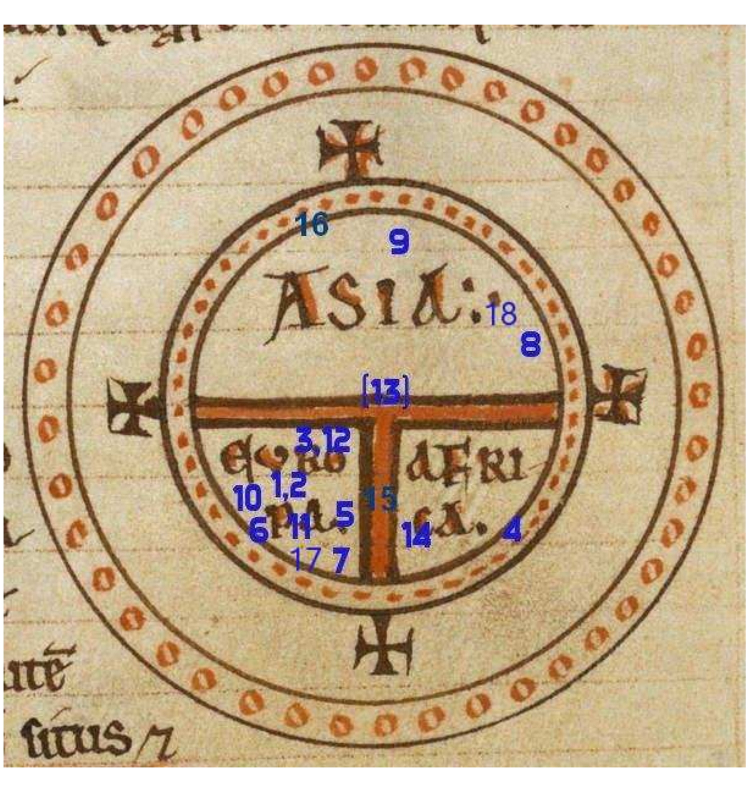
The world map of solvable potentials





What about *AMERICA*?

A word about PHHQP sites



1 Prague
2 Prague
3 Istanbul
4 Stellenbosch
5 Bologna
6 London
7 Benasque
8 Mumbai
9 Hangzhou
10 Dresden
11 Paris
12 Istanbul
13 Jerusalem
14 Sétif
15 Palermo
16 Kyoto
17 Bad Honnef
18 Bengaluru

The main territories in the map

${}_2F_1$: Natanzon class, solved by ${}_2F_1$ in general, by $P_n^{(\alpha,\beta)}(z)$ for bound states

${}_1F_1$: Natanzon confluent class, solved by ${}_1F_1$ in general, by $L_n^{(\alpha)}(z)$ for bound states

Shape-invariant: Natanzon (confluent) subclass, closed under a SUSY transformation

$\Sigma_2 F1$ and $\Sigma_1 F_1$: solutions in terms of the linear combination of several (confluent) hypergeometric functions

Non-SI SUSY partners of Natanzon (confluent) potentials

Potentials solved by exceptional orthogonal polynomials, a new type of SI

$\Longrightarrow$ *AMERICA?*

Solutions containing both independent solutions gen. Woods–Saxon

J_ν: potentials solved by Bessel functions

Σx^p: Quasi-exactly solvable potentials: exact solutions up to a finite n

Outside the world: solvable by semi-analytically solvable potentials, etc.

A reminder on the exact solutions of the Schrödinger equation

An old variable transformation method *Bhattacharjie and Sudarshan 1962*

Schrödinger eq. $\Longrightarrow$ differential equation of special function F

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 \qquad \text{insert} \qquad \psi(x) = \mathbf{f}(x)F(\mathbf{z}(x))$$

and compare with

$$\frac{\mathrm{d}^2F}{\mathrm{d}\mathbf{z}^2} + Q(\mathbf{z})\frac{\mathrm{d}F}{\mathrm{d}\mathbf{z}} + R(\mathbf{z})F(\mathbf{z}) = 0$$

to get

$$E - V(x) = \frac{\mathbf{z}'''(x)}{2\mathbf{z}'(x)} - \frac{3}{4}\left(\frac{\mathbf{z}''(x)}{\mathbf{z}'(x)}\right)^2 + (\mathbf{z}'(x))^2\left(R(\mathbf{z}(x)) - \frac{1}{2}\frac{\mathrm{d}Q(\mathbf{z})}{\mathrm{d}\mathbf{z}} - \frac{1}{4}Q^2(\mathbf{z}(x))\right) .$$

Schwartzian derivatve terms E and the main potential terms

Connection to SUSYQM:

$$W(x) = -\frac{1}{2}Q(z(x))z'(x) + \frac{z''(x)}{2z'(x)}$$

Apply the method to the Jacobi polynomials: $F(z) = P_n^{(\alpha,\beta)}(z)$

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{C}{\phi(z)}\left[s_I(1-z^2(x)) + s_{II} + s_{III}z(x)\right] .$$

$\mathbf{z}(x)$ obtained from

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \phi(z) \equiv \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \frac{p_{\mathrm{I}}(1-z^2)+p_{\mathrm{II}}+p_{\mathrm{III}}z}{(1-z^2)^2} = C .$$

by direct integration

$$\int \phi^{1/2}(\mathbf{z})\mathrm{d}\mathbf{z} = C^{1/2}x + \varepsilon .$$

ε: integration constant, coordinate shift, imaginary in $\mathcal{PT}$QM: $\varepsilon = -\mathrm{i}c$

The solutions are

$$\psi(x) \sim [\phi(z(x))]^{\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}}(1-z(x))^{\frac{\alpha}{2}}P_n^{(\alpha,\beta)}(z(x)) .$$

$$(n+\tfrac{1}{2}+\omega)^2 - \tfrac{1}{4} + s_I - p_I\frac{E_n}{C} = 0$$

$$(1-\omega^2-\rho^2) + s_{II} - p_{II}\frac{E_n}{C} = 0$$

$$-2\omega\rho + s_{III} - p_{III}\frac{E_n}{C} = 0$$

$\omega = (\alpha+\beta)/2$ and $\rho = (\alpha-\beta)/2$

Solving the problem: chose $p_i \Longrightarrow$ get $z(x) \Longrightarrow$ get $V(x) \Longrightarrow$ express E_n

The $\mathcal{PT}$-symmetrization of $z(x)$

...is related to the $\mathcal{PT}$-symmetrization of $z'(x) = C^{1/2}(1-z^2)[\phi(x)]^{-1/2}$...

... that is to that of $\phi(x) = p_{\rm I}(1-z^2) + p_{\rm II} + p_{\rm III} z(1-z^2)^2$...

...that depends on $p_{\rm I}$, $p_{\rm II}$ and $p_{\rm III}$

$$\phi(z(x)) \text{ is } \mathcal{PT}\text{−even if} \begin{cases} z(x) \text{ is } \mathcal{PT}\text{−even,} & \text{i.e. } p_i \text{ are real} \\ z(x) \text{ is } \mathcal{PT}\text{−odd,} & \text{i.e. } p_I \text{ and } p_{II} \text{ are real, } p_{III} \text{ imaginary} \end{cases}$$

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{C}{\phi(z)}\left[s_I(1-z^2(x)) + s_{II} + s_{III} z(x)\right]$$

$$V(x) \text{ is } \mathcal{PT}\text{−even if} \begin{cases} z(x) \text{ is } \mathcal{PT}\text{−even,} & \text{i.e. } s_i \text{ are real} \\ z(x) \text{ is } \mathcal{PT}\text{−odd,} & \text{i.e. } s_I \text{ and } s_{II} \text{ are real, } s_{III} \text{ imaginary} \end{cases}$$

Note: The Schwartzian derivative is $\mathcal{PT}$-even if $z(x)$ has definite $\mathcal{PT}$-parity

The $\mathcal{PT}$-transformation properties and normalizability of

$$\psi(x) \sim [\phi(z(x))]^{\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}}(1-z(x))^{\frac{\alpha}{2}} P_n^{(\alpha,\beta)}(z(x)) \ .$$

depend on $z(x)$, $\phi(z)$, α and β

The list of (real) shape-invariant potentials ($a = 1$, $C = \pm 1$)

$(z')^2 =$ (Class)	$V(x)$	$x \in$	Name
$C(1-z^2)$ (PI)	$(B^2 - A^2 - A)\mathrm{sech}^2(x) + B(2A+1)\mathrm{sech}(x)\tanh(x)$	$(-\infty, \infty)$	Scarf II
	$(B^2 + A^2 + A)\mathrm{cosech}^2(x) - B(2A+1)\mathrm{cosech}(x)\coth(x)$	$[0, \infty)$	gen. Pöschl–Teller
	$(B^2 + A^2 - A)\mathrm{cosec}^2(x) - B(2A-1)\mathrm{cosec}(x)\cot(x)$	$[0, \pi]$	Scarf I
	$A(A-1)\sec^2(x) + B(B-1)\mathrm{cosec}^2(x)$	$[0, \pi/2]$	(Pöschl–Teller I)
	$-A(A+1)\mathrm{sech}^2(x) + B(B-1)\mathrm{cosech}^2(x)$	$[0, \infty)$	(Pöschl–Teller II)
$C(1-z^2)^2$ (PII)	$-A(A+1)\mathrm{sech}^2(x) + 2B\tanh(x)$	$(-\infty, \infty)$	Rosen–Morse II
	$A(A-1)\mathrm{cosech}^2(x) - 2B\coth(x)$	$[0, \infty)$	Eckart
	$A(A+1)\mathrm{cosec}^2(x) - 2B\cot(x)$	$[0, \pi]$	Rosen–Morse I
Cz (LI)	$\frac{1}{4}\omega^2 x^2 + \frac{l(l+1)}{x^2} - (l + \frac{3}{2})\omega$	$[0, \infty)$	3d harmonic oscillator
C (LII)	$\frac{e^4}{4(l+1)^2} - \frac{e^2}{x} + \frac{l(l+1)}{x^2}$	$[0, \infty)$	Coulomb
Cz^2 (LIII)	$A^2 - B(2A+1)\exp(-x) + B^2\exp(-2x)$	$(-\infty, \infty)$	Morse
C (HI)	$-\frac{1}{2}\omega + \frac{1}{4}\omega^2 x^2$	$(-\infty, \infty)$	1d harmonic oscillator

Obtained by selecting certain single terms on the right handside of $E - V(x) = ...$

The Scarf II potential $p_I = 1,\ p_{II} = p_{III} = 0,\ s_I = 1/4,\ s_{II},\ s_{III}$

$$V(x) = -\frac{1}{\cosh^2 x}\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{\mathrm{i}\sinh x}{\cosh^2 x} 2\left(\frac{\beta+\alpha}{2}\right)\left(\frac{\beta-\alpha}{2}\right)$$

s_{II} s_{III}

$z(x) = \mathrm{i}\sinh(x)$ $\mathcal{PT}$**-even**

$\mathcal{PT}$ symmetry: $\Longrightarrow$ α, β are real **or** imaginary
$\alpha \leftrightarrow \beta$: $\Longrightarrow$ $V(x) \leftrightarrow V(-x)$
$V(x)$ invariant under $\alpha \leftrightarrow -\alpha$ $\Longrightarrow$ $q\alpha \equiv \pm\alpha$ quasi-parity

$$\psi_n^{(q)}(x) = C_n^{(q)}(1 - \mathrm{i}\sinh(x+\mathrm{i}\epsilon))^{\frac{q\alpha}{2}+\frac{1}{4}}(1 + \mathrm{i}\sinh(x+\mathrm{i}\epsilon))^{\frac{\beta}{2}+\frac{1}{4}} P_n^{(q\alpha,\beta)}(\mathrm{i}\sinh(x+\mathrm{i}\epsilon))$$

Normalizable if $n^{(q)} < -[\mathrm{Re}(q\alpha+\beta)+1]/2$

$$E_n^{(q)} = -\left(n + \frac{q\alpha+\beta+1}{2}\right)^2$$

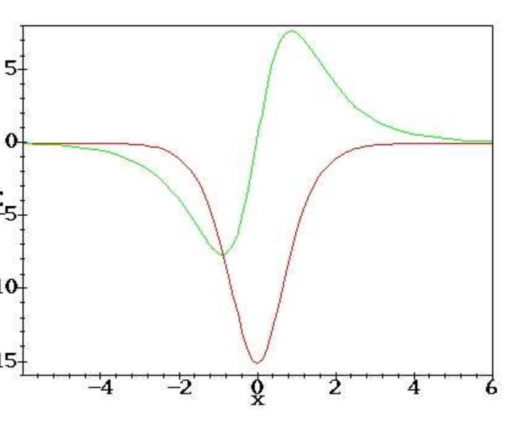
Complex conjugate pairs if α is imaginary Breakdown of $\mathcal{PT}$ symmetry

"Sudden" mechanism: all the $E_n^{(q)}$ turn complex at the same time

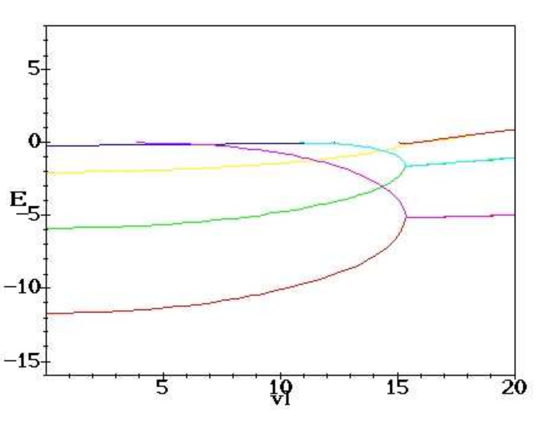
Note: spectral singularity for $\beta = -2n - 1$

The transition to complex energy eigenvalues

Reparametrize to $V(x) = -v_r V_R(x) + \mathrm{i} v_i V_I(x), \quad v_r$ fixed, v_i varied



$v_r = 15.1,\ v_i = 12.7$



$v_i = 0 \rightarrow 20$

At $v_i = 6.35,\ 11.35,\ 14.35.$

Complexification occurs for $|v_i| = v_r + 1/4 = 15.35$ for all n

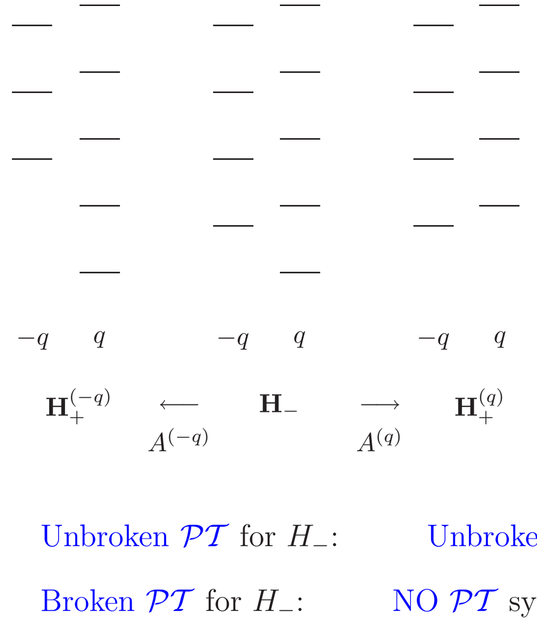
Sudden mechanism of $\mathcal{PT}$ symmetry breaking

Normalizability:

$\psi_0^{(-)}(x)$ for $v_i \geq 3.886$; $\psi_1^{(-)}(x)$, $v_i \geq 10.858$; $\psi_2^{(-)}(x)$, $v_i \geq 15.803$; $\psi_3^{(+)}(x)$, $v_i \leq 12.325$.

Sketch of the spectrum of the potentials for unbroken SUSY transformation

Lévai and Znojil 2002



Unbroken $\mathcal{PT}$ for H_-: Unbroken $\mathcal{PT}$ for $H_+^{(\pm q)}$

Broken $\mathcal{PT}$ for H_-: NO $\mathcal{PT}$ symmetry for $H_+^{(\pm q)}$

A Natanzon-class example: the generalized Ginocchio potential

G. Lévai et al. J. Phys. A **36** *(2003) 7611*

$p_I = -C/(4\gamma^4)$, $p_{II} = -p_{III} = -C(\gamma-1)/(2\gamma^4)$, $s_I = 1/4$, s_{III}, s_{III}

$$\begin{aligned} V(r) &= -\frac{\gamma^4(s(s+1)+1-\gamma^2)}{\gamma^2+\sinh^2 u} + \gamma^4\lambda(\lambda-1)\frac{\coth^2 u}{\gamma^2+\sinh^2 u} \\ &\quad -\frac{3\gamma^4(\gamma^2-1)(3\gamma^2-1)}{4(\gamma^2+\sinh^2 u)^2} + \frac{5\gamma^6(\gamma^2-1)^2}{4(\gamma^2+\sinh^2 u)^3} , \end{aligned}$$

where the Implicit $u(x)$ function is

$$\begin{aligned} r \equiv x - \mathrm{i}c = \frac{1}{\gamma^2} \quad & \Big[\tanh^{-1}\left((\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u\right) \\ & + (\gamma^2-1)^{\frac{1}{2}}\tan^{-1}\left((\gamma^2-1)^{\frac{1}{2}}(\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u\right)\Big] . \end{aligned}$$

$c \neq 0$ to avoid singularity at $x=0$ $\qquad$ $z(x) = \cosh(2u(x))$ is $\mathcal{PT}$**-even**

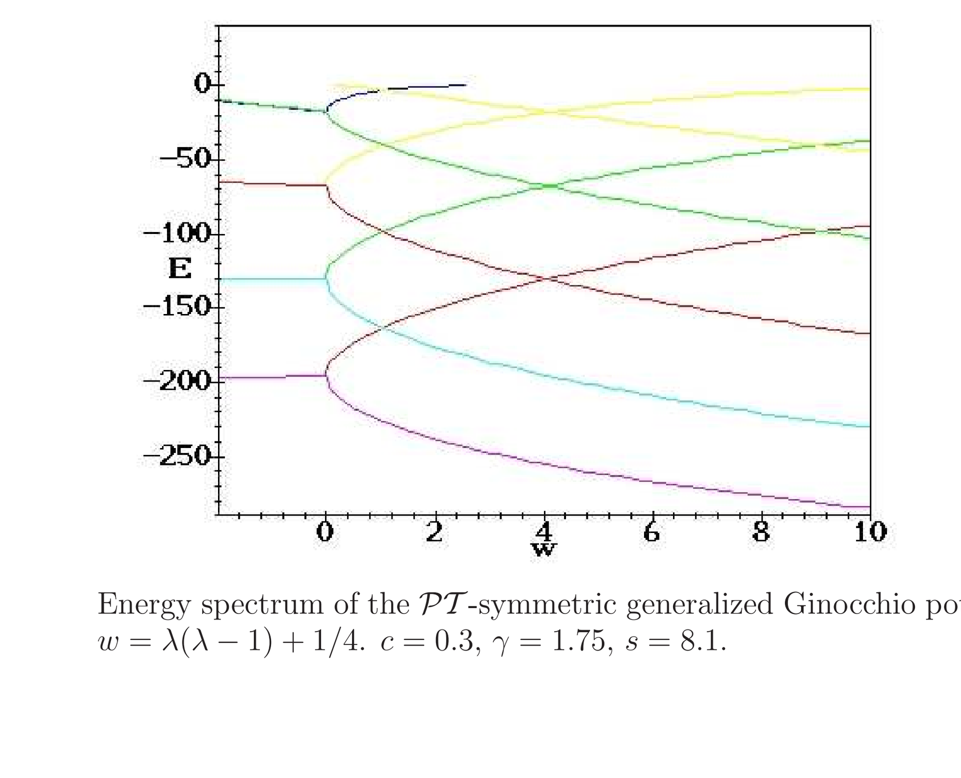
$\gamma = 1$: generalized Pöschl–Teller limit

The bound-state solutions depend on the $q = \pm 1$ quasi-parity

$$\begin{aligned} \psi_n^{(q)}(x) &= N_n^{(q)}(\gamma^2+\sinh^2 u)^{1/4}(\cosh u)^{-2n-1-\mu_{nq}-q(\lambda-\frac{1}{2})}(\sinh u)^{\frac{1}{2}+q(\lambda-\frac{1}{2})} \\ &\quad \times P_n^{(q(\lambda-\frac{1}{2}),-2n-1-\mu_{nq}-q(\lambda-\frac{1}{2}))}(\cosh(2u)) . \end{aligned}$$

Complexification of the energy eigenvalues $E_n^{(q)} = -\gamma^4 \mu_{nq}^2$

$$\mu_{nq} = \frac{1}{\gamma^2}\left[-\left(2n+1+q(\lambda-\frac{1}{2})\right) + \left[\gamma^2(s+\frac{1}{2})^2 + (1-\gamma^2)\left(2n+1+q(\lambda-\frac{1}{2})\right)^2\right]^{1/2}\right]$$

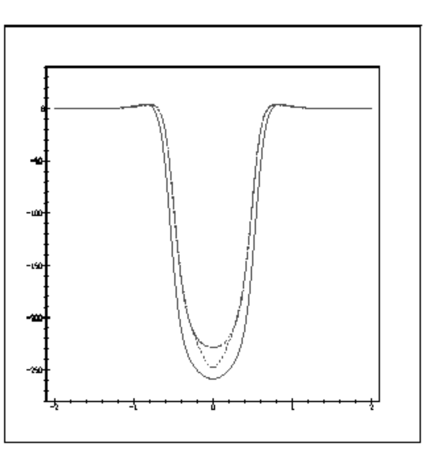


Energy spectrum of the $\mathcal{PT}$-symmetric generalized Ginocchio potential as the function of $w = \lambda(\lambda-1)+1/4$. $c = 0.3$, $\gamma = 1.75$, $s = 8.1$.

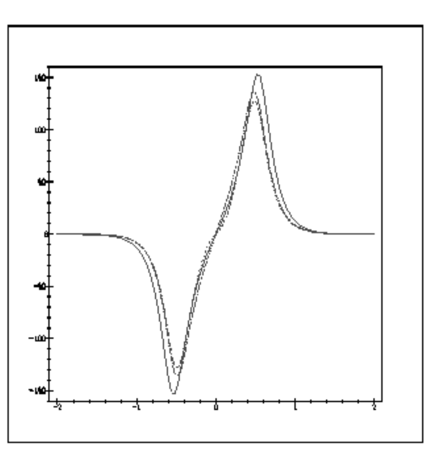
Unbroken SUSY, unbroken$\mathcal{PT}$ symmetry: real μ_{qn}, s, λ

$c = 0.3$, $\gamma = 1.75$, $s = 8.1$, $\lambda = 1.25$

$V_-(x)$: solid line, $V_+^{(q=+)}(x)$: dashed line, $V_+^{(q=-)}(x)$: dotted line



V_R



V_I

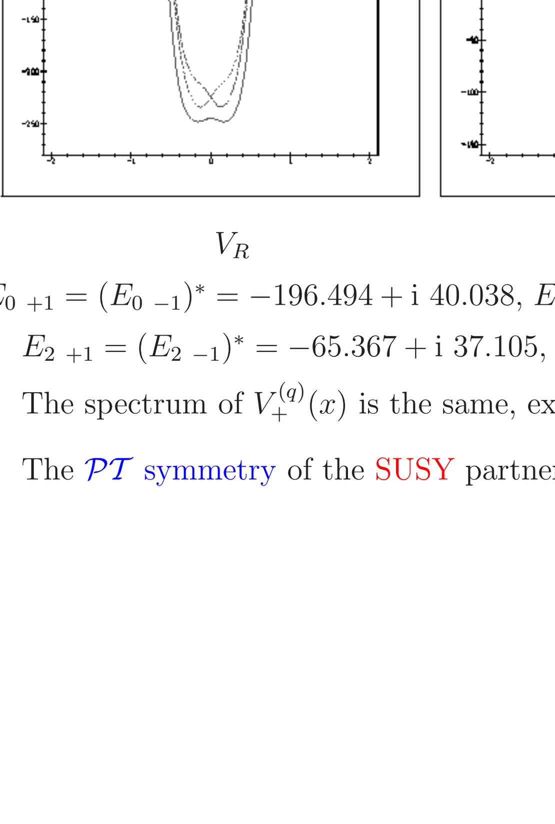
$E_0^{(-)} = -218.913, E_1^{(-)} = -154.978$, $E_2^{(-)} = -90.379$, $E_3^{(-)} = -33.993$, $E_4^{(-)} = -1.061$.
$E_0^{(+)} = -171.313$, $E_1^{(+)} = -106.160$, $E_2^{(+)} = -46.679$, $E_3^{(+)} = -5.666$;

The spectrum of $V_+^{(q)}(x)$ is the same, except for the missing $E_0^{(q)}$ level

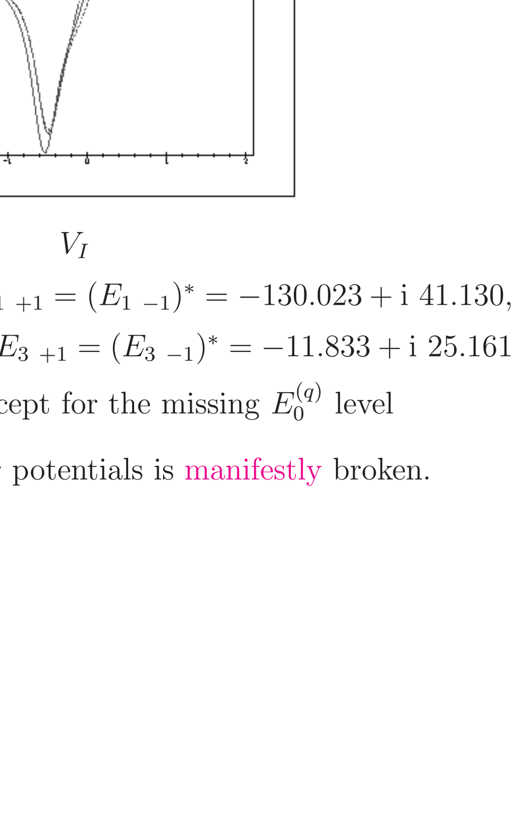
Unbroken SUSY, broken$\mathcal{PT}$ symmetry: complex μ_{qn}, $\lambda = \frac{1}{2} - \mathrm{i}\lambda$

$c = 0.3$, $\gamma = 1.75$, $s = 8.1$, $\lambda = 0.5 + 1.25\mathrm{i}$

$V_{-}(x)$: solid line, $V_{+}^{(q=+)}(x)$: dashed line, $V_{+}^{(q=-)}(x)$: dotted line



V_R



V_I

$E_{0\ +1} = (E_{0\ -1})^* = -196.494 + \mathrm{i}\ 40.038$, $E_{1\ +1} = (E_{1\ -1})^* = -130.023 + \mathrm{i}\ 41.130$,
$E_{2\ +1} = (E_{2\ -1})^* = -65.367 + \mathrm{i}\ 37.105$, $E_{3\ +1} = (E_{3\ -1})^* = -11.833 + \mathrm{i}\ 25.161$.

The spectrum of $V_{+}^{(q)}(x)$ is the same, except for the missing $E_0^{(q)}$ level

The $\mathcal{PT}$ symmetry of the SUSY partner potentials is manifestly broken.

Beyond the Natanzon-class: the rationally extended Scarf II potential

Solved by the X_1 type exceptional Jacobi polynomials *Gomez-Ullate et al. 2010*

$$\hat{P}_{n+1}^{(\alpha,\beta)}(z) = \left[-\frac{z}{2} + \frac{\beta+\alpha}{\beta-\alpha}\left(\frac{1}{2} + \frac{1}{\alpha+\beta+2n}\right)\right] P_n^{(\alpha,\beta)}(z) - \frac{P_{n-1}^{(\alpha,\beta)}(z)}{\alpha+\beta+2n}.$$

They satisfy a differential equation similar to ordinary Jacobi polynomials

They form an orthogonal basis, but start with degree $\nu = n+1 > 0$

Linear combinations of two Jacobi polynomials

The potential: *Quesne 2008, Bagchi et al. 2009*

$$\begin{aligned} V(x) \quad = \quad & -\frac{(\beta-\alpha)^2 + (\beta+\alpha)^2 + 1}{4\cosh^2 x} + \frac{\mathrm{i}(\beta+\alpha)(\beta-\alpha)\sinh x}{2\cosh^2 x} \\ & -\frac{2(\beta+\alpha)}{\beta+\alpha-\mathrm{i}(\beta-\alpha)\sinh x} + \frac{2[(\beta+\alpha)^2 - (\beta-\alpha)^2]}{(\beta+\alpha-\mathrm{i}(\beta-\alpha)\sinh x)^2}, \end{aligned}$$

Bound-state eigenfunctions:

$$\psi_n = N_n[1-\mathrm{i}\sinh(x)]^{\frac{\alpha}{2}-\frac{1}{4}}[1+\mathrm{i}\sinh(x)]^{\frac{\beta}{2}-\frac{1}{4}}[\alpha+\beta-\mathrm{i}(\beta-\alpha)\sinh x]^{-1}\hat{P}_{n+1}^{(\alpha,\beta)}[\mathrm{i}\sinh(x)]$$

Energy eigenvalues:

$$E_n = -\left[n + \frac{1}{2}(\alpha+\beta-1)\right]^2$$

This potential is related to the Scarf II potential by SUSYQM

And it can be $\mathcal{PT}$-symmetrized *Bagchi and Quesne 2010*

also, Mandal, Yadav, Kumari

Take the Scarf II potential and a SUSY transformation with

$$\chi(x) = [1 - \mathrm{i}\sinh(x)]^{-(\beta+\alpha-1)/4}[1 + \mathrm{i}\sinh(x)]^{-(\beta-\alpha)/4}(c + \mathrm{i}\sinh x).$$

and factorization energy

$$\epsilon = -\frac{1}{4}(\beta + \alpha - 3)^2$$

$V_+(x)$ becomes the rationally extended Scarf II potential for

$$c = -(\beta - \alpha)/(\beta + \alpha - 2)$$

The transformation introduces a new ground state at $E = \varepsilon$

It can be $\mathcal{PT}$-symmetrized as usual: α, β real

With $\alpha \Rightarrow q\alpha$ two SUSY partners are generated

In fact, $V_+(x)$ can be non-singular only in the $\mathcal{PT}$-symmetric setting:

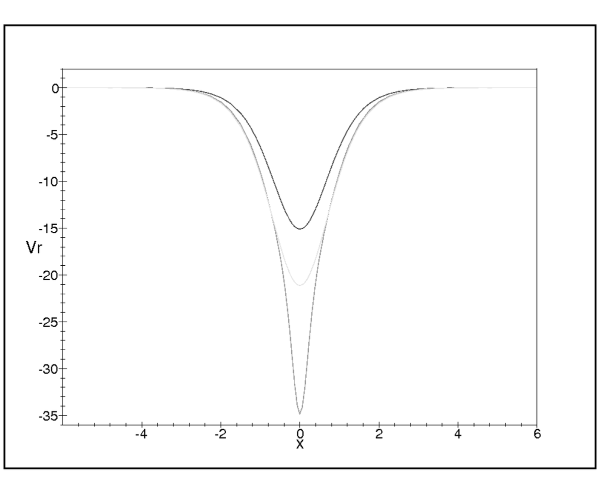
$\chi(x)$ cannot be nodeless in the real case

The rationally extended Scarf II potential from broken SUSY and unbroken$\mathcal{PT}$ symmetry:

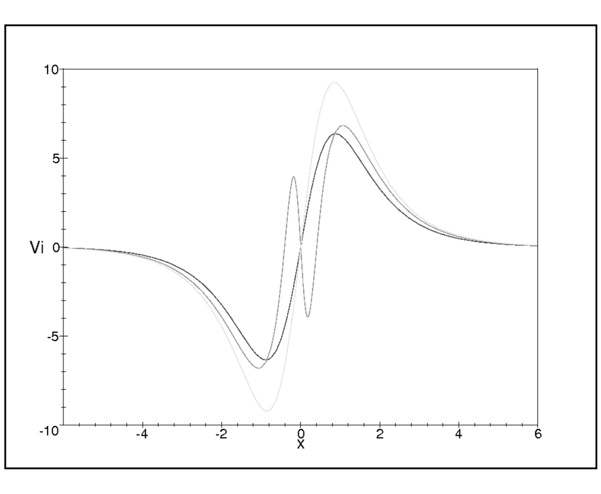
The $\mathcal{PT}$-symmetric Scarf II potential and its partners with $\alpha = -1.628$, $\beta = -5.296$

$V_+^{(+)}(x)$: New ground state at $E = -24.621$

$V_+^{(-)}(x)$: New ground state at $E = -11.116$



V_R



V_I

The case of broken$\mathcal{PT}$ symmetry also works

Then the spectral singularity of the Scarf II potential survives the SUSY transformation

The $\mathcal{PT}$-symmetric harmonic oscillator *Znojil 1999*

Extended to $x \in (-\infty, \infty)$ by an imaginary coordinate shift

$$V(x) = \frac{1}{4}\omega^2(x - \mathrm{i}c)^2 + (\alpha^2 - \frac{1}{4})(x - \mathrm{i}c)^{-2}.$$

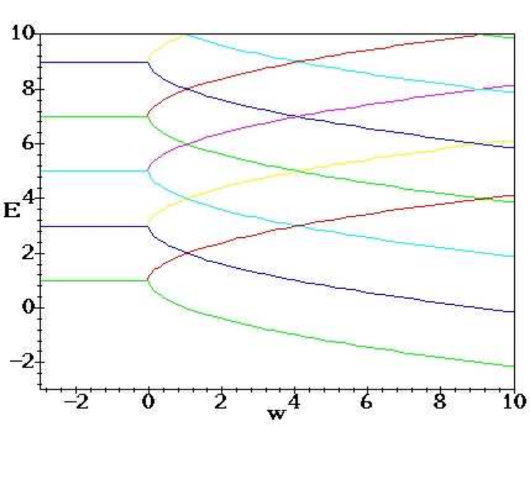
$\mathcal{PT}$-symmetric for real ω and α^2

Two sets of eigenstates discriminated by quasi-parity q

$$\psi_n^{(q)}(x) = N_n^{(q)} \exp\left[-\frac{1}{4}\omega(x - \mathrm{i}c)^2\right](x - \mathrm{i}c)^{q\alpha - 1/2} L_n^{(q\alpha)}\left[\frac{1}{2}(x - \mathrm{i}c)^2\right]$$

$$E_n^{(q)} = \omega(2n + 1 + q\alpha).$$

Unbroken/broken $\mathcal{PT}$ symmetry:
real/imaginary α $(w = \alpha^2)$



Unbroken *SUSY*: ground state $\psi_n^{(q)}$ eliminated

Transformation with broken SUSY adding a new ground state

Works for both unbroken and broken $\mathcal{PT}$ symmetry *Sinha, Lévai, Roy 2004*

With superpotential

$$W^{(q)}(x) = \frac{\omega}{2}(x-\mathrm{i}c) + \frac{q\alpha+1/2}{x-\mathrm{i}c} + \frac{2\omega(x-\mathrm{i}c)}{2q\alpha+2+\omega(x-\mathrm{i}c)^2}$$

$$v_+(x) \equiv V_+^{(q)}(x) - q\alpha\omega = \frac{\omega^2}{4}(x-\mathrm{i}c)^2 + \frac{\alpha^2-1/4}{(x-\mathrm{i}c)^2} + 3\omega$$

$$\begin{aligned} v_-^{(q)}(x) &\equiv V_-^{(q)}(x) - q\alpha\omega = \frac{\omega^2}{4}(x-\mathrm{i}c)^2 + \frac{(q\alpha+1/2)(q\alpha+3/2)}{(x-\mathrm{i}c)^2} + 2\omega \\ &+ \frac{4\omega}{2q\alpha+2+\omega(x-\mathrm{i}c)^2} - \frac{8\omega(2q\alpha+2)}{(2q\alpha+2+\omega(x-\mathrm{i}c)^2)^2}. \end{aligned}$$

Note that technically, the inverse transformation was used, with the $\mathcal{PT}$HO as target

But, the transformation corresponds to inserting new ground state to the $\mathcal{PT}$HO

Energy eigenvalues:

$$E_{n,+}^{(q)} = \omega(2n + 4 + q\alpha) \quad \text{and} \quad E_{n,-}^{(q)} = \omega(2n + 2 + q\alpha)$$

Bound-state wavefunctions, $n > 0$:

$$\begin{aligned} \psi_{n,-}^{(q)}(x) &= N_{n,-}^{(q)} e^{-\omega(x-\mathrm{i}c)^2/4}(x-\mathrm{i}c)^{q\alpha-\frac{1}{2}}\Big\{ -2(n+1)L_{n+1}^{(q\alpha)}[\frac{1}{2}\omega(x-\mathrm{i}c)^2] \\ &+ \left[2n + 2q\alpha + 4 - \frac{4(q\alpha+1)}{2q\alpha + 2 + \omega(x-\mathrm{i}c)^2}\right] L_n^{(q\alpha)}[frac12\omega(x-\mathrm{i}c)^2]\Big\}, \end{aligned}$$

The new ground state:

$$\begin{aligned} \psi_{0,-}^{(q)}(x) &= N_{0,-}^{(q)} \exp\left[-\int W^{(q)}(x)\mathrm{d}x\right] \\ &= N_{0,-}^{(q)} e^{-\omega(x-\mathrm{i}c)^2/4}(x-\mathrm{i}c)^{-q\alpha-\frac{1}{2}}[2q\alpha + 2 + \omega(x-\mathrm{i}c)^2]^{-1}. \end{aligned}$$

Similar to generating the rationally extended harmonic oscillator *Quesne 2008*

The wave functions there contain X_1 type exceptional Laguerre polynomials

(Unfortunately, these were not known in 2004)

Summary of the SUSY transformations of $\mathcal{PT}$-symmetric potentials

V_- Name	V_- Type	$\mathcal{PT}$	SUSY	V_+ Name	V_+ Type	$\mathcal{PT}$
Scarf II	SI (PI)	U	U	Scarf II	SI (PI)	U
		B	U		SI (PI)	NO
		U	B	RE Scarf II	Non-Nat. SI	U
		B	B		SI	NO
Gen. Ginocchio	Natanzon	U	U	"GG-SUSY"	Non-Nat.	U
		B	U			NO
HO	SI (LI)	U	U	HO	SI (LI)	U
		B	U		SI (LI)	NO
		U	B	RE HO	Non-Nat. SI	U
		B	B			NO

Summary and outlook

- The combination of $\mathcal{PT}$ symmetry and $SUSY$ leads to characteristic results
- The breakdown of $\mathcal{PT}$ symmetry in V_- destroys the $\mathcal{PT}$ symmetry of V_+
  $\Rightarrow$ There can be no broken $\mathcal{PT}$ symmetry in rationally extended potentials
- The isospectrality induced by $SUSYQM$ restricts $\mathcal{PT}$
  The $\mathcal{PT}$ symmetry of V_- restricts the $SUSY$ transformations
  Yet the results are inspiring
- $SUSYQM$ determines scattering too
  The spectrum can also be determined from $T(k)$ and $R(k)$
- Many more potentials could be considered
- $SUSYQM$ has many more realizations (matrix sizes, differential operators) that could be combined with $\mathcal{PT}$ symmetry
- Connection with algebras: e.g. potential algebra

The combination of symmetries is always fascinating