DENSITY OF EIGENVALUES IN A QUASI-HERMITIAN RANDOM MATRIX MODEL - THE CASE OF INDEFINITE METRIC

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OUTLINE

- *Quasi-Hermitian (QH) vs. Strictly Quasi-Hermitian (SQH) Matrices*
- *Strictly Quasi-Hermitian Matrices - a simple mechanical example*
- *Strictly Quasi-Hermitian Random Matrices*
- *Indefinite Metric - Motivation*
- *QH Random Matrices with a fixed deterministic metric*
- *Interlude: analyzing the spectra of non-hermitian random matrices, Feynman diagrams and the Method of Hermitization*
- *Example: the spectrum for the indefinite metric* $B = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots - 1)$

STRICTLY QUASI-HERMITIAN MATRICES

$\mathbf{M}, \mathbf{H}$ $N \times N$ *matrices*

$\mathbf{M}$ *hermitian* **positive metric**

the complex matrix $\mathbf{H}$

is **strictly quasi-hermitian** *with respect to the metric* $\mathbf{M}$

if it fulfills the **intertwining** *relation*

$$\mathbf{H}^{\dagger}\mathbf{M} = \mathbf{M}\mathbf{H}$$

($\mathbf{M} = \mathbf{1}$ *reduces to ordinary hermiticity)*

The intertwining relation $\mathbf{H}^\dagger\mathbf{M} = \mathbf{M}\mathbf{H}$

simply means that $\mathbf{H}$ *is hermitian with respect to the metric* $\mathbf{M}$

Inner product with respect to metric $\mathbf{M}$

$$\langle \mathbf{u_1} | \mathbf{u_2} \rangle_\mathbf{M} = \langle \mathbf{u_1} | \mathbf{M}\mathbf{u_2} \rangle$$

hermiticity with respect to this inner product

$$\langle \mathbf{u_1} | \mathbf{H}\mathbf{u_2} \rangle_\mathbf{M} = \langle \mathbf{H}\mathbf{u_1} | \mathbf{u_2} \rangle_\mathbf{M}$$

that is

$$\langle \mathbf{u_1} | \mathbf{M}\,\mathbf{H}\mathbf{u_2} \rangle = \langle \mathbf{H}\mathbf{u_1} | \mathbf{M}\mathbf{u_2} \rangle = \langle \mathbf{u_1} | \mathbf{H}^\dagger\,\mathbf{M}\mathbf{u_2} \rangle$$

the **general** *solution of the intertwining relation* $\mathbf{H}^\dagger \mathbf{M} = \mathbf{M}\mathbf{H}$ *is*

$$\mathbf{H} = \mathbf{A}\,\mathbf{M}\,, \quad \mathbf{A}^\dagger = \mathbf{A}$$

(***see e.g., Y.N. Joglekar and W.A. Karr, PRE83(2011)031122***)

Thus, given $\mathbf{M}$*, there are* N^2 *independent (strictly) quasi-hermitian matrices with respect to* $\mathbf{M}$

The spectrum of $\mathbf{H}$ *is real, since it is* **similar** *to the hermitian matrix*

$$\mathbf{h} = \sqrt{\mathbf{M}}\,\mathbf{A}\sqrt{\mathbf{M}}$$

$$\mathbf{h} = \sqrt{\mathbf{M}}\,\mathbf{H}\frac{1}{\sqrt{\mathbf{M}}}$$

Numerical investigation of level spacing statistics in SQH-RM models was carried in

Y.N. Joglekar and W.A. Karr, ***PRE83(2011)031122***

T. Deguchi, P.K. Gosh & K. Kudo, ***PRE80(2009)026213***

I should also mention in this context the recent paper

S. Kumar and A. Ahmed, ***PRE96(2017)022157***

QUASI-HERMITIAN MATRICES

relax positivity of $\mathbf{M}$ *, but keep it invertible*

thus, $\mathbf{M}$ **is allowed to have either positive or negative eigenvalues, but not zeros**

then the intertwining relation $\mathbf{H}^\dagger\mathbf{M} = \mathbf{M}\mathbf{H}$

defines $\mathbf{H}$ *merely as a* **quasi-hermitian** *matrix.*

$$\mathbf{H} = \mathbf{A}\,\mathbf{M}\,, \quad \mathbf{A}^\dagger = \mathbf{A}$$

remains the general solution of the intertwining relation

however, $\sqrt{\mathbf{M}}$ *is no longer hermitian, and therefore*

$$\mathbf{h} = \sqrt{\mathbf{M}}\,\mathbf{A}\sqrt{\mathbf{M}}$$

can *possibly be non-hermitian as well*

Quasi-Hermitian (**QH**) *matrices, as opposed to Strictly-Quasi-Hermitian* (**SQH**) *matrices, may have pairs of complex-conjugate eigenvalues*

the intertwining relation and invertibility of $\mathbf{M}$

imply that the characteristic polynomial $P(z) = \det\left(z - H\right)$

has real coefficients, namely, $P^*(z) = P(z^*)$

A SIMPLE PHYSICAL EXAMPLE OF SQH MATRICES

a mechanical system of s-degrees of freedom executing small oscillations about a stable equilibrium state

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$$

$\mathbf{M}$ mass matrix

$\mathbf{K}$ matrix of spring constants

$\mathbf{M}, \mathbf{K}$ strictly positive hermitian sxs matrices

$$L = \frac{1}{2}\dot{\mathbf{q}}^T \, \mathbf{a}(\mathbf{q}) \, \dot{\mathbf{q}} - U(\mathbf{q})$$

strictly positive definite matrix

$\mathbf{q_0}$ equilibrium point

$$\mathbf{x} = \mathbf{q} - \mathbf{q_0}$$

$$\mathbf{M} = \mathbf{a}(\mathbf{q}_0)$$

$$\mathbf{K}_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j}_{|\mathbf{q_0}}$$

Hessian at minimum

Harmonic eigenmodes

$$\mathbf{x} = \mathbf{A}e^{i\omega t}$$

amplitude vector

$$\left(-\omega^2\mathbf{M} + \mathbf{K}\right)\mathbf{A} = 0$$

eigenmode equation

Eigenfrequencies are roots of the characteristic polynomial

$$P_s\left(\omega^2\right) = \det\left(-\omega^2\mathbf{M} + \mathbf{K}\right)$$

small oscillations about a stable minimum, so all frequencies must be real, and therefore all roots ω^2 *of the polynomial must be positive, which is the case, since in a given mode,*

$$\omega^2 = \frac{\mathbf{A}^\dagger\mathbf{K}\mathbf{A}}{\mathbf{A}^\dagger\mathbf{M}\mathbf{A}}$$

is the ratio of two positive numbers

another way to establish positivity of eigenfrequencies:

rewrite the eigenmode equation as

$$\frac{1}{\sqrt{\mathbf{M}}}\mathbf{K}\frac{1}{\sqrt{\mathbf{M}}}\left(\sqrt{\mathbf{M}}\mathbf{A}\right) = \omega^2\left(\sqrt{\mathbf{M}}\mathbf{A}\right)$$

$\sqrt{\mathbf{M}}$ the all-positive root (among the 2^s roots of $\mathbf{M}$)

it is of course hermitian

thus $\mathbf{h}\tilde{\mathbf{A}} = \omega^2\tilde{\mathbf{A}}$

$$\mathbf{h} = \frac{1}{\sqrt{\mathbf{M}}}\mathbf{K}\frac{1}{\sqrt{\mathbf{M}}} \quad , \quad \tilde{\mathbf{A}} = \sqrt{\mathbf{M}}\mathbf{A}$$

the matrix $\mathbf{h}$ *is hermitian and strictly positive* *yet another way to write the eigenmode equation:* $\frac{1}{\mathbf{M}}\mathbf{K}\,\mathbf{A} = \omega^2\mathbf{A}$

that is, $\mathbf{H}\mathbf{A} = \omega^2\mathbf{A}$ *where*

$\mathbf{H} = \frac{1}{\mathbf{M}}\mathbf{K}$ *and* $\mathbf{H}^\dagger = \mathbf{K}\,\frac{1}{\mathbf{M}}$

clearly, $\mathbf{H}^\dagger \neq \mathbf{H}$ **however,**

$\mathbf{H}^\dagger\mathbf{M} = \mathbf{M}\mathbf{H} \quad (= \mathbf{K})$ **intertwining relation**

or, equivalently, $\mathbf{H}^\dagger = \mathbf{M}\,\mathbf{H}\,\frac{1}{\mathbf{M}}$; $\mathbf{H}$ *and* $\mathbf{H}^\dagger$ *are similar*

$\mathbf{H}$ is an example of a (strictly) ***quasi-hermitian*** matrix, with respect to a metric $\mathbf{M}$

and it is similar to a (positive) hermitian matrix: $\mathbf{H} = \frac{1}{\sqrt{\mathbf{M}}}\,\mathbf{h}\,\sqrt{\mathbf{M}}$

hence its spectrum is real (and positive)

SQH Random Matrix Theory

For very large mechanical systems, with high connectivity (all mechanical d.o.f.‘s are coupled to each other), this problem naturally lends itself to analysis in terms of random matrices.

Draw the matrices M and K from sensible uncorrelated probability distributions of positive definite matrices, and compute the resulting eigenvalue statistics of

$$H = \frac{1}{M} K$$

Free probability theory offers a natural approach to this mathematical problem (S-transforms)

(J. Feinberg - to appear)

Indefinite Metric

Physical and Mathematical Motivation

1. Imagine the matrix M in our mechanical example acquires some negative eigenvalues. In other words, the (real symmetric) metric M becomes indefinite. Some of the eigenvalues ω^2 will be negative, with purely imaginary frequencies.

A more likely physical situation is that some of the mechanical springs become unstable. That is, the Hessian K develops some negative modes.

We can think of K as the metric, writing the spectral problem as $M\mathbf{A} = \frac{1}{\omega^2}K\mathbf{A}$

2. Pesudo-anti-hermiticity of the infinitesimal generators of the classical non-compact groups

indefinite metric $M = \mathrm{diag}(+1, +1, \ldots, +1, -1, -1, \ldots, -1)$

n m

orthogonal groups $O(n,m)$

$g \in O(n,m)$ $g^T M g = M$

infinitesimal generators $g = \mathbf{1} + \epsilon$

intertwining relation $\epsilon^T M + M\epsilon = 0 \quad \rightarrow \quad \epsilon = AM\,, \quad A^T = -A$

unitary groups $U(n,m)$

$g \in U(n,m)$ $g^\dagger M g = M$

nfinitesimal generators $g = \mathbf{1} + \epsilon$

ntertwining relation $\epsilon^\dagger M + M\epsilon = 0 \quad \rightarrow \quad \epsilon = AM \quad A^\dagger = -A$

Quasi/Pseudo-Hermitian RMT: draw a generator at random. What will be the distribution of eigenvalues in the complex plane?

3. Studying the PT-symmetry breaking transition in a quantum system, defined in a finite dimensional Hilbert space. As some parameters in the model vary, the metric develops negative eigenvalues. Similarly - studying the phase of broken PT-symmetry (gain-loss balance) in optics

Quasi-Hermitian Random Matrix Theory: the Case of Fixed Deterministic Metric

(J. Feinberg & R. Reiser - to appear)

$N \times N$ random matrix φ

fixed deterministic metric $B\,, \qquad B^\dagger = B$

intertwining relation $\qquad \varphi^\dagger B = B\varphi$

general solution of this relation: $\varphi = AB\,, \qquad A^\dagger = A$

draw A from a probability ensemble invariant under unitary rotations, e.g., $P(A) = \frac{1}{\mathcal{Z}}\exp(-N\mathrm{tr}V(A))$

For simplicity, it is enough to consider the GUE ensemble

$$P(A) = \frac{1}{\mathcal{Z}}\exp(-\frac{Nm^2}{2}\mathrm{tr}A^2)$$

With no loss of generality, we can take the metric B to be diagonal, since its unitary diagonalizing matrix can be always absorbed into a unitary rotation of A without changing the statistics, due to unitary-rotational invariance of the GUE.

Goal: calculate the average eigenvalue density of the QH random matrix φ, given the fixed metric B (in the large-N limit)

The latter can be obtained from the resolvent of $\varphi = AB$

$$G(w) = \langle \frac{1}{N} \mathrm{Tr} \frac{1}{w - AB} \rangle$$

averaged over the GUE ensemble of A

Difficulties in spectral analysis of non-hermitian matrices & the Method of Hermitization (an interlude)

The matrix $\varphi = AB$ is typically non-hermitian.

Its eigenvalues are complex, and in the large-N limit, they may occupy a two-dimensional domain in the complex plane.

Therefore, the resolvent (a.k.a. Green's function)

$$G(w) = \langle \frac{1}{N} \mathrm{Tr} \frac{1}{w - AB} \rangle$$

is NOT an analytic function of the complex variable w

For example, if the $N \times N$ complex matrix φ is drawn from Ginibre's ensemble

$$P(\varphi) = \frac{1}{\mathcal{Z}} \exp(-N \mathrm{tr} \varphi^{\dagger} \varphi)$$

then it's resolvent is (in the large-N limit)

$$G(w, w^*) = \langle \frac{1}{N} \, \mathrm{Tr} \, \frac{1}{w - \varphi} \rangle = \begin{cases} w^* \, , & |w| \leq 1 \\ \frac{1}{w} \, , & |w| > 1 \end{cases}$$

which is manifestly a non-holomorphic function of w

One can infer from this Green's function that the eigenvalues of φ uniformly occupy the unit disk in the complex w plane.

interpretation in terms of 2d electrostatics

$G(w, w^*)$ is the electrostatic field generated by the "charge" (eigenvalue) density

$$\rho(w, w^*) = \frac{1}{\pi} \frac{\partial}{\partial w^*} G(w, w^*)$$ (Gauss Law)

$G(w, w^*)$ is non-holomorphic precisely where the eigenvalues condense

This non-analyticity of the Green's function poses serious difficulties to analyzing spectra of non-hermitian matrices, because most methods in our toolbox are tailored for analyzing hermitian matrices, whose eigenvalues are restricted to the real axis, rendering their resolvents analytic functions in the complex plane, off the real axis.

The Method of Hermitization

J.F. and A. Zee (1997)

and also, independently:

M. Nowak et al (1997)

J. Chalker and J. Wang (1997)

K. Efetov (1997)

offers a way to overcome these difficulties, by reducing the problem of determining the spectrum of a nonhermitian matrix, to that of determining the spectrum of a hermitian one:

$$H = \begin{pmatrix} 0 & \varphi \\ \varphi^{\dagger} & 0 \end{pmatrix}$$

More precisely, one computes the doubled $2N \times 2N$ resolvent

$$\hat{\mathcal{G}}(\eta, w, w^*) = \langle \frac{1}{\hat{\mathcal{G}}_0^{-1} - H} \rangle$$

where

$$\hat{\mathcal{G}}_0^{-1} = \begin{pmatrix} \eta & w \\ w^* & \eta \end{pmatrix}$$

is the bare inverse propagator

(where each $N \times N$ block is proportional to the unit matrix),

by expanding it in inverse powers of the complex variable η

The resulting series converges to a unique analytic function of η

(in the complex η plane, cut appropriately along the real axis)

At the end of computation we set $\eta = 0$

and thus obtain

$$\hat{\mathcal{G}}(0, w, w^*) = \left\langle \begin{pmatrix} 0 & \frac{1}{(w-\varphi)^\dagger} \\ \frac{1}{w-\varphi} & 0 \end{pmatrix} \right\rangle$$

The desired Green's function

$$G(w, w^*) = \langle \frac{1}{N} \mathrm{Tr} \frac{1}{w - \varphi} \rangle$$

is obtained by tracing the lower left block of $\hat{\mathcal{G}}(0, w, w^*)$

a few words on the diagrammatic expansion

Each term in the expansion of $\hat{\mathcal{G}}(\eta, w, w^*)$in inverse powers of η has a diagrammatic representation in terms of certain Feynman diagrams, in which propagators of matrices are represented by oriented double lines, since a matrix carries two indices (G. 't Hooft 1974)

Each diagram is weighted by an inverse power of N, N^χ, where

$$\begin{aligned}\chi &= \#\text{vertices} - \#\text{edges (propagators)} + \#\text{faces (index loops)} \\ &= 2(1-G) - \#\text{boundaries (holes)}\end{aligned}$$

is the Euler character of the graph

Thus the perturbative diagrammatic expansion is organized according to topology of the diagrams.

Borrowing terminology of particle physics, we may think of the Green's function

$$\hat{G}_{ij} = \langle q_i \bar{q}_j \rangle = \langle \left(\frac{1}{w - \varphi} \right)_{ij} \rangle$$

as the propagator of a "quark" q_i(fundamental representation - a single index) in the color field of "gluons" φ_{ij}

So that $G(w, w^*) = \frac{1}{N} \hat{G}_{ii} = \langle \frac{1}{N} \mathrm{Tr} \frac{1}{w - \varphi} \rangle$

closes an unpaired index loop

Thus, the leading diagrams are planar ($G = 0$) with a single boundary (a quark loop at the perimeter $b = 1$)

so that the leading large-N behavior of $G(w, w^*) = \frac{1}{N} \hat{G}_{ii}$

is $\frac{1}{N} N^{\chi} = N^{-1+2-1} = N^0$

In this work we restrict ourselves to the leading contribution in the $\frac{1}{N}$ expansion. That is, planar diagrams (with a single quark loop at the exterior perimeter).

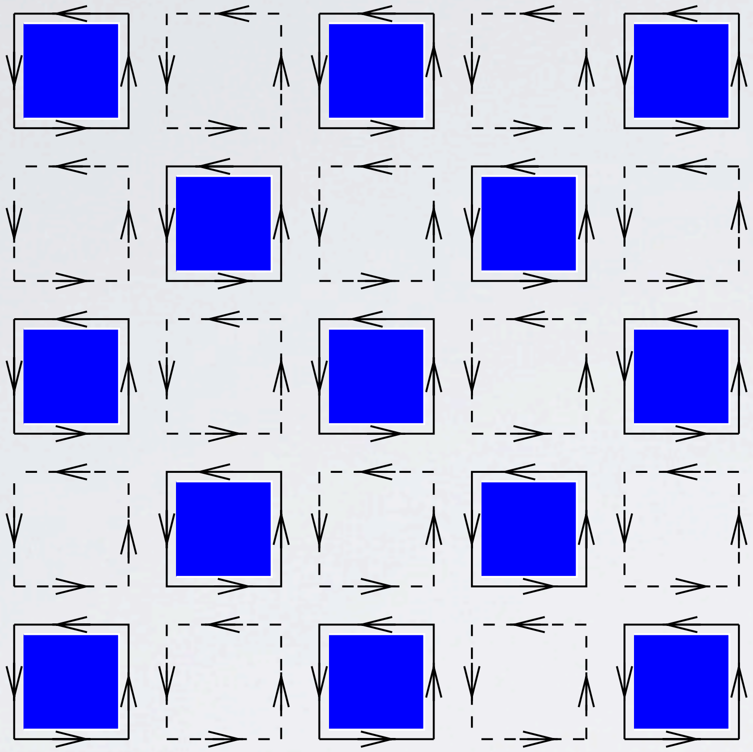


Figure 3. An example of a simply connected checkerboard surface which contains, in addition to quark–gluon vertices, only quartic gluon self-interaction vertices. There is a single quark loop which forms the edge. This diagram has 16 quark–gluon vertices plus 16 gluon quartic vertices ($V = 32$), 25 index loops (faces, $F = 25$), and 40 gluon propagators and 16 quark propagators (edges, $E = 56$). As explained below (see (32)), this diagram scales like N.

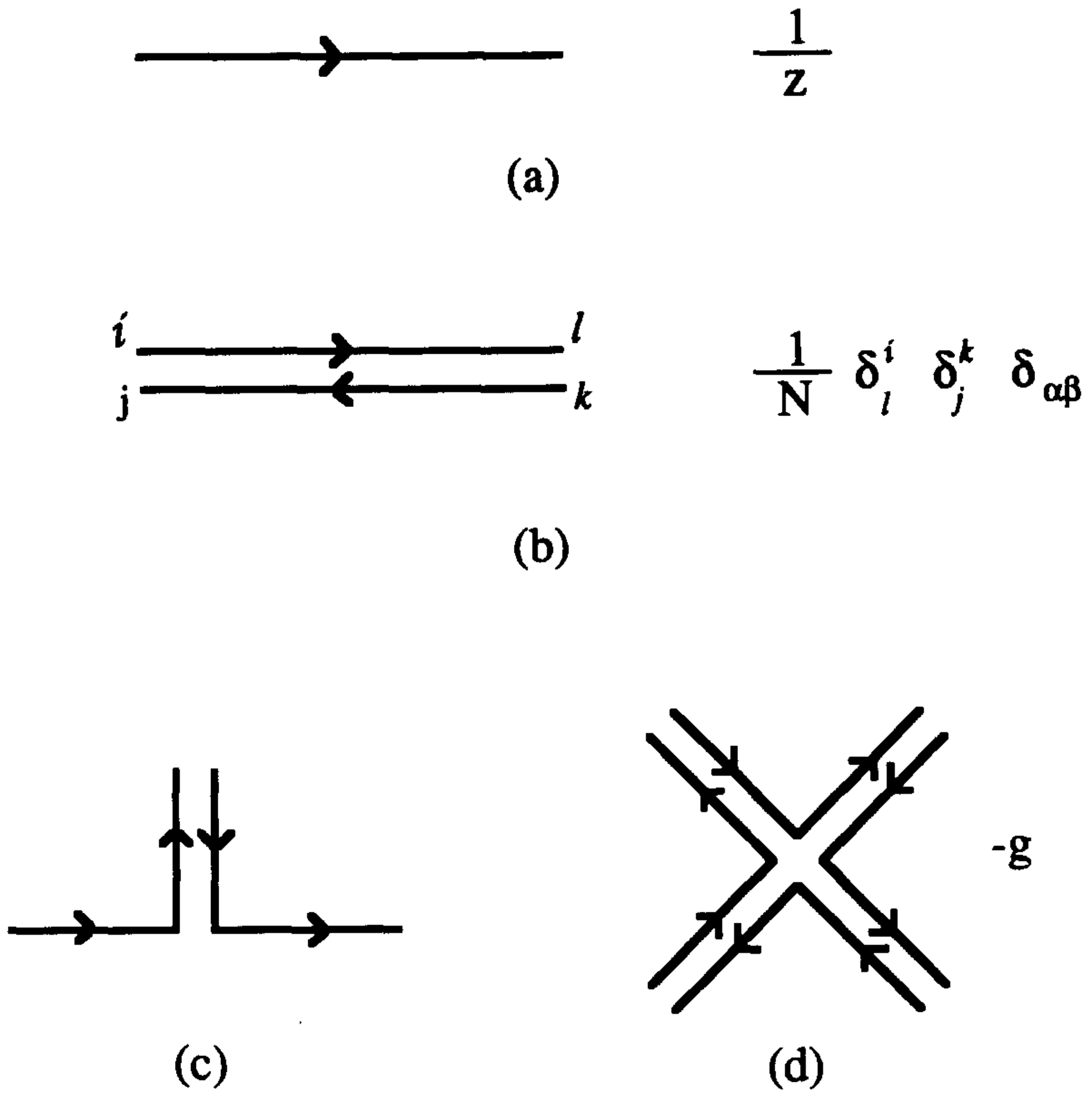


Fig. 1. Feynman rules: (a) quark propagator, (b) gluon propagator, (c) quark gluon vertex, (d) gluon interaction, illustrated here with a $g\varphi^4$ vertex.

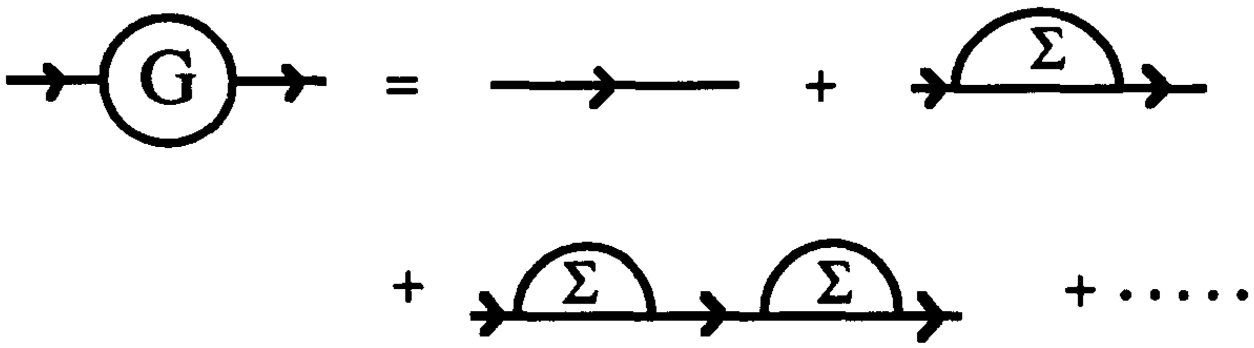


Fig. 2. Quark propagator and one-particle irreducible self-energy.

A. Zee, NPB474(1996)726-744

Hermitization is essential to carrying the diagrammatic expansion of the Green's function!

e.g., in Ginibre's case, $\langle \frac{1}{N} \mathrm{Tr} \varphi^n \rangle = \delta_{n,0}$

due to rotational invariance

Thus, resumming all moments (which is equivalent to carrying out the diagramatic expansion) will result in

$$G(w) = \frac{1}{w}$$

which is just the holographic branch of $G(w)$, outside the condensate of eigenvalues in the unit disk.

Quasi-Hermitian Random Matrix Theory: the Case of Fixed Deterministic Metric (continued)

our aim is to compute $G(w) = \langle \frac{1}{N} \mathrm{Tr} \frac{1}{w - AB} \rangle$

which is quadratic in the matrices.

It is obviously desirable to linearize the problem, i.e., to avoid the product of matrices AB. This will render averaging over A easier.

This can be achieved by the following trick, due to Burda et al.

(Z. Burda, R.A. Janik & M.A. Nowak, Phys. Rev. E84(2011)061125)

consider $$H = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

This trick actually has physical meaning. Recall our earlier mechanical problem. If we assign

$$A = M^{-1} \quad \text{and} \quad B = -K$$

then

$$H = \mathcal{L} = \begin{pmatrix} 0 & M^{-1} \\ -K & 0 \end{pmatrix}$$

is the Liouvillian of the system:

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{pmatrix} = \mathcal{L} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix}$$

it's resolvent is

$$\frac{1}{z-H} = \begin{pmatrix} z & -A \\ -B & z \end{pmatrix}^{-1} = \begin{pmatrix} \frac{z}{z^2-AB} & \frac{1}{z^2-AB}A \\ \frac{1}{z^2-BA}B & \frac{z}{z^2-BA} \end{pmatrix}$$

thus, the desired Green's function is obtained from the upper left block of this resolvent (after dividing by z and substituting $w = z^2$)

Alas, we are not done yet, since

$$\begin{pmatrix} z & -A \\ -B & z \end{pmatrix}$$

is not hermitian! We have to carry out the hermitization procedure.

We end up with this $4N \times 4N$ resolvent

$$\hat{\mathcal{G}} = \left[\begin{pmatrix} \eta & z \\ z^* & \eta \end{pmatrix} - \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix} \right]^{-1}$$

$2N \times 2N$

blocks

$$= \begin{pmatrix} \eta & 0 & z & -A \\ 0 & \eta & -B & z \\ z^* & -B & \eta & 0 \\ -A & z^* & 0 & \eta \end{pmatrix}^{-1}$$

$N \times N$

blocks

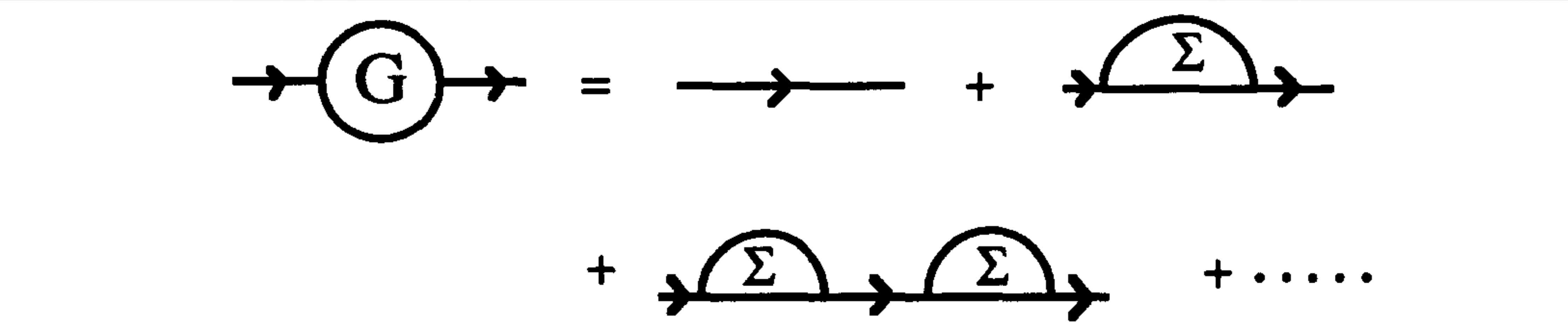
The procedure from now on is to expand the resolvent $\hat{\mathcal{G}}$

in powers of the bare propagator

$$\hat{\mathcal{G}}_0 = \begin{pmatrix} \eta & 0 & z & 0 \\ 0 & \eta & -B & z \\ z^* & -B & \eta & 0 \\ 0 & z^* & 0 & \eta \end{pmatrix}^{-1}$$

which includes the deterministic matrix B (the metric), and average over the random GUE matrix A

we then rearrange the perturbative expansion (the Born series) in terms of the self energy $\hat{\Sigma}$

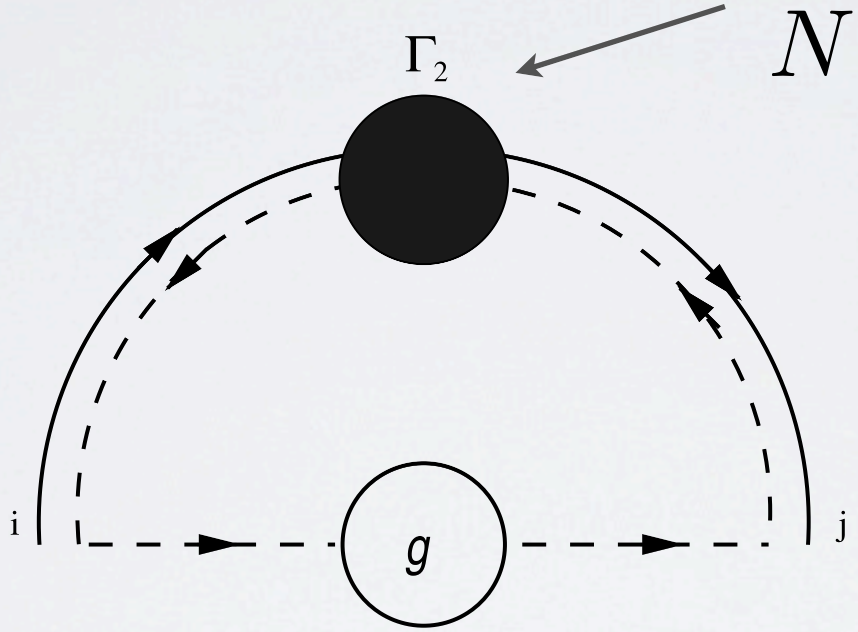


$$\langle \hat{\mathcal{G}} \rangle = \frac{1}{\hat{\mathcal{G}}_0^{-1} - \hat{\Sigma}} = \sum_{n=0}^{\infty} \left(\hat{\mathcal{G}}_0 \hat{\Sigma} \right)^n \hat{\mathcal{G}}_0$$

where $\hat{\Sigma}$ is the sum over all 1-quark irreducible diagrams

and may be obtained in terms of the connected cumulants of the distribution of A and the full propagator $\langle \hat{\mathcal{G}} \rangle$

Since the random matrix A is Gaussian, there is only one connected cumulant, and in the large-N limit (that is, summing only over planar diagrams) we arrive at this expression for the self-energy:



$$\frac{1}{Nm^2}$$

Figure 7. Contribution of the quadratic cumulant Γ_2 to self-energy.

We end up with

$$\hat{\Sigma} = \frac{1}{m^2}\begin{pmatrix} 4\bar{4} & 0 & 0 & 4\bar{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1\bar{4} & 0 & 0 & 1\bar{1} \end{pmatrix}$$

where $\alpha\bar{\beta} = \frac{1}{N}\langle \mathrm{Tr}\, \hat{\mathcal{G}}_{\alpha\beta}\rangle\,, \qquad \alpha, \beta := 1, 2, 3, 4$

The sparse texture of $\hat{\Sigma}$ arises because the random matrix appears only in the 14 and 41 blocks of the $4N \times 4N$ hermitized hamiltonian.

It is now safe to set $\eta = is \to 0$

and determine the blocks $\alpha\bar{\beta} = \frac{1}{N}\langle \mathrm{Tr}\,\hat{\mathcal{G}}_{\alpha\beta}\rangle$

self-consistently from the Schwinger-Dyson equation

$$\langle\hat{\mathcal{G}}\rangle = \frac{1}{\hat{\mathcal{G}}_0^{-1} - \hat{\Sigma}}$$

In this way we obtain four self-consistent “gap equations” which determine the four blocks $1\bar{1}\,,\, 1\bar{4} = 4\bar{1}^*\,,\, 4\bar{4}$ uniquely

All the other blocks of $\langle\hat{\mathcal{G}}\rangle$ are determined in terms of these four quantities. In particular, the object of our interest, namely, the resolvent of AB is given by

$$\langle\frac{1}{N}\,\mathrm{Tr}\,\hat{\mathcal{G}}_{31}\rangle = \langle\frac{1}{N}\,\mathrm{Tr}\,\frac{z}{z^2 - AB}\rangle =$$

$$\frac{m^2 z}{N}\,\mathrm{Tr}\,\frac{B^{-1}\left(1\bar{4} - m^2 w^* B^{-1}\right)}{1\bar{1}\cdot 4\bar{4} - \left(4\bar{1} - m^2 w B^{-1}\right)\left(1\bar{4} - m^2 w^* B^{-1}\right)} \qquad w = z^2$$

$$\langle\frac{1}{N}\,\mathrm{Tr}\,\frac{1}{w-AB}\rangle=\frac{m^2}{N}\,\mathrm{Tr}\,\frac{B^{-1}\left(\bar{1}4-m^2w^*B^{-1}\right)}{\bar{1}1\cdot\bar{4}4-\left(\bar{4}1-m^2wB^{-1}\right)\left(\bar{1}4-m^2w^*B^{-1}\right)}$$

General Properties of the Solution

Either both $\bar{1}1\,,\bar{4}4$ vanish, or both are different from zero

The Holomorphic Solution

If $\bar{1}1=\bar{4}4=0$, then $\bar{4}1$ is a holomorphic function of w

in which case it follows that

$$\langle\frac{1}{N}\,\mathrm{Tr}\,\frac{1}{w-AB}\rangle=-\frac{m^2}{N}\,\mathrm{Tr}\,\frac{B^{-1}}{\bar{4}1-m^2wB^{-1}}$$

is also a holomorphic function of w. It therefore accounts for eigenvalues coalescing only on the real axis, i.e, for the real part of the spectrum of AB .

The Non-Holomorphic Solution

This solution arises only when $\bar{1}1 \cdot \bar{4}4 \neq 0$

In this case, all quantities are non-holomorphic functions of w

In particular, $\langle \frac{1}{N} \mathrm{Tr} \frac{1}{w - AB} \rangle$ is non-holomorphic, and therefore accounts for the eigenvalues of AB which coalesced in the complex plane, off the real axis.

The boundary of the two-dimensional domain in the complex plane, occupied by these eigenvalues is obtained from the self-consistent gap equations by tuning $\bar{1}1 \cdot \bar{4}4 \to 0$

The Case of Positive-Definite Metric B

It can be proved that for any positive-definite B, only the holomorphic solution exists. Thus, as expected, all eigenvalues reside on the real axis. Our solution provides the average Green's function, and therefore, the average eigenvalue density of a positive matrix multiplied by a non-positive hermitian matrix. This goes beyond what free probability theory can give, since the S-transform formalism of free probability theory is designed only for positive-definite matrices.

Example: the case of indefinite metric

$$B = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots - 1)$$

k $N - k$

Analytical Results

We consider values of k which are a finite fraction of N $\lambda = \frac{k}{N}, \quad 0 \leq \lambda \leq$

For $0 < \lambda < 1$ the holomorphic and non-holomorphic solutions co-exist.

This means that on average, there are always real eigenvalues along with complex ones.

The complex eigenvalues occupy two compact blobs, which are symmetric with respect to the real axis.

The average density of eigenvalues in these blobs is uniform, and drops sharply to zero at the boundary.

This density ρ is independent of λ

The boundary of these blobs is determined by

$$r^4 - \frac{r^2}{m^2} + \left(\frac{\sin\theta_0}{2m^2 \sin\theta} \right)^2 = 0$$

where r, θ are polar coordinates and $\sin\theta_0 = 2\lambda - 1$

Positivity of the discriminant implies $\sin^2\theta \geq \sin^2\theta_0$

which fixes one sector in the upper half-plane, and its mirror image in the lower half-plane

The two roots are

$$r_\pm = \frac{1}{m\sqrt{2}} \sqrt{1 \pm \sqrt{1 - \left(\frac{\sin\theta_0}{\sin\theta} \right)^2}}$$

$r_+(\theta)$ is the part of the boundary farther from the origin, and $r_-(\theta)$ is the closer one. They are sewed together at the two points where

$$sin\theta = \pm \sin \theta_0$$

The gap between the two blobs is given by

$$2r_{-,min} = 2r_-(\frac{\pi}{2}) = \frac{2}{m} \sin\left(\frac{\theta_0}{2}\right)$$

and it closes, that is, the two blobs kiss, when $\theta_0 = 0\,, \text{i.e.,}\, \lambda = \frac{1}{2}$

In this case, all N eigenvalues occupy a uniform disk of radius $r_+ = \frac{1}{m}$ in the complex plane, exactly as for Ginibre's ensemble. (They do not form a one-dimensional measure concentrated on the real axis.)

The density (which is independent of λ) is therefore $\rho = \frac{1}{\pi r_+^2} = \frac{m^2}{\pi}$

The area of one of these blobs, say the upper one, is (for $\lambda \leq \frac{1}{2}$, i.e., $\pi < \theta_0 < 2\pi$)

$$S = \frac{1}{2} \int\limits_{\theta_0 - \pi}^{2\pi - \theta_0} \left(r_+^2(\theta) - r_-^2(\theta) \right) \, d\theta =$$

$$\frac{\pi}{2m^2}(1 + \sin\theta_0) = \frac{\pi\lambda}{m^2}$$

Therefore, the fraction of eigenvalues which reside in the two blobs is

$$2S\rho = \frac{2\pi\lambda}{m^2}\frac{m^2}{\pi} = 2\lambda = \frac{2k}{N}$$

This result holds for $\lambda \leq \frac{1}{2}$.

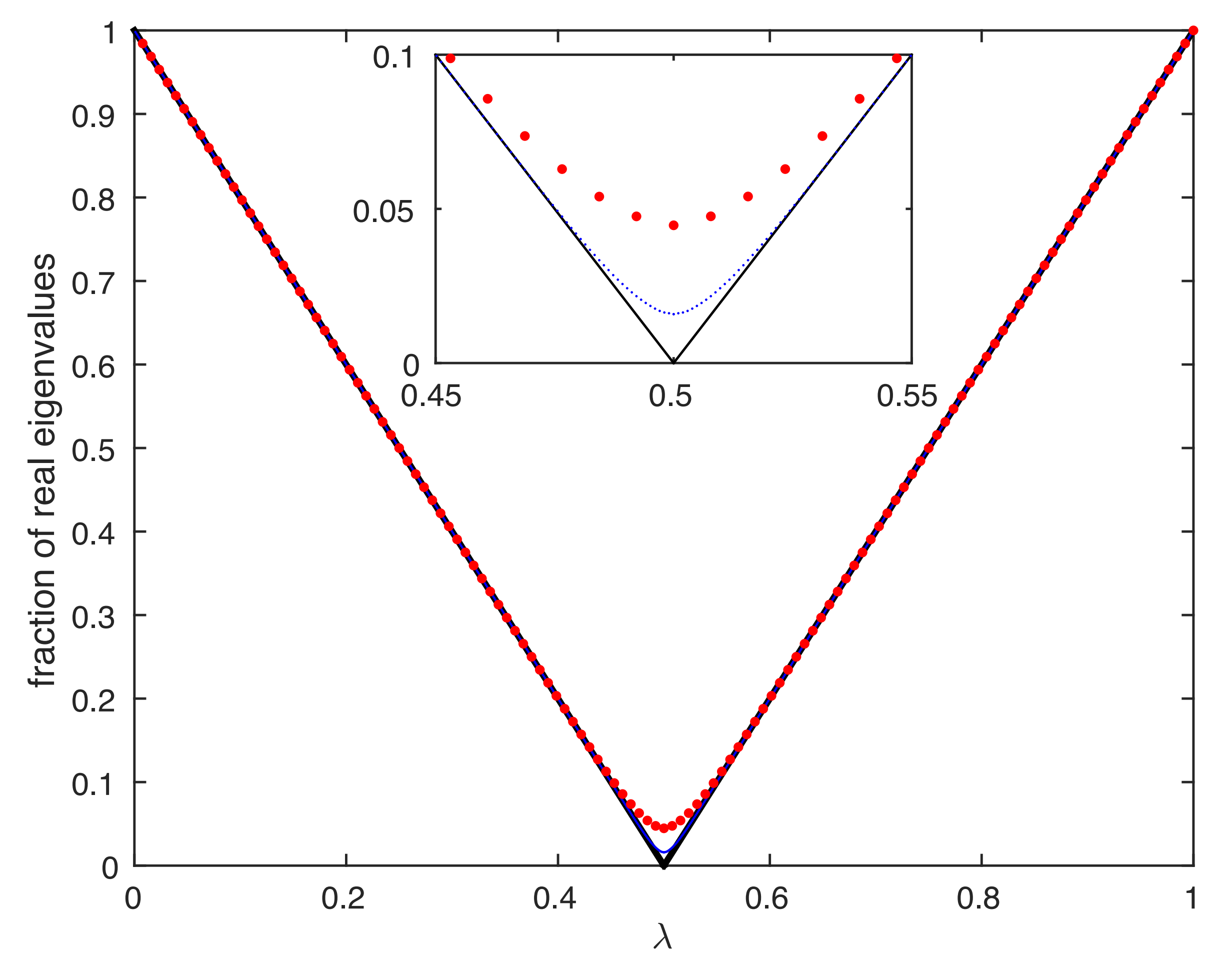
For $\frac{1}{2} < \lambda \leq 1$ we should replace in the last formula $\lambda \to 1 - \lambda$

That is $k \to N - k$, because the intertwining relation $\varphi^\dagger B = B\varphi$

is invariant under $B \to -B$

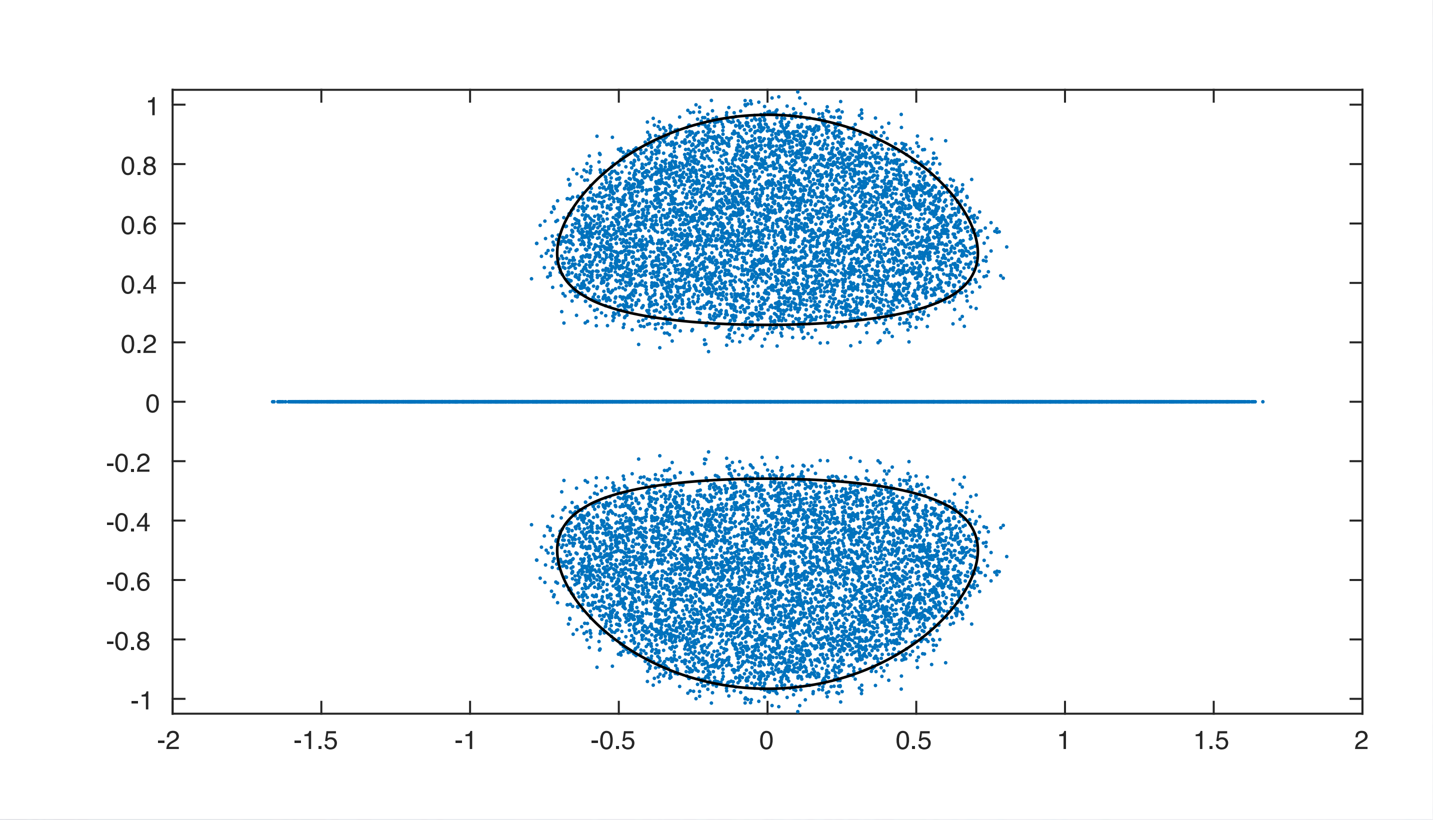
Numerical Results

Fraction of real eigenvalues as function of λ

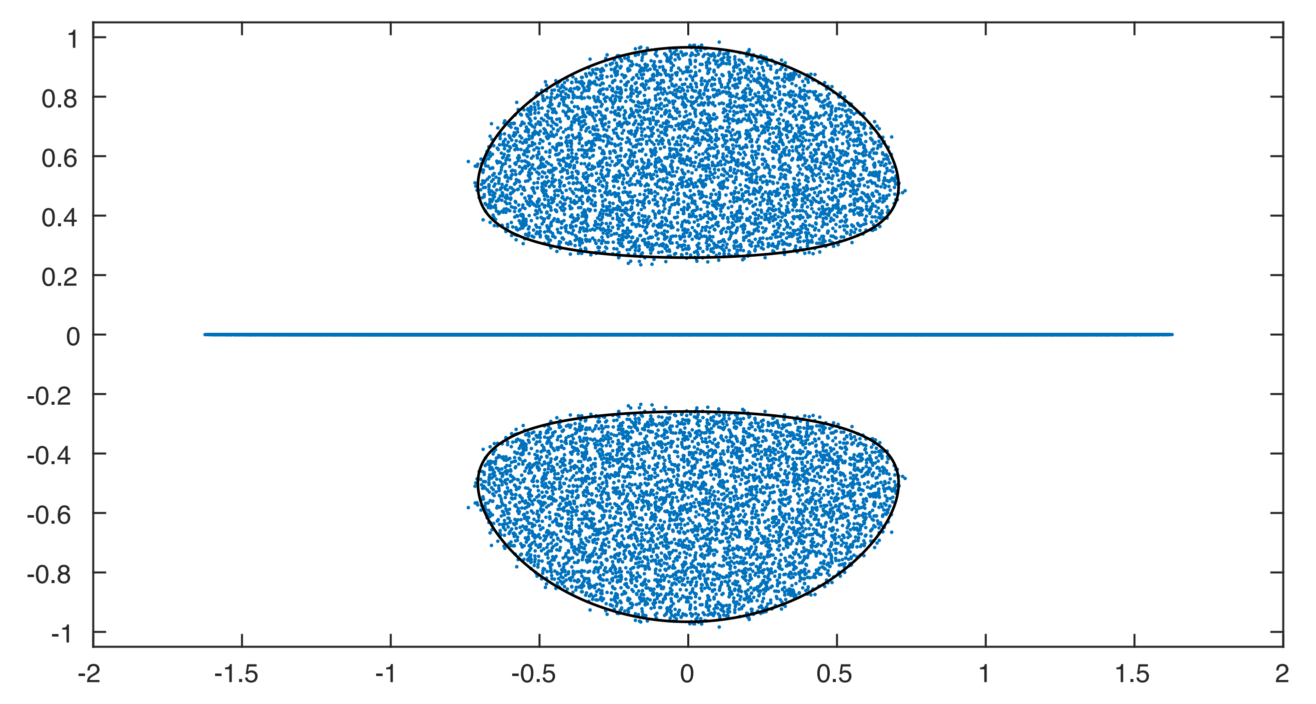


● N=128, ● N=1024

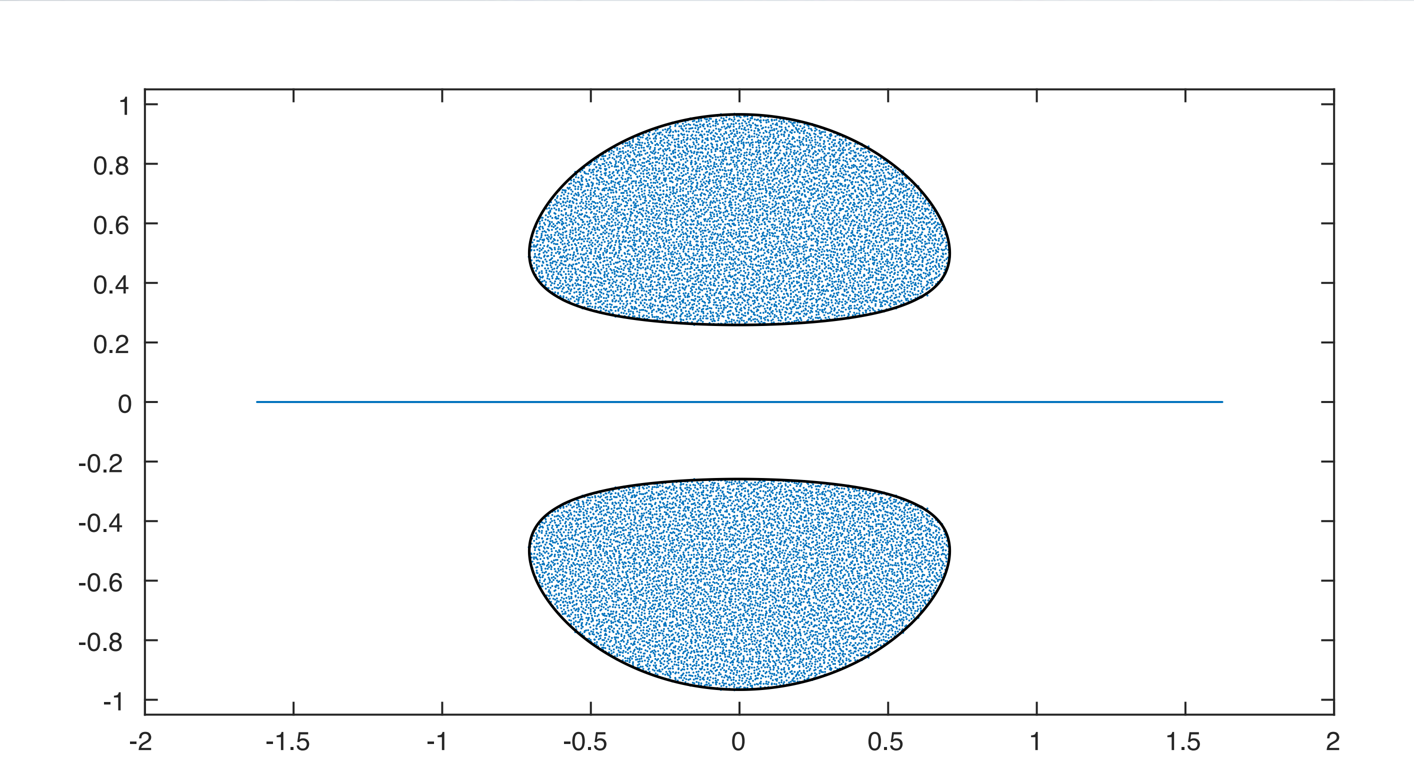
scatter plots for $\lambda = 0.25$



N=128, k=32, $\lambda = 0.25$ 200 samples

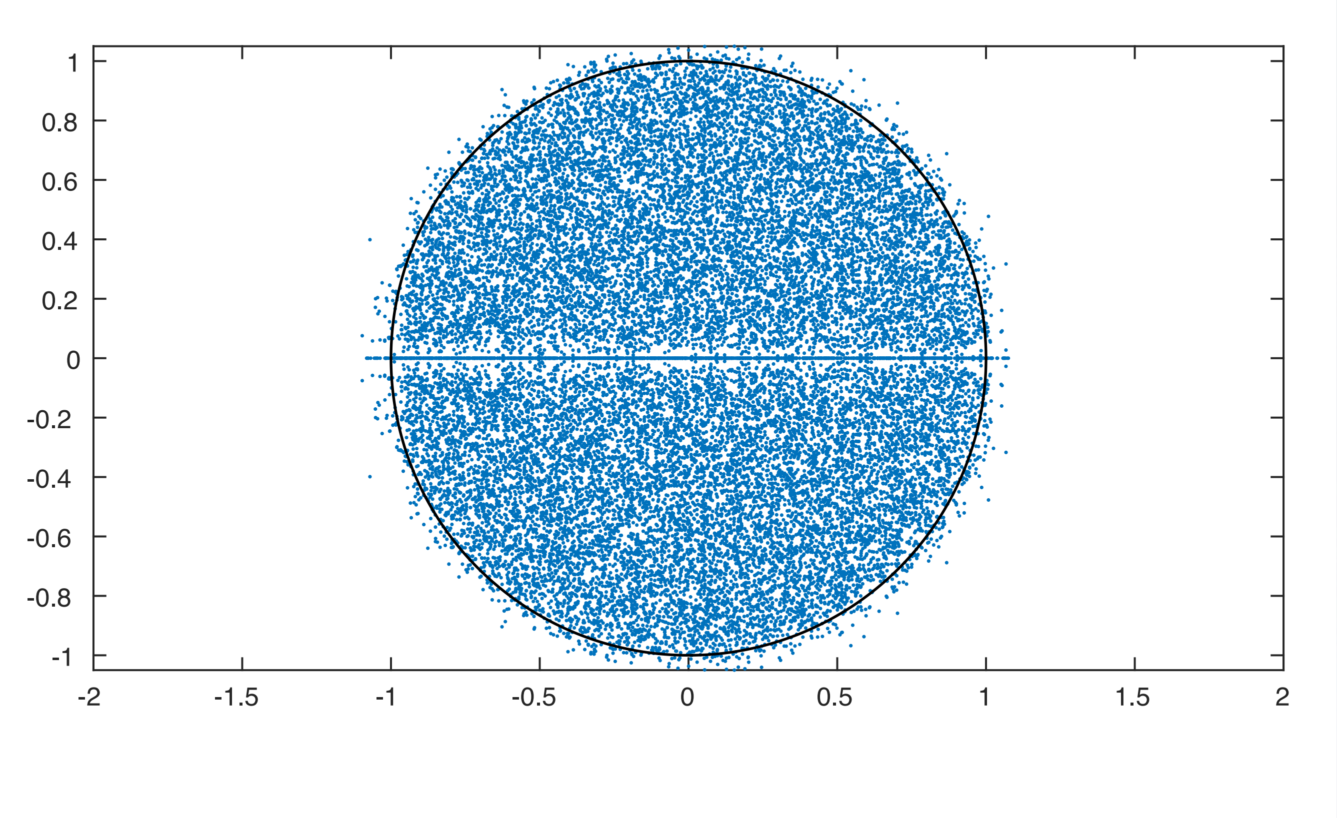


N=1024, k=256, $\lambda = 0.25$ 20 samples

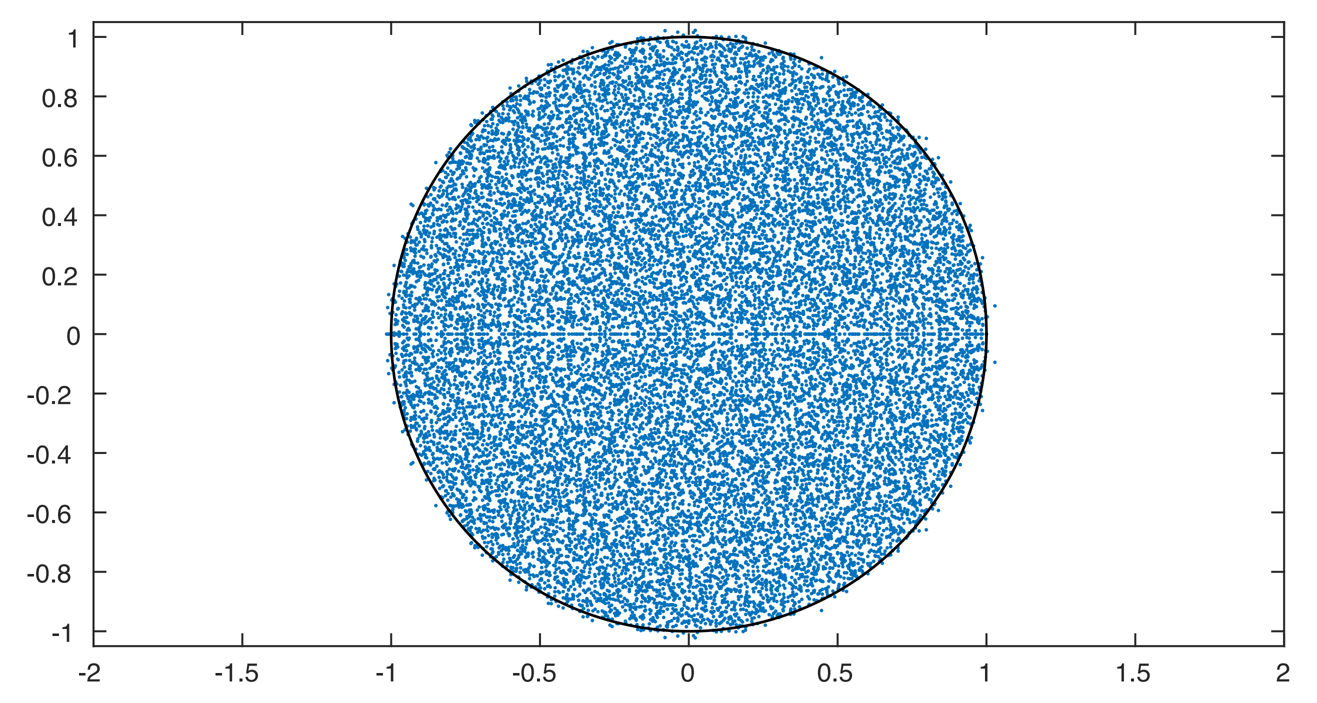


N=32768, k=8192, $\lambda = 0.25$ 1 sample

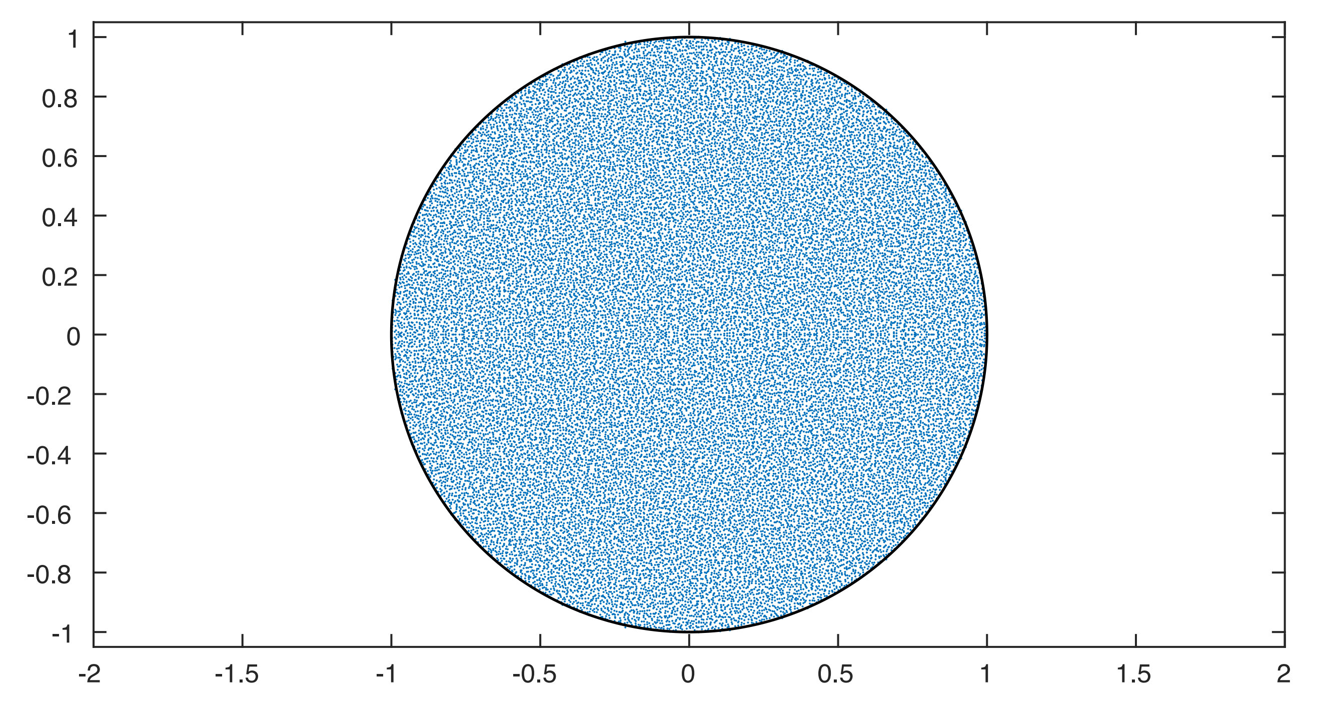
scatter plots for $\lambda = 0.5$



N=128, k=32, $\lambda = 0.5$ 200 samples

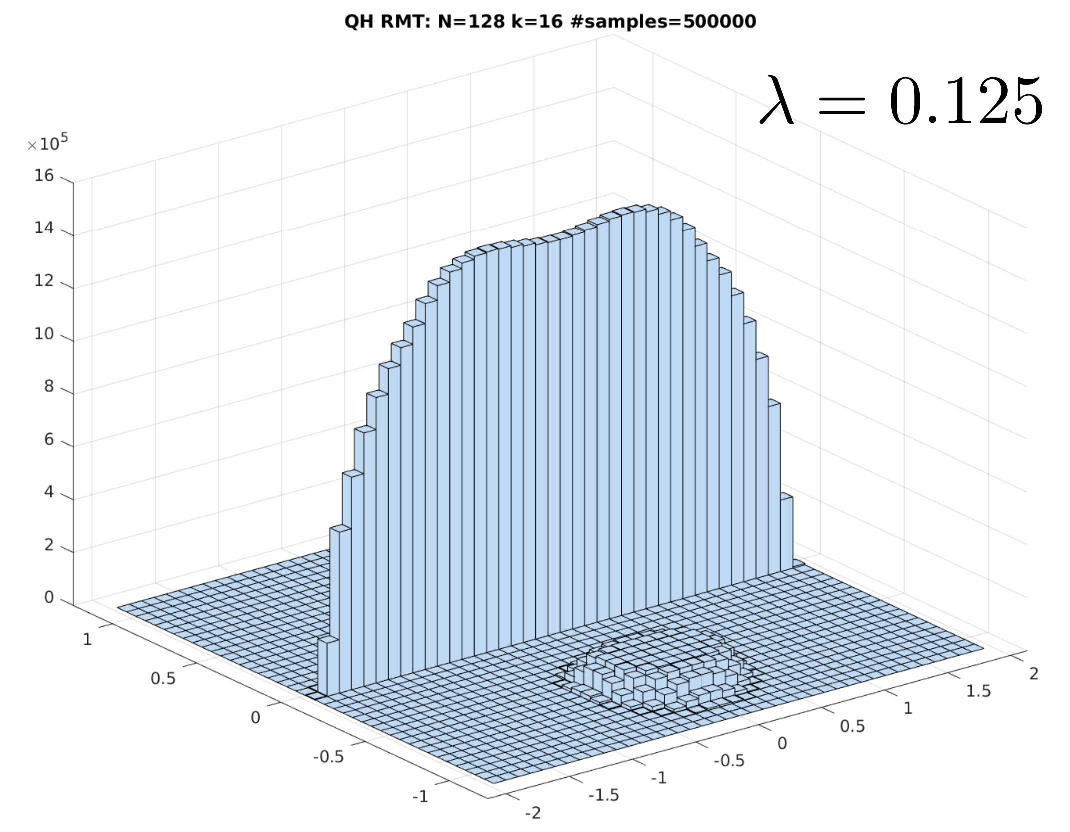


N=1024, k=256, $\lambda = 0.5$ 20 samples

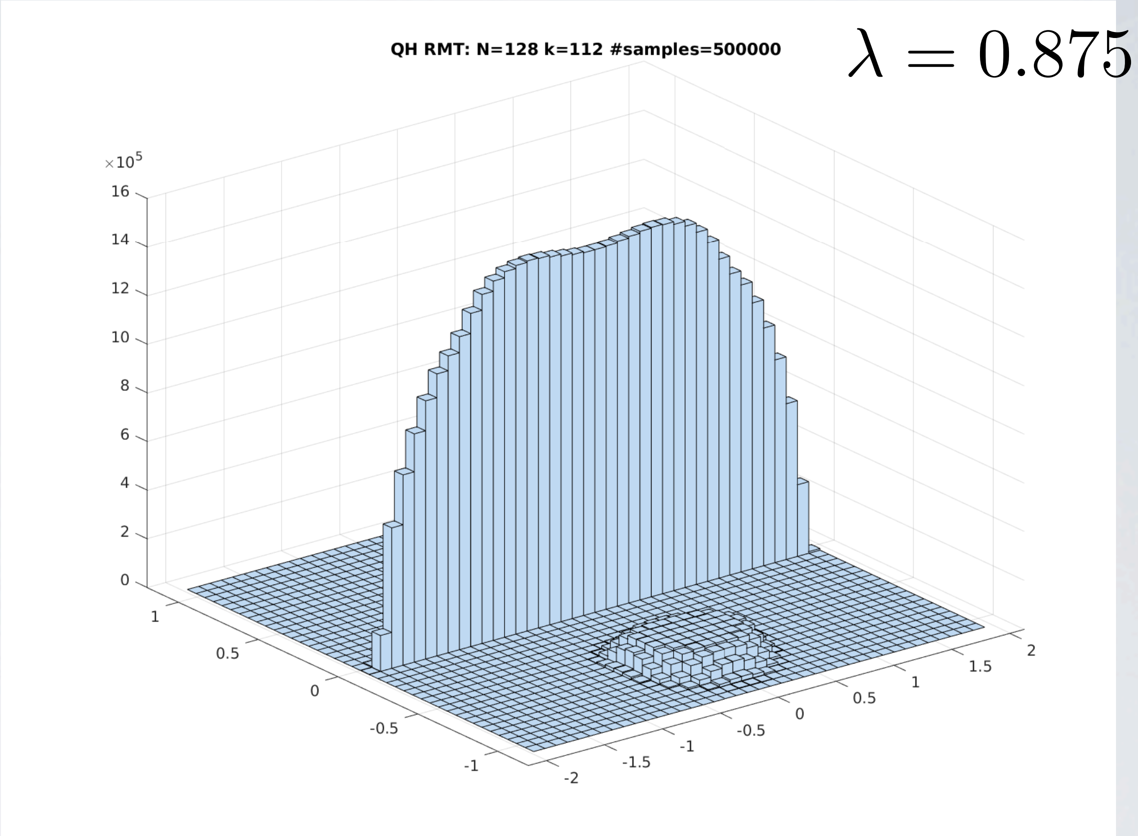


N=32768, k=16384, $\lambda = 0.5$ 1 sample

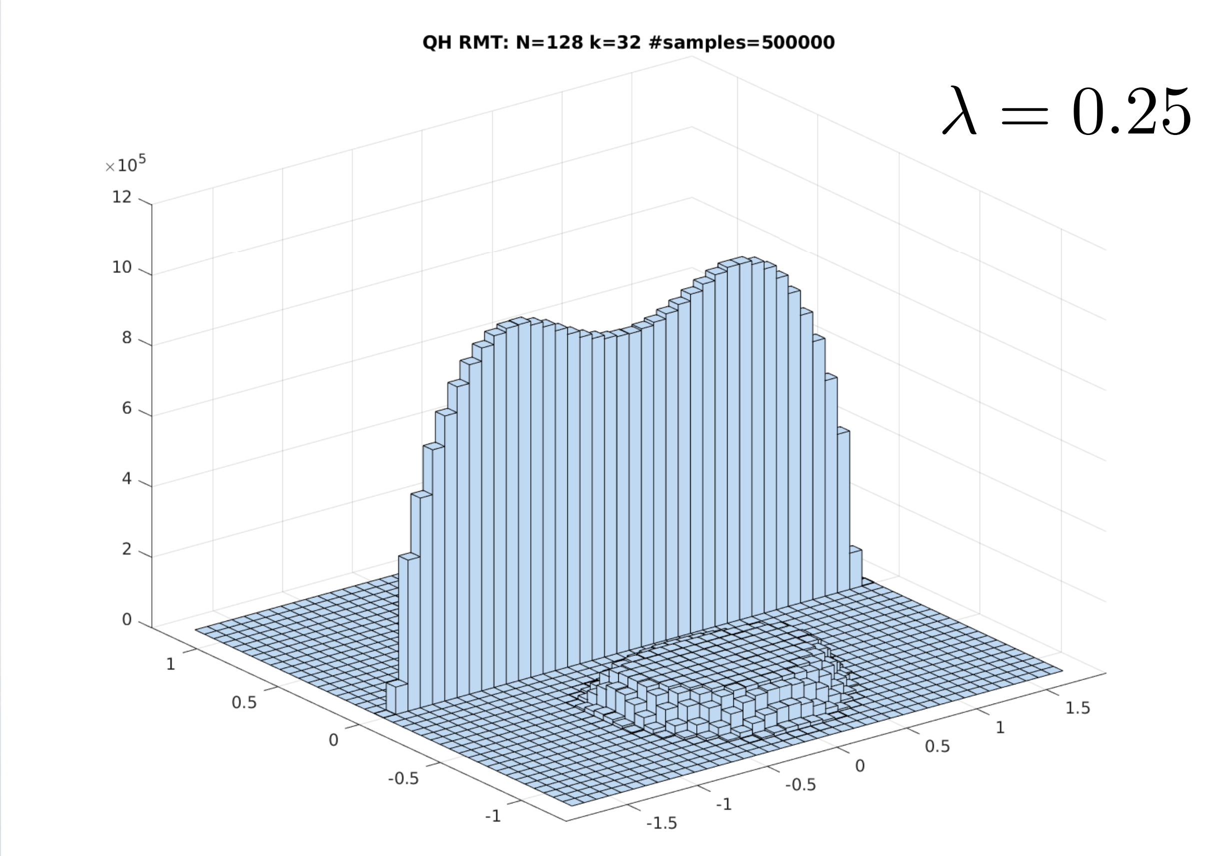
3d histograms (N=128)


QH RMT: N=128 k=16 #samples=500000

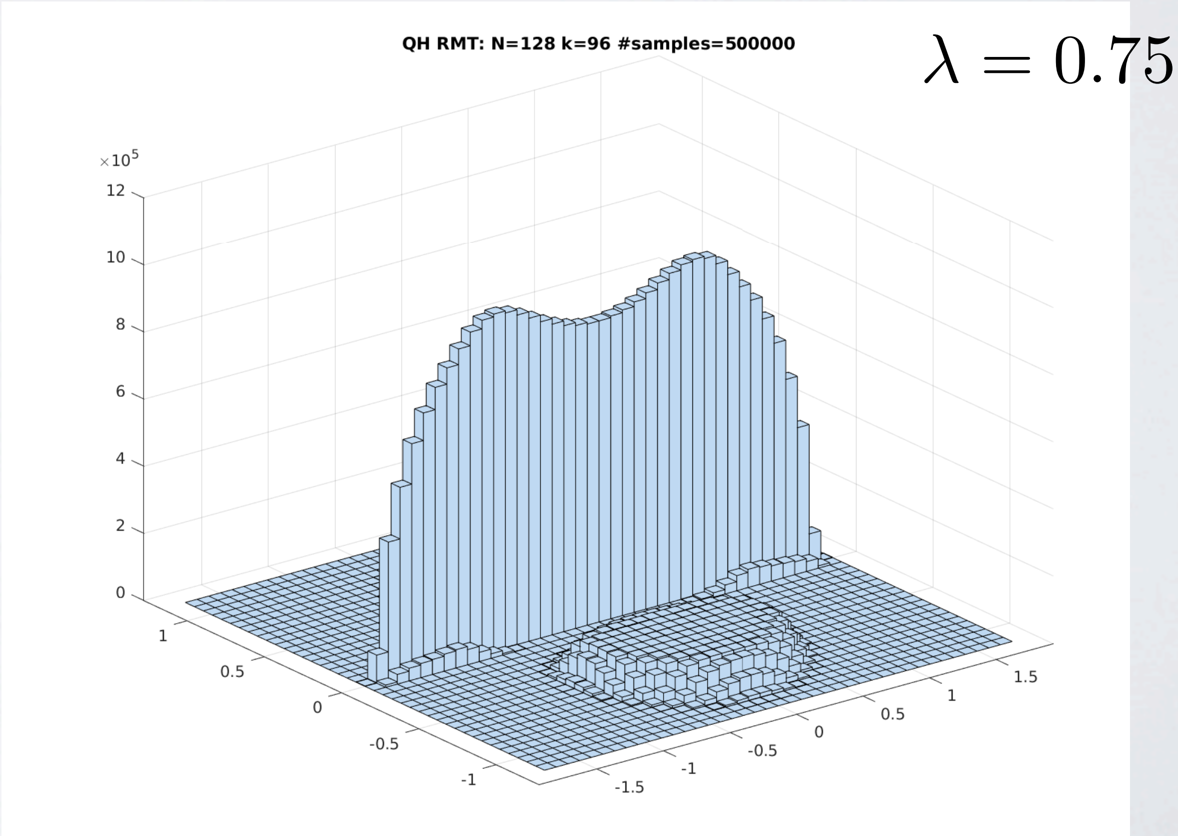
$\lambda = 0.125$


QH RMT: N=128 k=112 #samples=500000

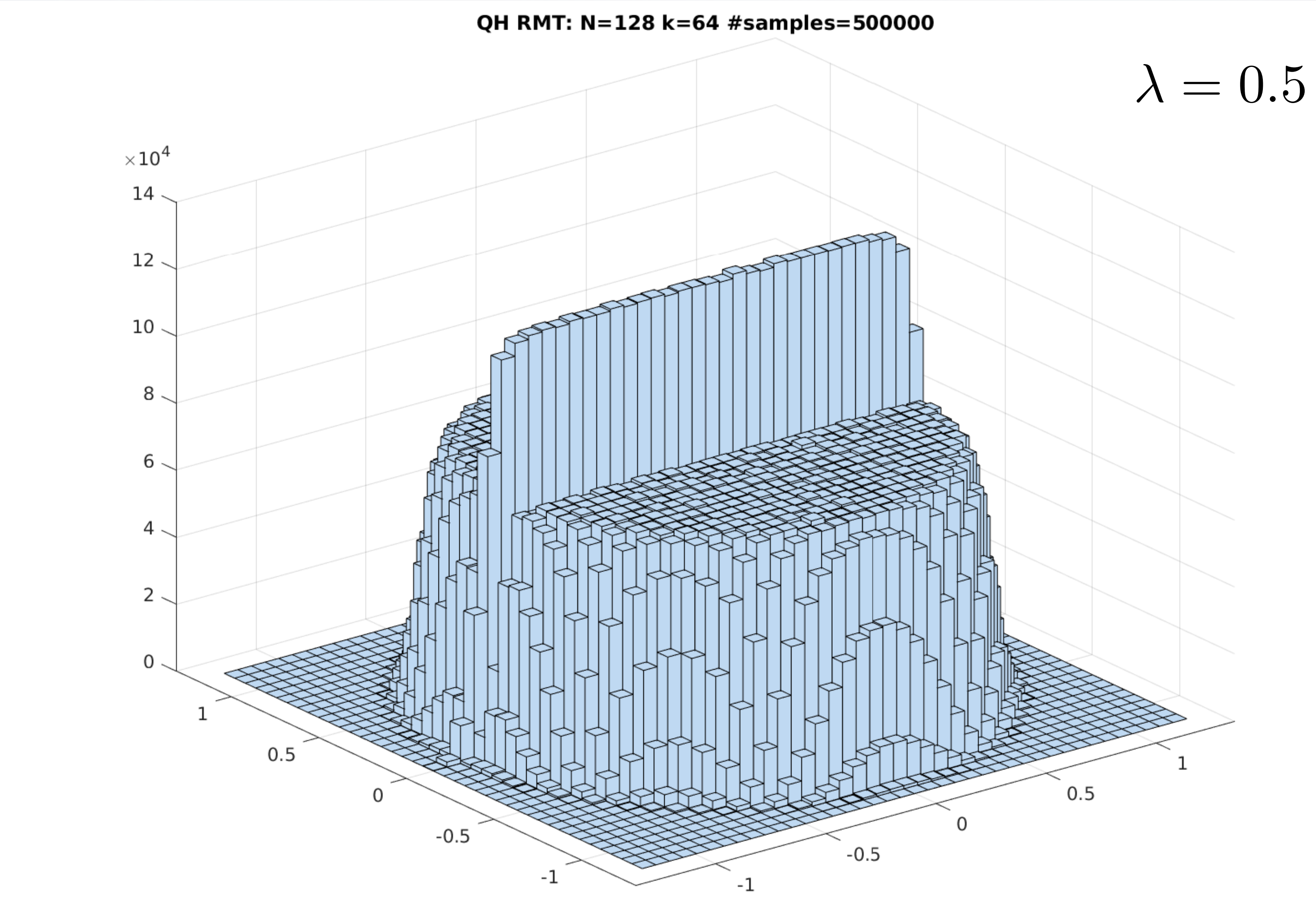
$\lambda = 0.875$


QH RMT: N=128 k=32 #samples=500000

$\lambda = 0.25$

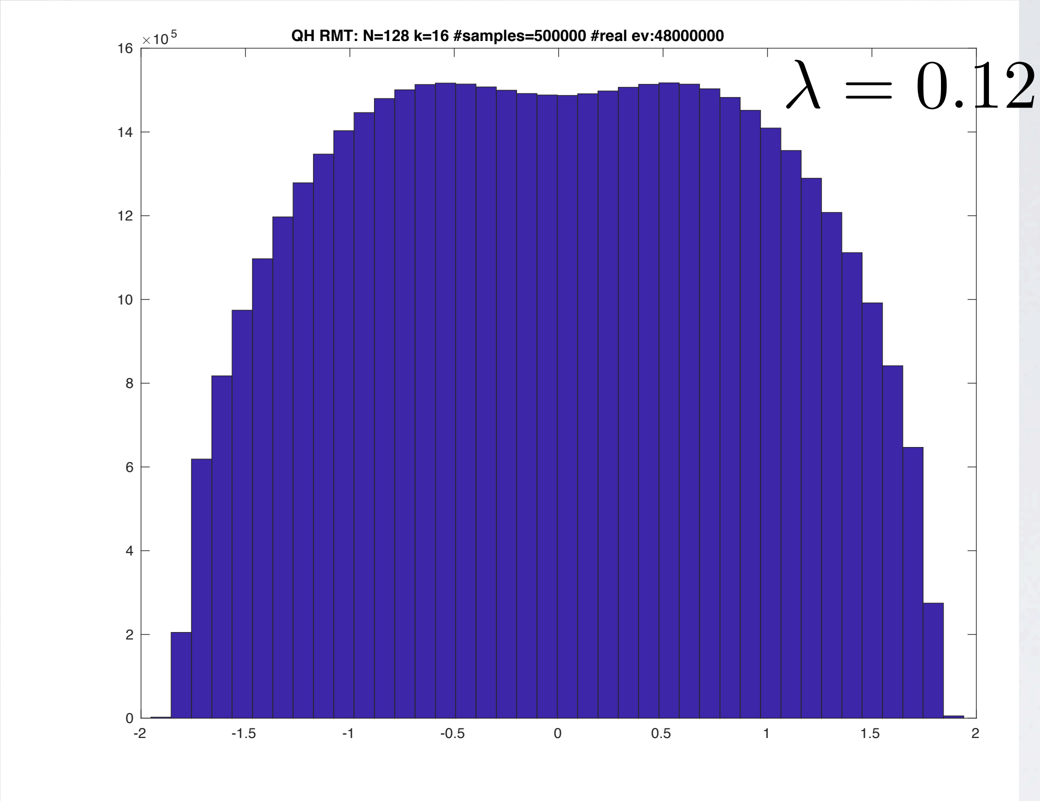

QH RMT: N=128 k=96 #samples=500000

$\lambda = 0.75$

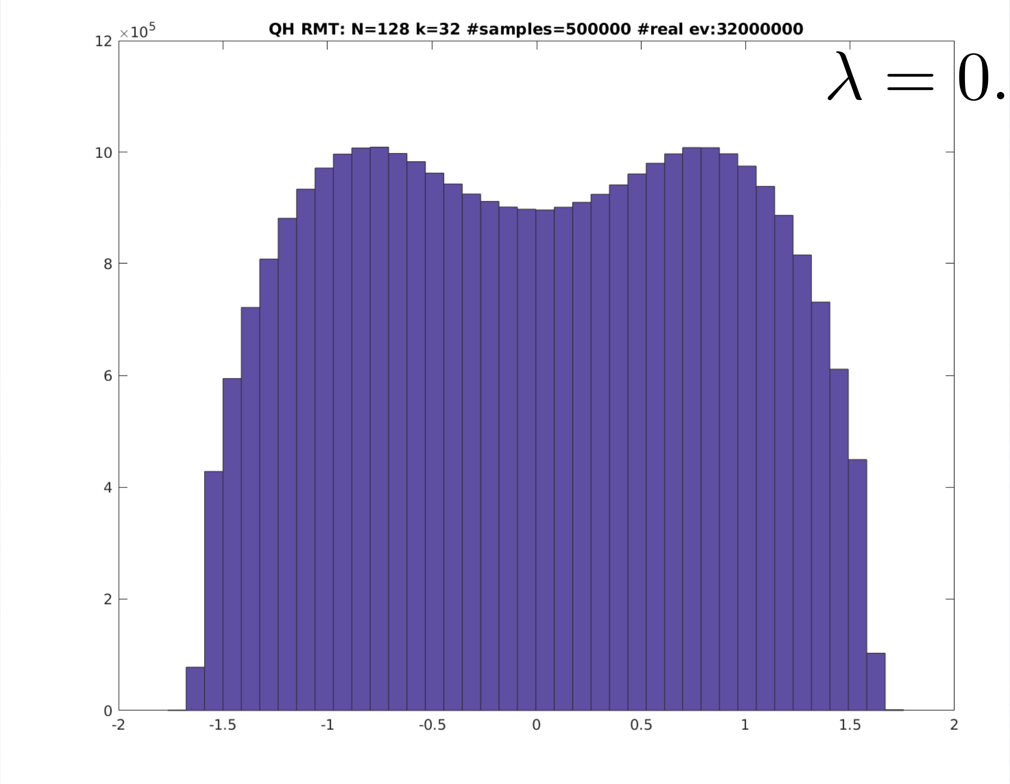

QH RMT: N=128 k=64 #samples=500000
×10^4
14
12
10
8
6
4
2
0
1
0.5
0
-0.5
-1
1
0.5
0
-0.5
-1

$\lambda = 0.5$

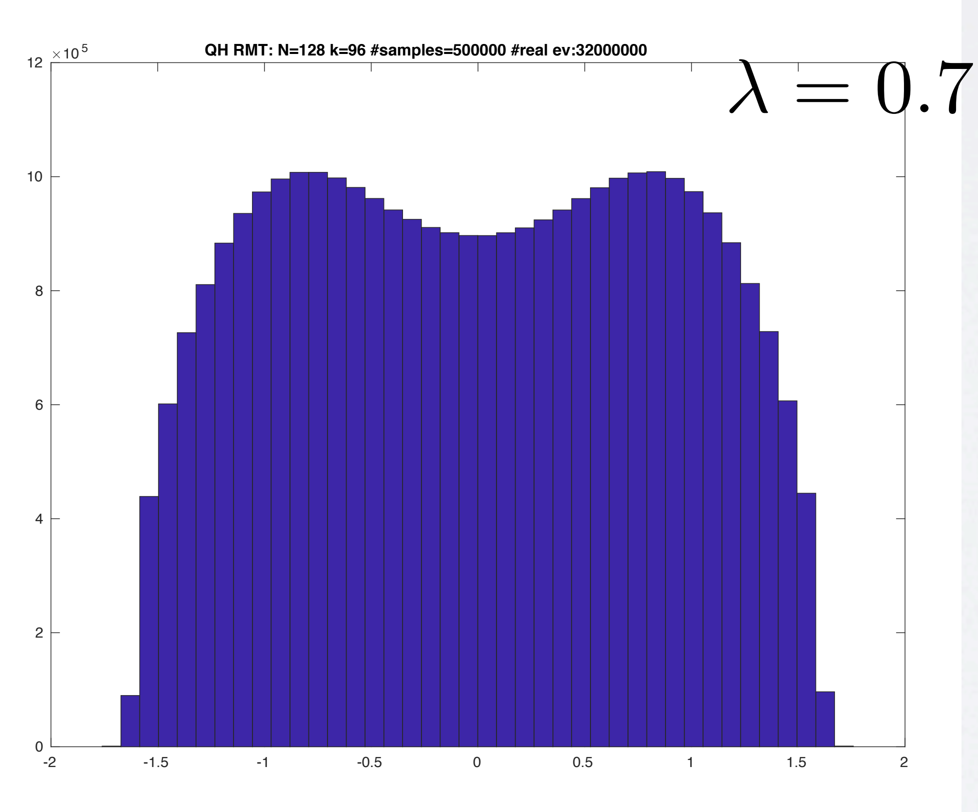
histograms of real eigenvalues for N=128



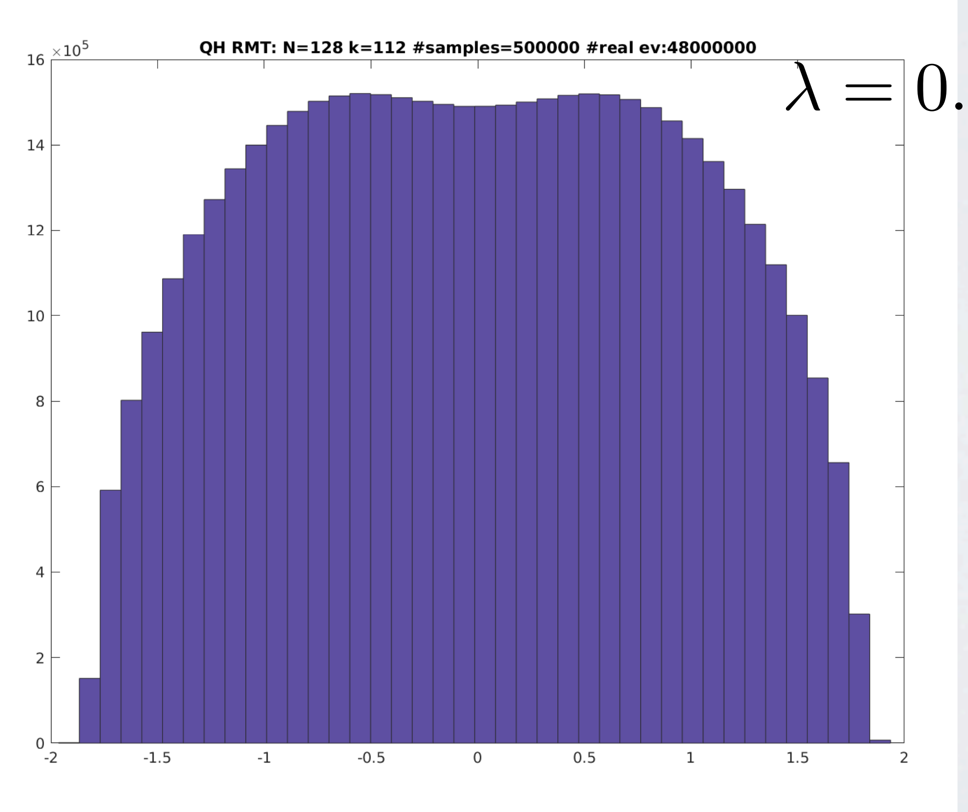
$\lambda = 0.125$



$\lambda = 0.25$



$\lambda = 0.75$



$\lambda = 0.875$

semicircle (GUE)