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Non Hermiticity and Universality

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complexity and non-hermiticity

Random non-hermitian operators govern the dynamics of many real world systems

Examples

- transport phenomenon through disorder media
- dissipative quantum systems (Feinberg and Zee, PRE, (1999))
- chaotic quantum scattering (Fyodorov and Sommers, JMP, (1997))
- neural network dynamics (Sommers et al, PRL, (1988))
- statistics of flux lines in dirty superconductors (Hatano and Nelson, (1996))
- classical diffusion in random media (Chalker and Wang, PRL, (1997))
- biological growth problems (Nelson and Shnerb)

effect of complexity

Physical properties fluctuate due to complexity (even in a single sample). Knowledge of their average behaviour is no longer sufficient. One needs to know the distribution

- Search of a common mathematical formulation for physical properties of a wide range of non-hermitian systems,
- Single parametric formulation if possible....

a possible tool

- Study of the distributions of the eigenfunctions and eigenvalues of the generator of the system (quantum/classical)
- Common mathematical formulation

why system-dependent random matrix ensemble?

- Lack of detailed knowledge about interactions manifests itself by randomization (partial / full) of various operators of the system.
- Determination of some/ all matrix elements $\Rightarrow$ within an uncertainty $\Rightarrow$ matrix elements are at best described by a probability distribution.
- Uncertainty associated with various matrix element can be of different types $\Rightarrow$ matrix elements can have different distributions.
- Generator $\Rightarrow$ a random matrix (**some/ all matrix elements randomly distributed**)
- Generator of complex system $\Rightarrow$ represented by an ensemble of matrices, each of them equally probable representative of system.
- Choice of a suitable random matrix model of a complex system is sensitive to nature of the complexity.
- Analysis of a wide range of random matrices required.

what is ensemble density?

random matrix H: some matrix elements randomly distributed

- distribution of one random matrix element: $\rho_{kl}(H_{kl})$
- distribution of non-random matrix element: $\delta(H_{ij} - t_{ij})$
- ensemble density: joint prob. density of all matrix elements

$$\rho(H) \propto \rho_{11}(H_{11})\, \rho_{12}(H_{12}) \ldots \rho_{kl}(H_{kl}) \ldots \rho_{NN}(H_{NN})$$

- distribution of eigenvalues $\lambda_1, \ldots \lambda_n$:

$$P(e_1, \ldots, e_N) = \int \prod_{k=1}^{N} \delta(e_k - \lambda_k)\, \rho(H)\, \mathrm{D}H$$

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How constraints affect nature of ensemble?

How to determine matrix element distribution?

- Maximization of information entropy under existing constraints $\Rightarrow$ distribution $\rho(H)$ of matrix elements of H
- Nature of the ensemble depends on global and local constraints.

How constraints affect nature of ensemble?

How to determine matrix element distribution?

- Maximization of information entropy under existing constraints
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- Nature of the ensemble depends on global and local constraints.

- **Global constraints** (matrix constraints)
  which affect nature of each matrix, their transformation rules and relation among the elements e.g. conservation laws, symmetries etc.
  Hamiltonian with time-reversal symmetry is a real-symmetric matrix in a symmetry-preserving basis. May or may not have disorder or chaos.

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- Nature of the ensemble depends on global and local constraints.

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- **Local constraints** (ensemble constraints)
  which affect distribution parameters of the matrix elements: disorder, dimensionality, boundary conditions, nature of dynamics (chaotic or non-chaotic), interactions among subunits of system etc.

Examples

Examples: zero trace, elements with zero mean and different variances

$$H = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

eigenvalues $E_{1,2} = \sqrt{a^2 + b^2}$

ensemble density $$\rho(H) = \rho(a,b) = \frac{1}{2\pi\alpha\beta} \exp\left[-\frac{a^2}{2\alpha^2} - \frac{b^2}{2\beta^2}\right]$$

ev-spacing distribution $$P(S) = \int \delta\left(S - 2\sqrt{a^2+b^2}\right) \rho(a,b)\, da\, db$$

under conditions $$\int P(S)\, dS = 1 \qquad \int S\, P(S)\, dS = 1$$

$$P(S) = 2h^2 S\, e^{-(1+\Lambda^2)h^2 S^2/4\Lambda}\, I_0\left(\frac{h^2(1-\Lambda^2)}{4\Lambda} S^2\right)$$

$$h(\Lambda) = \sqrt{\frac{2}{\pi\Lambda}}\, \mathrm{E}(1-\Lambda^2)$$

single parameter $$\Lambda = \frac{\alpha}{\beta}$$

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Examples: Hermitian with zero trace, different mean and same variances

$$H = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

eigenvalues $E_{1,2} = \sqrt{a^2 + b^2}$

ensemble density

$$\rho(H) = \rho(a,b) = \frac{1}{2\pi\alpha^2} \exp\left[-\frac{(a-a_0)^2}{2\alpha^2} - \frac{(b-b_0)^2}{2\alpha^2} \right]$$

ev-spacing distribution

$$P(S) = \int \delta\left(S - 2\sqrt{a^2 + b^2}\right) \rho(a,b)\, da\, db$$

normalization conditions

$$\int P(S)\, dS = 1 \qquad \int S\, P(S)\, dS = 1$$

$$P(S) = h(\Lambda) e^{-\Lambda/2} S\, e^{-h(\Lambda) S^2/2} I_0\left(\sqrt{\Lambda\, h(\Lambda)}\, S\right)$$

$$h(\Lambda) = \frac{\pi}{8}\, e^{-\Lambda/2} \left[(2+\Lambda) I_0\left(\frac{\Lambda}{4}\right) + \Lambda I_1\left(\frac{\Lambda}{4}\right) \right]$$

Single parameter

$$\Lambda = \frac{a_0^2 + b_0^2}{\alpha^2}$$

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Examples: Non-Hermitian, zero trace, zero mean and different variances

$$H = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

eigenvalues $E_{1,2} = \sqrt{a^2 + bc}$

ensemble density
$$\rho(H) = \rho(a,b,c) = \frac{\sqrt{\pi}^3}{\alpha\beta\gamma\sqrt{2}} \exp\left[-\frac{a^2}{2\alpha^2} - \frac{b^2}{2\beta^2} - \frac{c^2}{2\gamma^2} \right]$$

E-spacing distribution
$$P(S) = \int \delta\left(S - 2\sqrt{a^2 + bc} \right) \rho(a,b,c)\, da\, db\, dc$$

normalization conditions
$$\int P(S)\, dS = 1 \qquad \int S\, P(S)\, dS = 1$$

$$P(S) = \frac{\pi S}{4} f(\Lambda) \int dx \left(\sqrt{1+4ix}\, \sqrt{1+\Lambda^2 x^2} \right)^{-1} e^{ix S^2 f^2 / 2}$$

Single parameter $\Lambda = \dfrac{\beta\gamma}{\alpha^2}$

multi-parametric Gaussian ensemble

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an ensemble of $N \times N$ non-Hermitian matrices H defined by a Gaussian measure

$$\tilde{\rho} \propto \exp\Big[-\sum_{\mathbf{s=1}}^{\beta}\sum_{\mathbf{k,l}}(\mathbf{y_{kl;s}H^2_{kl;s}} + \mathbf{x_{kl;s}H_{kl;s}H_{lk;s}})\Big] = \mathbf{C}\rho(\mathbf{H},\mathbf{y},\mathbf{x}) \qquad (1)$$

y, x : sets of the variances and covariances of matrix elements.

examples

$$y_{kl} = \frac{N\gamma}{(1-\tau^2)}, \qquad x_{kl;s} = x_{lk;s} = \frac{(-1)^{s-1}\gamma\tau N}{(1-\tau^2)}$$

- $\tau = 1$: Gaussian ensemble of complex Hermitian matrices (GUE),
- $\tau = 0$: Gaussian ensemble of Complex matrices (Ginibre or GBE)
- $\tau = -1$: Gaussian ensemble of complex anti-symmetric matrices (GASE),

Distribution of the eigenvalues of

$\tilde{\rho} \propto \exp[-\sum_{s=1}^{\beta}\sum_{k,l}(y_{kl;s}H^2_{kl;s} + x_{kl;s}H_{kl;s}H_{lk;s})] = C\rho(H,y,x)$

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- eignvalues of H: $\lambda_1, \ldots, \lambda_N$
- eigenvector matrices of H: U, V
- JPDF of the eigenvalues of $\tilde{\rho}(H)$

$$P(z_1 \ldots, z_N) = C \int \prod_{i=1}^{N} \delta(z_i - \lambda_i)\, \delta(z_i^* - \lambda_i^*)\, \rho(H,\ x,\ y)\, \mathrm{d}H$$

consider an evolution of ρ with parameters y, x

$$S \equiv \sum_{s=1}^{\beta}\sum_{k,l}\left[A_{kl;s}\frac{\partial\rho}{\partial y_{lk;s}} + B_{kl;s}\frac{\partial\rho}{\partial x_{kl;s}}\right]$$

$$A_{kl;s} = y_{kl;s}[\gamma + 2(-1)^s x_{lk;s}]$$
$$B_{kl;s} = [\gamma x_{kl;s} + (-1)^s x_{kl;s}x_{lk;s} + (-1)^s y_{kl;s}y_{lk;s}]$$

what parametric derivatives lead to?

$$\rho = \exp[-\sum_{s=1}^{\beta} \sum_{k,l} (y_{kl;s} H_{kl;s}^2 + x_{kl;s} H_{kl;s} H_{lk;s})]$$

$$\frac{\partial \rho}{\partial y_{kl;s}} = -H_{kl;s}^2 \ \rho$$

$$\frac{\partial \rho}{\partial x_{kl;s}} = -H_{kl;s} \ H_{lk;s} \ \rho$$

$$\frac{\partial \rho}{\partial H_{kl;s}} = -2 \ y_{kl;s} \ H_{kl;s} \ \rho - x_{kl;s} \ H_{lk;s} \ \rho$$

$$H_{kl;s} \frac{\partial \rho}{\partial H_{kl;s}} = 2 \ y_{kl;s} \ \frac{\partial \rho}{\partial y_{kl;s}} + x_{kl;s} \ \frac{\partial \rho}{\partial x_{kl;s}}$$

what parametric derivatives lead to?

$$\rho = \exp[-\sum_{s=1}^{\beta} \sum_{k,l} (y_{kl;s} H_{kl;s}^2 + x_{kl;s} H_{kl;s} H_{lk;s})]$$

$$\frac{\partial \rho}{\partial y_{kl;s}} = -H_{kl;s}^2 \ \rho$$

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$$\sum_{k,l;s} \left[A_{kl;s} \frac{\partial \rho}{\partial y_{lk;s}} + B_{kl;s} \frac{\partial \rho}{\partial x_{kl;s}} \right] + C_1 \rho = \sum_{k,l;s}^{N} \frac{\partial}{\partial H_{kl;s}} \left[\frac{\partial}{\partial H_{lk;s}} + \gamma \ H_{kl;s} \right] \rho$$

Single parametric formulation

Define $\rho_1 = C_2 \, \rho$ with $C_2 = \exp\left[\int C_1 \, \mathrm{d}Y\right]$,

$$\sum_{k,l;s} \left[A_{kl;s} \frac{\partial \rho}{\partial y_{lk;s}} + B_{kl;s} \frac{\partial \rho}{\partial x_{kl;s}} \right] + C_1 \rho = \sum_{k,l;s}^{N} \frac{\partial}{\partial H_{kl;s}} \left[\frac{\partial}{\partial H_{lk;s}} + \gamma \, H_{kl;s} \right] \rho$$

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Single parametric formulation

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Q: can LHS be written as a single parameteric derivative?

$$\frac{\partial \rho_1}{\partial Y} = \sum_{k,l;s}^{N} \frac{\partial}{\partial H_{kl;s}} \left[\frac{\partial}{\partial H_{lk;s}} + \gamma \; H_{kl;s}\right] \rho_1$$

Single parametric formulation

Define $\rho_1 = C_2\ \rho$ with $C_2 = \exp\left[\int C_1\ \mathrm{d}Y\right]$,

$$\sum_{k,l;s}\left[A_{kl;s}\frac{\partial\rho}{\partial y_{lk;s}} + B_{kl;s}\frac{\partial\rho}{\partial x_{kl;s}}\right] + C_1\rho = \sum_{k,l;s}^{N}\frac{\partial}{\partial H_{kl;s}}\left[\frac{\partial}{\partial H_{lk;s}} + \gamma\ H_{kl;s}\right]\rho$$

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$$\frac{\partial\rho_1}{\partial Y} = \sum_{k,l;s}^{N}\frac{\partial}{\partial H_{kl;s}}\left[\frac{\partial}{\partial H_{lk;s}} + \gamma\ H_{kl;s}\right]\rho_1$$

Clearly Y should satisfy the condition that

$$\sum_{s=1}^{\beta}\sum_{k,l}\left[A_{kl;s}\frac{\partial\rho}{\partial y_{lk;s}} + B_{kl;s}\frac{\partial\rho}{\partial x_{kl;s}}\right] = \frac{\partial\rho}{\partial Y}.$$

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Define $\rho_1 = C_2 \ \rho$ with $C_2 = \exp\left[\int C_1 \ \mathrm{d}Y\right]$,

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Q: can LHS be written as a single parameteric derivative?

$$\frac{\partial \rho_1}{\partial Y} \quad = \quad \sum_{k,l;s}^{N} \frac{\partial}{\partial H_{kl;s}} \left[\frac{\partial}{\partial H_{lk;s}} + \gamma \ H_{kl;s} \right] \rho_1$$

Clearly Y should satisfy the condition that

$$\sum_{s=1}^{\beta} \sum_{k,l} \left[A_{kl;s} \frac{\partial \rho}{\partial y_{lk;s}} + B_{kl;s} \frac{\partial \rho}{\partial x_{kl;s}} \right] = \frac{\partial \rho}{\partial Y}.$$

equivalently, $\sum_{s=1}^{\beta} \sum_{k,l} \left[A_{kl;s} \frac{\partial Y}{\partial y_{lk;s}} + B_{kl;s} \frac{\partial Y}{\partial x_{kl;s}} \right] = 1.$

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$$\text{condition} \quad \sum_{s=1}^{\beta} \sum_{k,l} \left[A_{kl;s} \frac{\partial Y}{\partial y_{lk;s}} + B_{kl;s} \frac{\partial Y}{\partial x_{kl;s}} \right] = 1 \qquad \longrightarrow$$

$$\longrightarrow \qquad \frac{\mathrm{d}y_{11;1}}{A_{11;1}} = \frac{\mathrm{d}x_{11;1}}{B_{11;1}} = \ldots = \frac{\mathrm{d}y_{kl;s}}{A_{kl;s}} = \frac{\mathrm{d}x_{kl;s}}{B_{kl;s}} = \frac{\mathrm{d}Y}{1}$$

.

$$Y = \frac{1}{\beta N^2} \sum_{k,l;s} (-1)^s \int \frac{\mathrm{d}y_{kl;s}}{y_{kl;s}\sqrt{\gamma^2 + 4y_{kl;s}(c_{kl;s}y_{kl;s} + (-1)^s \tilde{c}_{kl;s})}}$$

- Y_0 given by the initial conditions.
- $c_{kl;s}$ and $\tilde{c}_{kl;s}$ given by relations $y_{lk;s} = c_{kl;s} y_{kl;s}$ and $x_{kl;s}^2 + (-1)^s \gamma x_{kl;s} - c_{kl;s} y_{kl;s}^2 - (-1)^s \tilde{c}_{kl;s} y_{kl;s} = 0.$

a general parametric transformation

In general, a transformation of the M independent variables $\{t_{kl}\}$ leads to another set $\{y_1, y_2, .., y_M\}$ of independent variables;

$$t_{kl} = t_{kl}(y, y_2, .., y_M)$$

$$\Rightarrow \quad \rho(\{t_{kl}\}) \longrightarrow \rho(y_1, y_2, ..y_M)$$

.

$$\text{desired} \quad \sum_{k \le l} \frac{\partial \rho}{\partial t_{kl}} = \frac{\partial \rho}{\partial y_1} - -- \longrightarrow \sum_{k \le l} \frac{\partial y_n}{\partial t_{kl}} \frac{\partial \rho}{\partial y_n} = \frac{\partial \rho}{\partial y_1}$$

$$\text{fulfilled if} \quad \frac{\partial y_j}{\partial t_{kl}} = \delta_{n1}$$

$$\mathbf{dt_{11} = dt_{12} = ... = dt_{NN} = ... \frac{dy_n}{\delta_{n1}}}$$

$$\mathbf{y_1 = \frac{1}{M} \sum_{k,l} t_{kl}, \qquad y_n = constant}$$

constants of the evolution

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$$\mathbf{y_1 = \frac{1}{M} \sum_{k,l} t_{kl}, \qquad y_n = constant}$$

Question: it is possible to define a transformation such that $y_j \equiv y_j(\{t_{kl}\})$ for $j > 1$ remain constant as ρ evolves due to any changes in system conditions?

the evolution in M dimensional y-space must have $M - 1$ constants which therefore should be given by the initial state

Claim: always possible if the basis is fixed !

As the matrix-element dynamics is studied in a fixed basis, these constants can be chosen in terms of the basis constants and initial conditions (*global constraints*). The statistics during the transition is then governed by $Y \equiv y_1$ only.

Distribution of the eigenvalues of

$\tilde{\rho} \propto \exp[-\sum_{s=1}^{\beta} \sum_{k,l} (y_{kl;s} H_{kl;s}^2 + x_{kl;s} H_{kl;s} H_{lk;s})] = C\rho(H, y, x)$

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- eignvalues of H: $\lambda_1, \ldots, \lambda_N$
- eigenvector matrices of H: U, V
- JPDF of the eigenvalues of $\tilde{\rho}(H)$

$$P_1(z_1 \ldots, z_N) = \int \prod_{i=1}^{N} \delta^2(z_i - \lambda_i)\ \rho_1(H,\ x,\ y)\ \mathrm{d}H$$

$$\frac{\partial P_1}{\partial Y} = \int \prod_{i=1}^{N} \delta^2(z_i - \lambda_i)\ \frac{\partial \rho_1}{\partial Y}\ \mathrm{d}H$$

$$\frac{\partial P_1}{\partial Y} = \sum_{k,l;s}^{N} \int \prod_{i=1}^{N} \delta^2(z_i - \lambda_i) \frac{\partial}{\partial H_{kl;s}} \left[\frac{\partial}{\partial H_{lk;s}} + \gamma\ H_{kl;s} \right] \rho_1\ \mathrm{d}H$$

dynamics of eigenvalues and eigenvectors with varying matrix elements

$$H = U \, \Lambda \, V$$

$$\frac{\partial \lambda_n}{\partial H_{kl;s}} = i^{s-1} U_{nk} V_{ln}$$

$$\sum_{k,l;s} \frac{\partial \lambda_n}{\partial H_{kl;s}} H_{kl;s} = \lambda_n$$

$$\sum_{k,l;s} (-1)^{s-1} \frac{\partial \lambda_n}{\partial H_{kl;s}} \frac{\partial \lambda_m}{\partial H_{lk;s}} = \beta \delta_{mn}$$

$$\sum_{k,l;s} (-1)^{s-1} \frac{\partial^2 \lambda_n}{\partial H_{kl;s} H_{lk;s}} = \sum_m \frac{2\beta}{\lambda_n - \lambda_m}$$

$$\frac{\partial U_{nr}}{\partial H_{kl;s}} = \sum_{m \neq n} \frac{U_{nk}}{\lambda_n - \lambda_m} \frac{\partial \lambda_m}{\partial H_{rl;s}},$$

$$\frac{\partial V_{rn}}{\partial H_{kl;s}} = \sum_{m \neq n} \frac{V_{ln}}{\lambda_n - \lambda_m} \frac{\partial \lambda_m}{\partial H_{kr;s}}$$

Diffusion of eigenvalue distribution

$$\frac{\partial P_1}{\partial Y} = \sum_{r=1}^{2} \sum_{n=1}^{N} \frac{\partial}{\partial z_{nr}} \left[\frac{\partial}{\partial z_{nr}} - \beta \frac{\partial \ln|\Delta_N(z)|}{\partial z_{nr}} + \gamma z_{nr} \right] P_1$$

- steady state $P_s \equiv |Q_N|^2 = |\Delta_N(z)|^2 \mathrm{e}^{-\frac{\gamma}{2}\sum_k |z_k|^2}$
- corresponds to $\frac{\partial P}{\partial Y} \to 0$, $Y \to \infty$ and $\Delta_N = \prod_{m<n} |z_m - z_n|$
- a choice of almost all $y_{kl;s} \to \frac{N\gamma}{(1-\tau^2)}$ and $x_{kl;s} \to \frac{(-1)^{s-1} N\gamma\tau}{(1-\tau^2)}$ with $\gamma = (1-\tau^2)$ fulfills the steady state condition for $\tau \to 0, \pm 1$.
- leads to three different types of steady states:
  Ginibre ($\tau = 0$), GUE ($\tau = 1$), GASE ($\tau = -1$).

Eigenvalues undergo a diffusion process governed by a single parameter Y

Y : a measure of complexity of the system $\to$ complexity parameter

ensemble averaged spcetral density R_1

$$R_1 = N \int P(z_1, \ldots, z_N; Y) \prod_{n=2}^{N} \mathrm{D}^2 z_n$$

.

$$\frac{\partial P_1}{\partial Y} = \sum_{r=1}^{2} \sum_{n=1}^{N} \frac{\partial}{\partial z_{nr}} \left[\frac{\partial}{\partial z_{nr}} - \beta \frac{\partial \ln |\Delta_N(z)|}{\partial z_{nr}} + \gamma z_{nr} \right] P_1$$

evolution equation

Integration of diffusion equation for P, followed by the change $z_1 \to e$ gives[1]

$$\frac{\partial \mathbf{R_1}}{\partial \mathbf{Y}} \approx \sum_{\mathbf{r=1}}^{\mathbf{2}} \frac{\partial}{\partial \mathbf{e_r}} \left(\gamma \mathbf{e_r} - \mathbf{2P} \int \frac{\mathbf{e_r} - \mathbf{e'_r}}{|\mathbf{e} - \mathbf{e'}|^{\mathbf{2}}} \; \mathbf{R_1}(\mathbf{e'}) \; \mathrm{D}^{\mathbf{2}} \mathbf{e'} \right) \mathbf{R_1}$$

---

[1] terms containing 2^{nd} order cluster functions and the diffusion term neglected, both being $O(N)$ (or more) smaller than other terms.

Density Correlations $\mathbf{R}_n(z_1,..,z_n;Y)$

$$\mathbf{R}_n = \frac{N!}{(N-n)!} \int P(z,Y) \prod_{k=n+1}^{N} \mathrm{D}^2 z_k .$$

evolution equation in limit $N \to \infty$

$\mathbf{R}_n$ for $n > 1$ evolve rapidly, should be rescaled before limit $N \to \infty$:

$$R_n(e_1,..,e_n;\Lambda) = \lim_{N\to\infty} \frac{\mathbf{R}_n(z_1,..,z_n;Y)}{\mathbf{R}_1(z_1;Y)...\mathbf{R}_1(z_n;Y)}$$

with $e = \int^e \sqrt{\mathbf{R}_1(z;Y)}\, \mathrm{d}z$

Integration of diffusion equation for P now gives (with $\gamma = 1$) $\longrightarrow$:

$$\frac{\partial R_n}{\partial \Lambda} = \sum_{r,n} \frac{\partial}{\partial e_{jr}} \left[|\Delta_n|^\beta \frac{\partial}{\partial e_{jr}} \frac{R_n}{|\Delta_n|^\beta} - \beta \int_{-\infty}^{\infty} \mathrm{d}e_{n+1} R_{n+1} \frac{\partial \ln|e_j - e_{n+1}|}{\partial e_{jr}} \right]$$

- the transition for R_n occurs on the scales determined by $Y \approx R_1^{-1}$,
- for R_1, the corresponding scale is given by $Y \approx N\, R_1^{-1}$.
- indicates a clear separation of the scales of the global and local behaviour of the density.

spectral complexity parameter

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Joint probability distribution of eigenvalues $P_e(e_1, e_2, .., e_N; Y)$

$$\frac{\partial P_1}{\partial Y} = \sum_{r=1}^{2} \sum_{n=1}^{N} \frac{\partial}{\partial z_{nr}} \left[\frac{\partial}{\partial z_{nr}} - \beta \frac{\partial \ln |\Delta_N(z)|}{\partial z_{nr}} + \gamma z_{nr} \right] P_1 \qquad (2)$$

Eigenvalues undergo a diffusion process governed by the complexity parameter

Y_0: complexity parameter for the initial state.

Spectral statistics depends on system information only through $\Lambda_e = \frac{Y - Y_0}{\Delta_{local}}$**.**

Δ_{local}: local mean level spacing (depends on localization length, dimensionality)

$\Lambda_e = \Lambda_e(N)$: function of size N

$\lim_{N\to\infty} \Lambda_e \to \infty$: system approaches steady state

$\lim_{N\to\infty} \Lambda_e \to 0$: system approaches initial state

$\lim_{N\to\infty} \Lambda_e \neq 0, \infty$: **critical state**

Implictaions of single parameteric diffusion

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Implictaions of single parameteric diffusion

- Distributions of eigen correlations of a complex system described by a multiparametric Gaussian ensemble of non-Hermitian matrices with independent elements is a non-stationary state of a diffusion governed by the complexity parameter.

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- Distributions of eigen correlations of a complex system described by a multiparametric Gaussian ensemble of non-Hermitian matrices with independent elements is a non-stationary state of a diffusion governed by the complexity parameter.

- Systems with different system-conditions (local constraints), will have same eigenvalue statistics if their Λ_e are same (PS, Phys.Rev. Lett., 2001, Phys. Rev. B 2007, 2014). **(valid for same global constraint class)**.

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  critical state of the statistics if Λ remains finite i.e size-independent in lim $N \rightarrow \infty$

connection to Calogero Hamiltonian

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$$\Psi = \frac{P_1}{|Q_N|^{\beta/2}} \quad - - \longrightarrow \quad \frac{\partial \Psi}{\partial Y} = \hat{H} \Psi$$

$$|Q_N|^2 = |\Delta_N(z)|^2 \, \mathrm{e}^{-\frac{\gamma}{2} \sum_k |z_k|^2}$$

'Hamiltonian' $\hat{H}$: a variant of CS Hamiltonian in two dimensions (with $\gamma = 1$):

$$\hat{H} = \sum_i \frac{\partial^2}{\partial r_i^2} - g \sum_{i,j;i<j} \frac{1}{r_{ij}^2} - G \sum_{i,j,k;i<j,i,j\neq k} \frac{\mathbf{r_{ki}}.\mathbf{r_{kj}}}{r_{ki}^2 r_{kj}^2} - \sum_i r_i^2$$

where $\mathbf{r}_i \equiv z_i$, $\mathbf{r}_{ki} \equiv z_k - z_i$ and $r_{ki} \equiv |\mathbf{r}_{ki}|$.

- $P_1(e, Y|e_0, Y_0) = |Q_N|^{\beta/2} \sum_n c_n \phi_n \, \mathrm{e}^{-iYE_n}$
  $P_1(e, Y) = \int P(e, Y|e_0, Y_0) \, P(e_0, Y_0) \, \mathrm{D}e_0$
- The bosonic radial eigenstates and the eigenvalues of $\hat{H}$ for case $g = G$:

$$\phi_n = \prod^N |\mathbf{r_i} - \mathbf{r_j}|^{\Lambda} \mathrm{e}^{-\frac{1}{2} \sum_k |r_k|^2} L_n \qquad E_n = [4n + N(N-1)\sqrt{G/2} + 2N]/2$$

Universal Hamiltonian

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$$\hat{H} = \sum_i \frac{\partial^2}{\partial r_i^2} - g \sum_{i,j;i<j} \frac{1}{r_{ij}^2} - G \sum_{i,j,k;i<j,i,j\neq k} \frac{\mathbf{r_{ki}}.\mathbf{r_{kj}}}{r_{ki}^2 r_{kj}^2} - \sum_i r_i^2$$

- $G = g = 1$ for non-hermitian case with all real eigenvalues ($D > 1$),
- $G = g = 2$ for the complex non-hermitian case ($D > 1$),
- $G = 0, g = 1$ the complex hermitian case ($D = 1$ Calogero-Sutherland H),
- inverse square term does not drop out for complex non-Hermitian case.
- particles in non-hermitian case are bosons (as $g = G$) instead of fermions (as in hermitian case).
- particle positions in classical ground state for a specific g, G corresponds to zeros of Hermite Polynomial (G.Date, PK Ghosh and MVN Murthy, PRL, 1998)
- vicinal surfaces show long distance, universal behavior where microscopic detail like the specific material, lattice structure, surface reconstruction if any do not play a direct role. The Hamiltonian for the interacting steps in continuum limit is Calogero-Sutherland Hamiltonian.
  (S. M, Bhattarcharjee and S. Mukherjee, PRL, 1999).
- correlated N-electron state of a confined electron gas in two dimenson, subject to an external magnetic field of arbitrary strength can be effectively mapped to the classical ground state of the Calogero Hamiltonian.

  (N.F. Johnson and L. Quiroga, PRL, 1995).

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Example: Gaussian orthogonal ensemble (GOE), power law ensemble of real matrices, and, time-reversal Anderson ensemble belong to same "global constraints" universality class although their complexity parameters, in general, are not equal (infinity for GOE and finite in the other two cases) . However for the system parameters leading to same finite value of the complexity parameter, the Anderson ensemble and power law ensemble show same statistics.

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- Energy dependence of Λ varies from system to system. Thus two systems in general may have same statistics for some energy range but need not for entire energy range.

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- **The formulation suggests a deep rooted universality and a web of connection hidden underneath the world of complex systems.**

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Thank you