$\mathcal{PT}$ symmetric non-unitary quantum walk
--- how to make it ---

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Topological Materials Science

Non-Hermitian Physics - PHHQP XVIII
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In collaboration with

- Theory:

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- Experiment:

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  L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang,
  X. Zhang, X. P. Wang, J. Li

  Wei Yi (U. of Science and Technology of China, China)
  Barry Sanders (U. of Calgary, Canada)

unitary QW:

PRB 84, 195139 (2011).
PRB 88, 121406(R) (2013).
PRB 92, 045424 (2015).

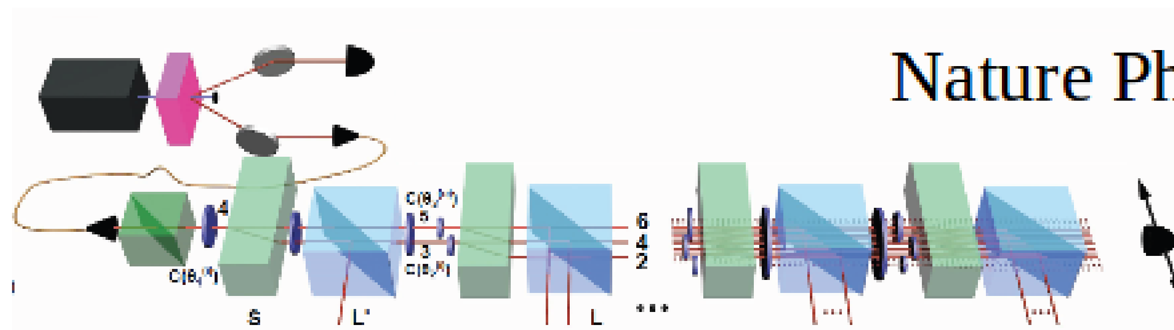
$\mathcal{PT}$ symmetric non-unitary QW:

PRA 93, 062116 (2016).
IIS 23, 95 (2017) [arXiv:1608.00719]
arXiv:1609.09650.
Nature Physics 13, 1117 (2017).

Why are QWs important for PT symmetry?

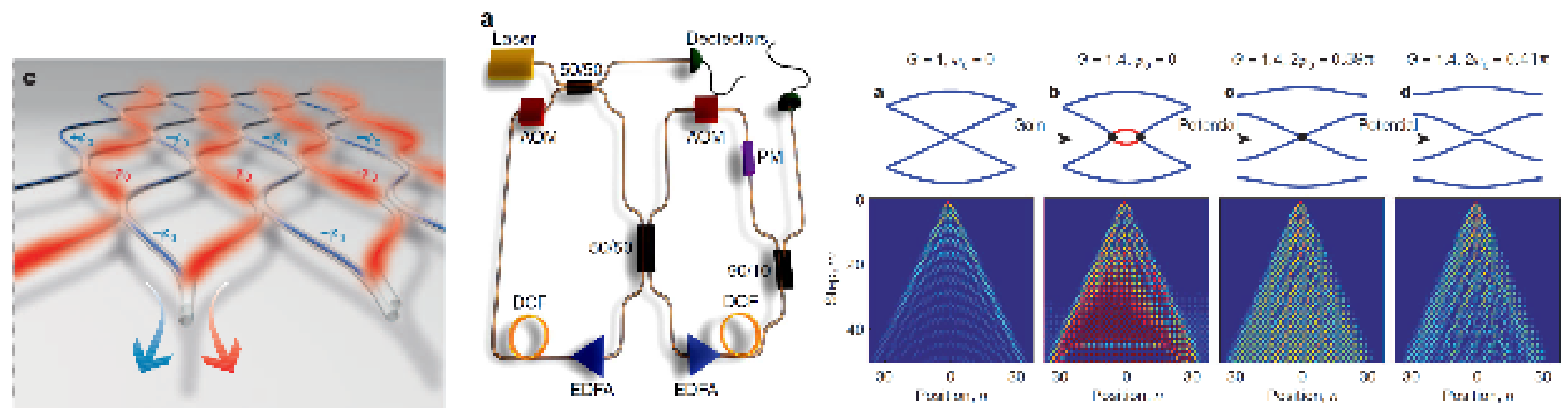
Why are QWs important for PT symmetry?

- Experiment for PT symmetric QWs is possible in quantum regime. (entanglement exists!)



Nature Physics 13, 1117 (2017).

- By photonic PT symmetric QWs with lasers, large scale experiment is possible in classical regime.



Regensburger et al., Nature 488, 167 (2012)

- PT symmetry can be easily defined for time-dependent systems (periodically driven Floquet systems).

$$(\mathcal{PT})\, U\, (\mathcal{PT})^{-1} = U^{-1}$$

$$\eta\, U^{-1}\, \eta^{-1} = U^{\dagger}$$

Outline

- Introduction of quantum walks (QWs)
- How to identify symmetry of unitary QWs
- How to construct PT symmetric non-unitary QWs
- Experimental result

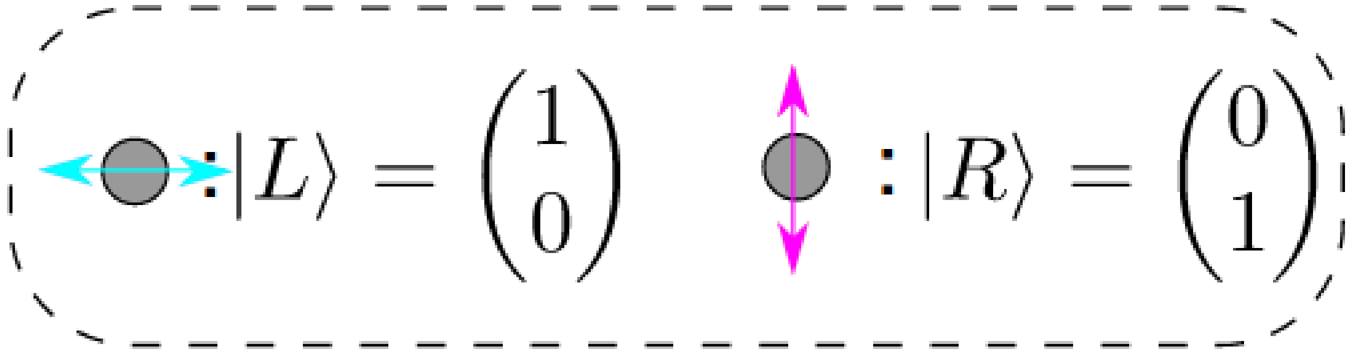
Quantum Walk

Definition of Discrete-Time QW in 1D

1.Basis:

- position space $\otimes$ internal states

$$|x\rangle \otimes |s\rangle \qquad x \in \mathbb{Z}$$
$$s = L, R$$

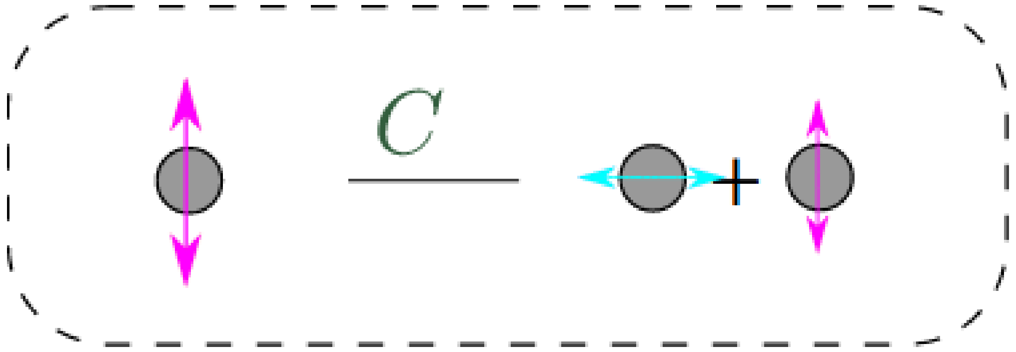


2. Defining elemental unitary operators:

- Coin operator: C

$$C[\theta(x)] = \sum_x |x\rangle\langle x| \otimes \mathcal{R}[\theta(x)]$$

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = e^{-i\theta\sigma_2}$$



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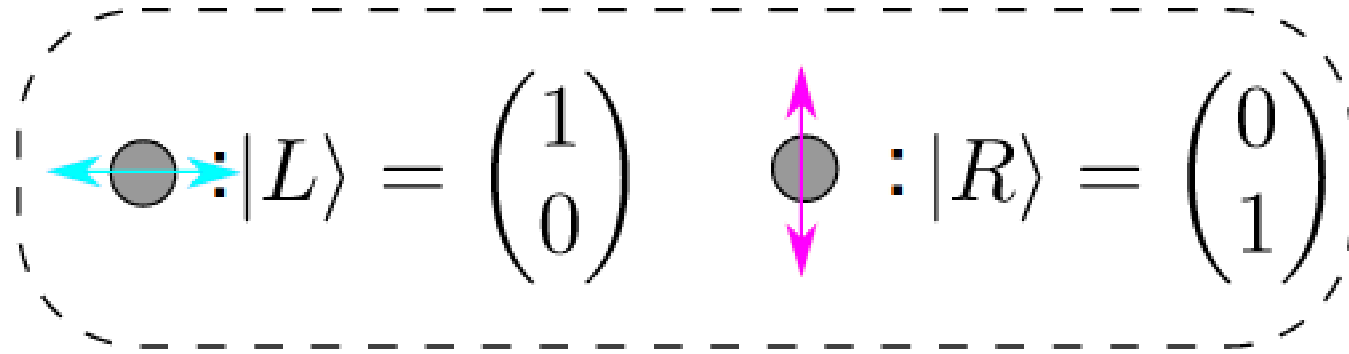

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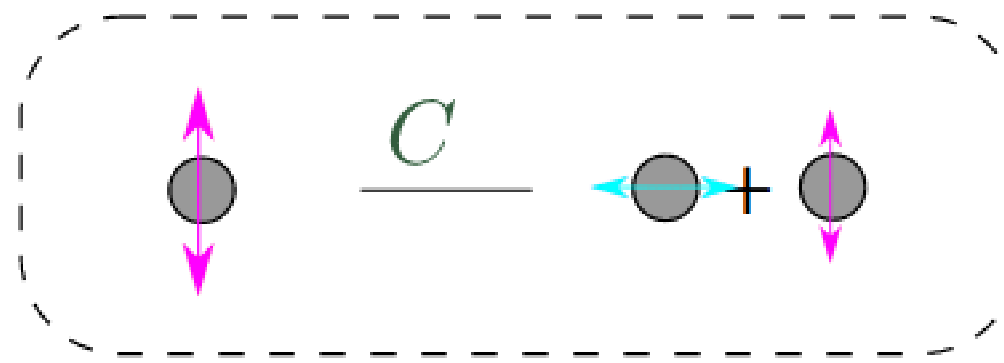
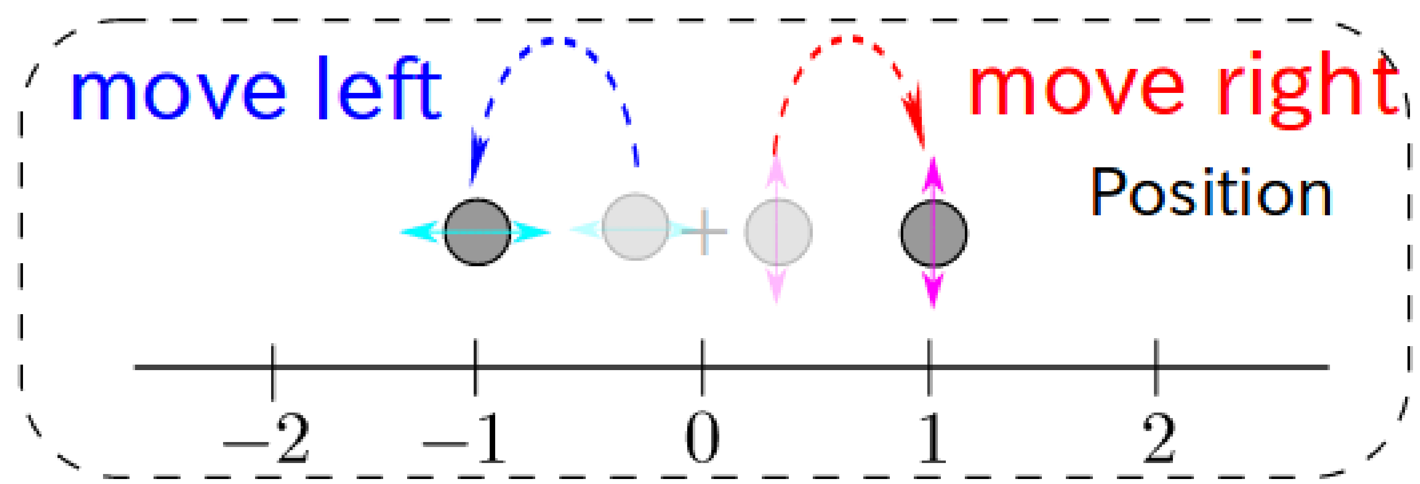


- Shift operator : S

$$S = \sum_x \Big(|x+1\rangle\langle x| \otimes |R\rangle\langle R| + |x-1\rangle\langle x| \otimes |L\rangle\langle L| \Big)$$

$$\xrightarrow{\mathcal{F}} \int dk |k\rangle\langle k| \otimes e^{ik\sigma_3}$$



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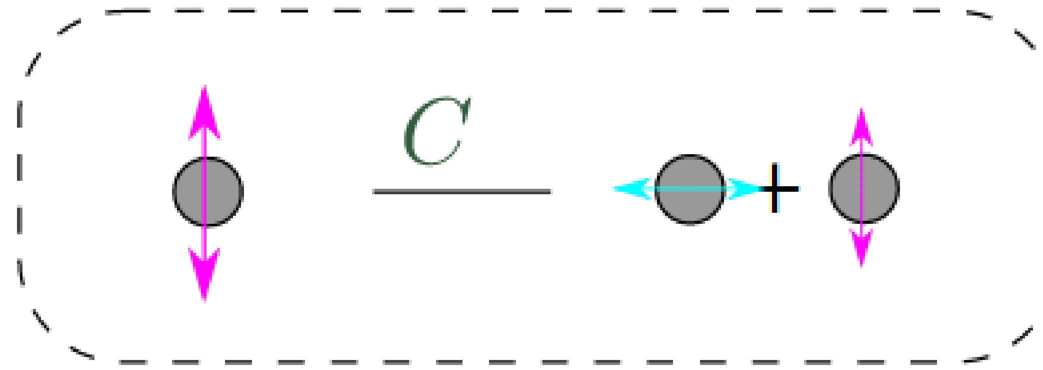

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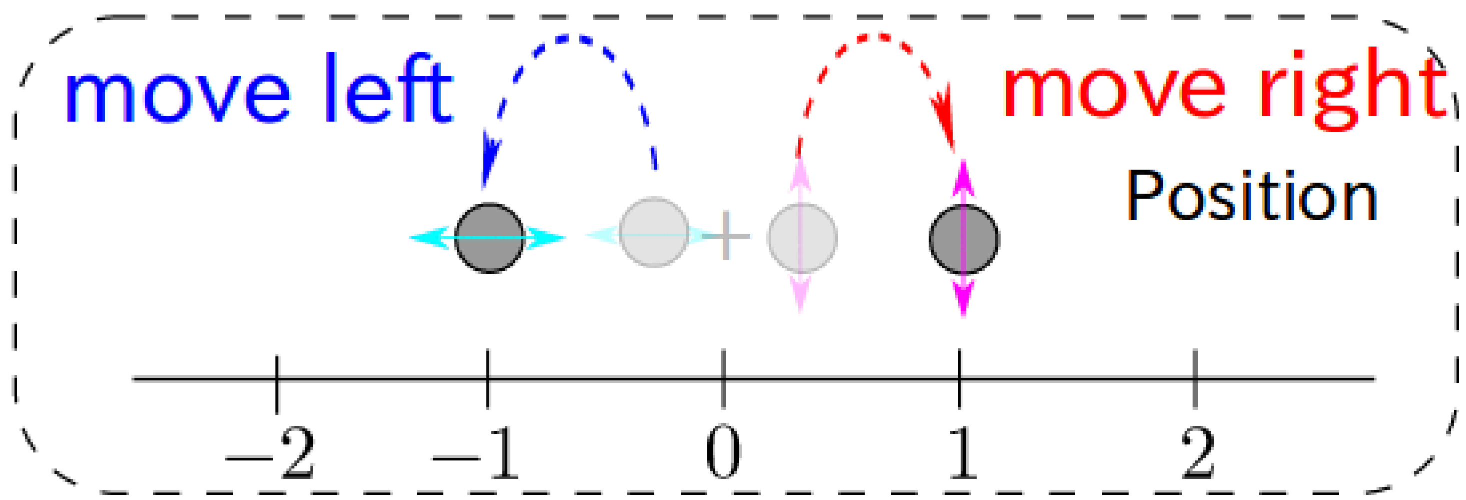


3. Building up time-evolution operators by products of elemental operators:

- Time-evo. operator for single-time step :

single-step: $U = SC(\theta)$

two-step: $U = SC(\theta_2) \cdot SC(\theta_1)$

- quantum state at time t $(\in \mathbb{Z}^+)$

$$|\psi(t)\rangle = U^t |\psi(0)\rangle$$

periodically and stroboscopically driven Floquet systems

- Time-evo. operator for single-time step :

single-step: $U = SC(\theta)$

two-step: $U = SC(\theta_2) \cdot SC(\theta_1)$

Quantum dynamics can be defined by U.

➡ No Hamiltonian is explicitly defined.

$$U = e^{-iH_F}$$

- Continuum limit

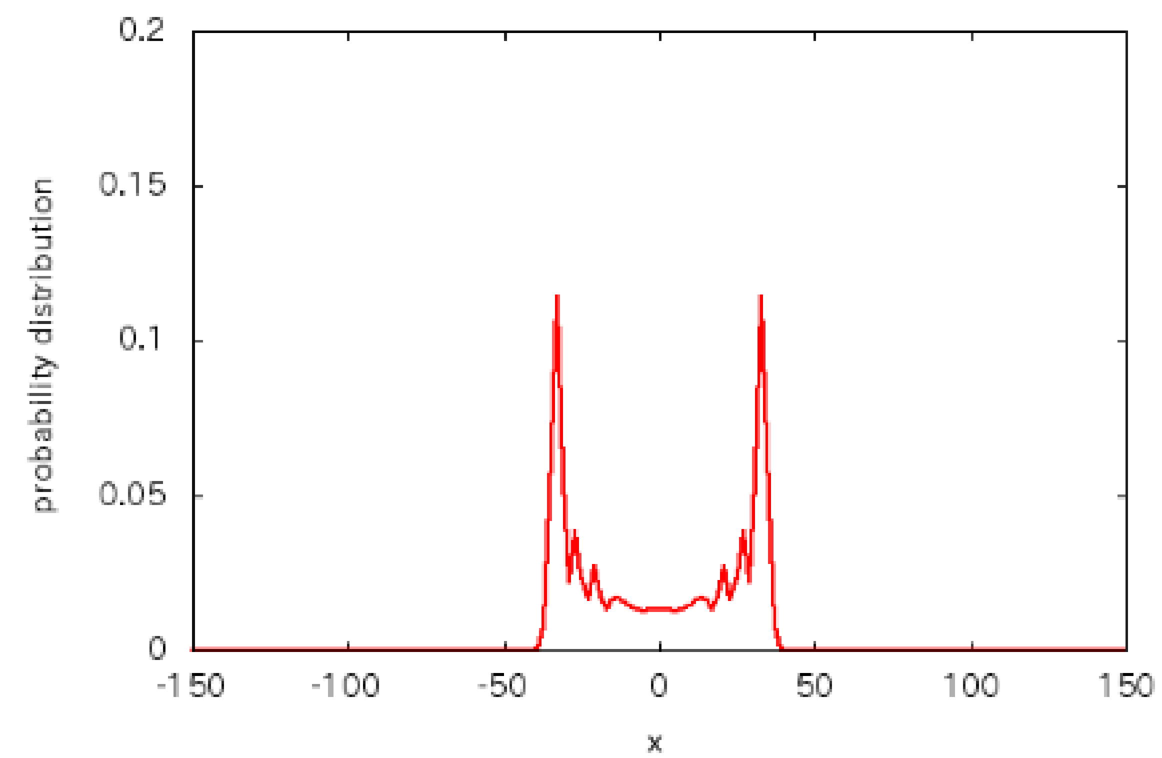
Dirac operator

$$U = SC(\theta) \quad \longrightarrow \quad H_F = \hat{p}\sigma_3 + \theta(x)\sigma_2$$

- quantum state at time $t \ (\in \mathbb{Z}^+)$

$$|\psi(t)\rangle = U^t|\psi(0)\rangle$$

periodically and stroboscopically driven Floquet systems



Quasi-energy

- Stationary states (eigen states of U):

$$U|\psi\rangle = \lambda|\psi\rangle \qquad U = e^{-iH_F}$$

$$\lambda = e^{-i\varepsilon}$$

ε : quasi-energy

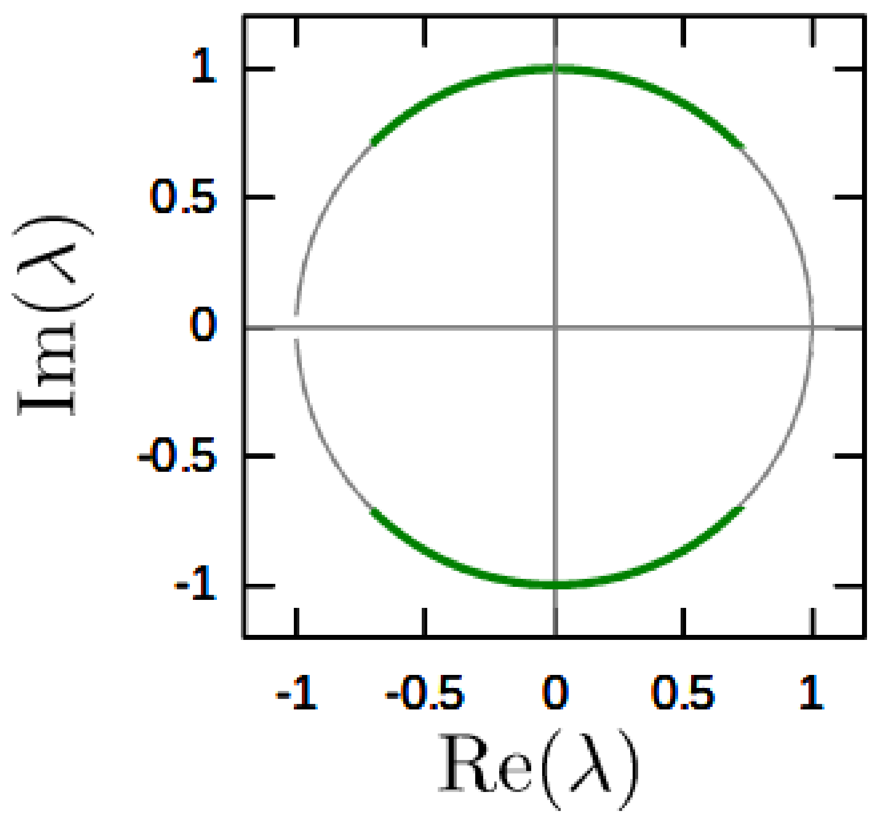
(2π periodicity)

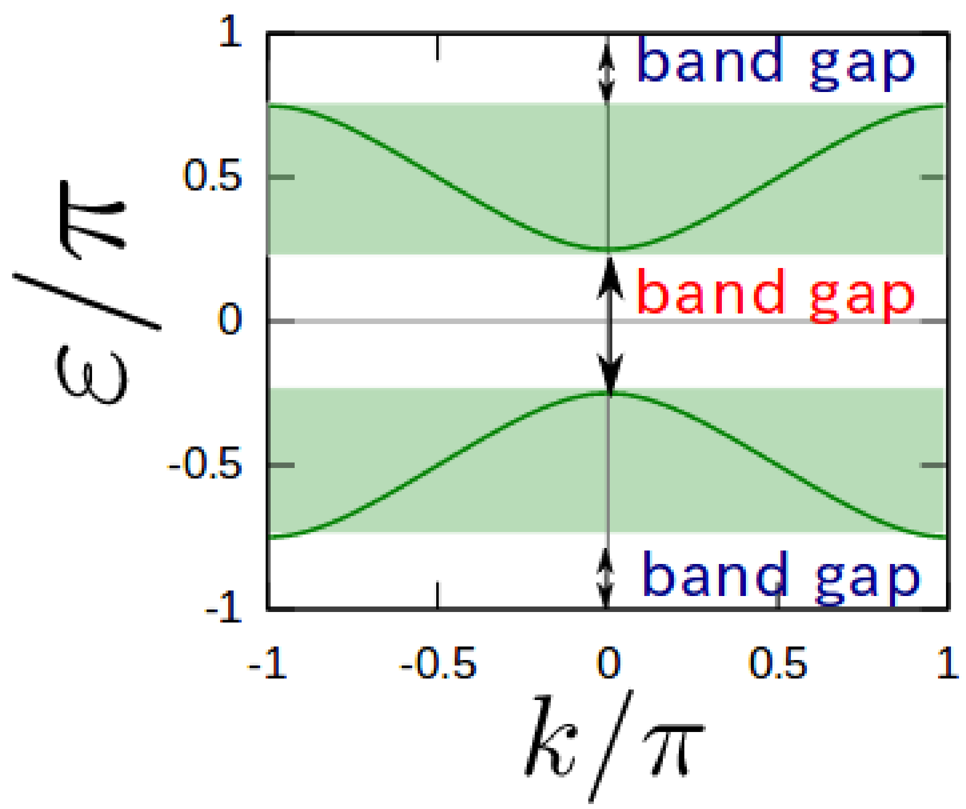
- For unitary QWs,

$$|\lambda| = 1 \Rightarrow \varepsilon \in \mathbb{R}$$

- For non-unitary QWs,

$$\cancel{|\lambda| = 1} \Rightarrow \varepsilon \in \mathbb{C}$$





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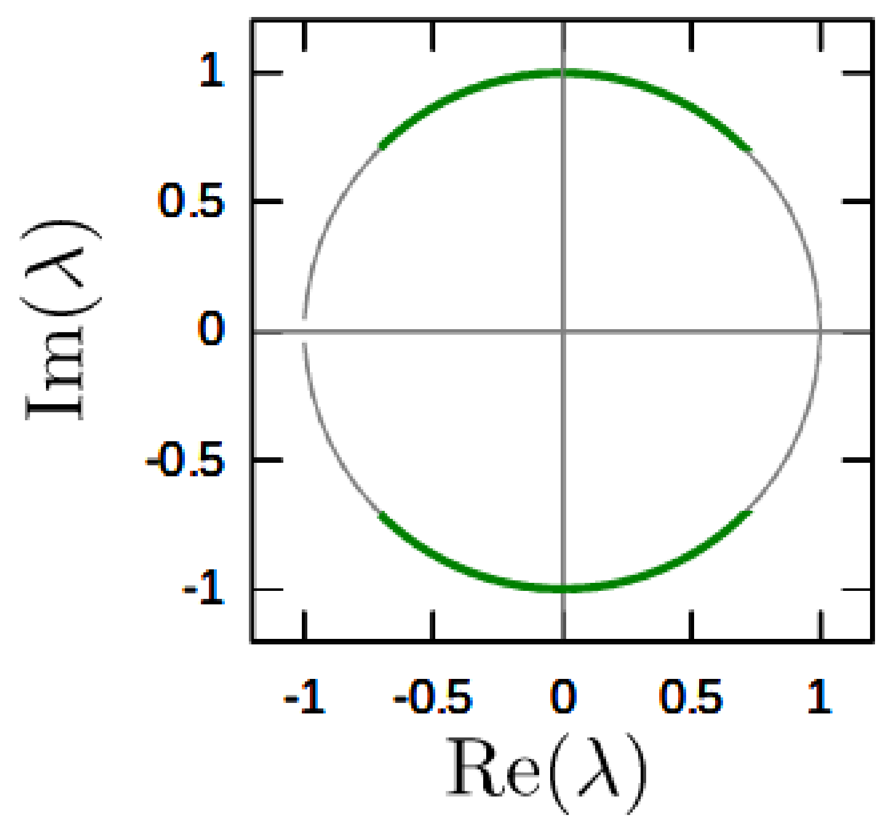
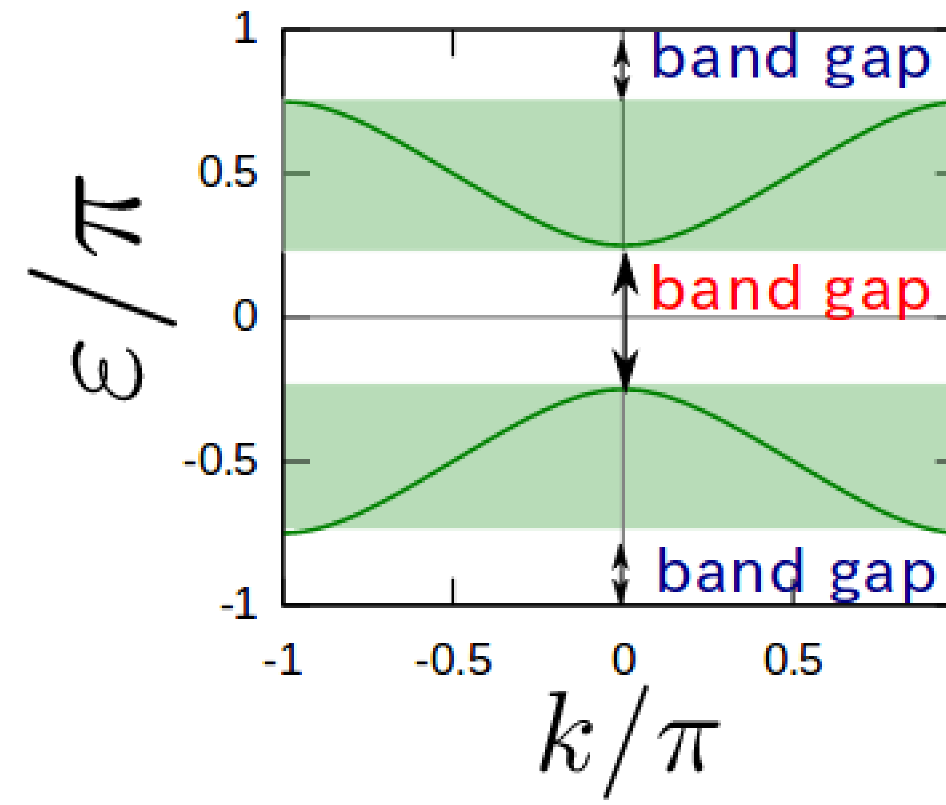




- For non-unitary QWs,

$$\cancel{|\lambda| = 1} \Rightarrow \varepsilon \in \mathbb{C}$$

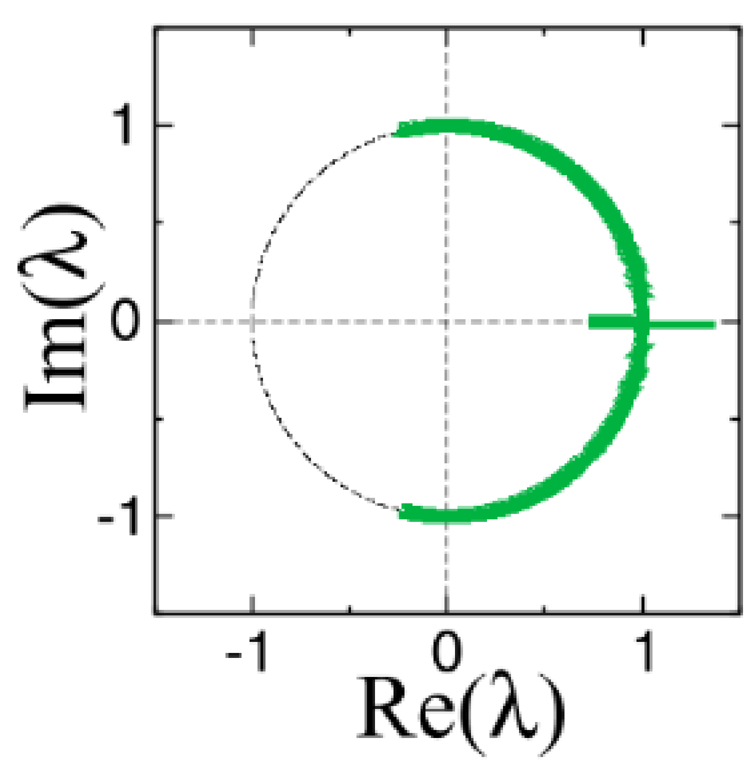
- For PT symmetric non-unitary QWs,

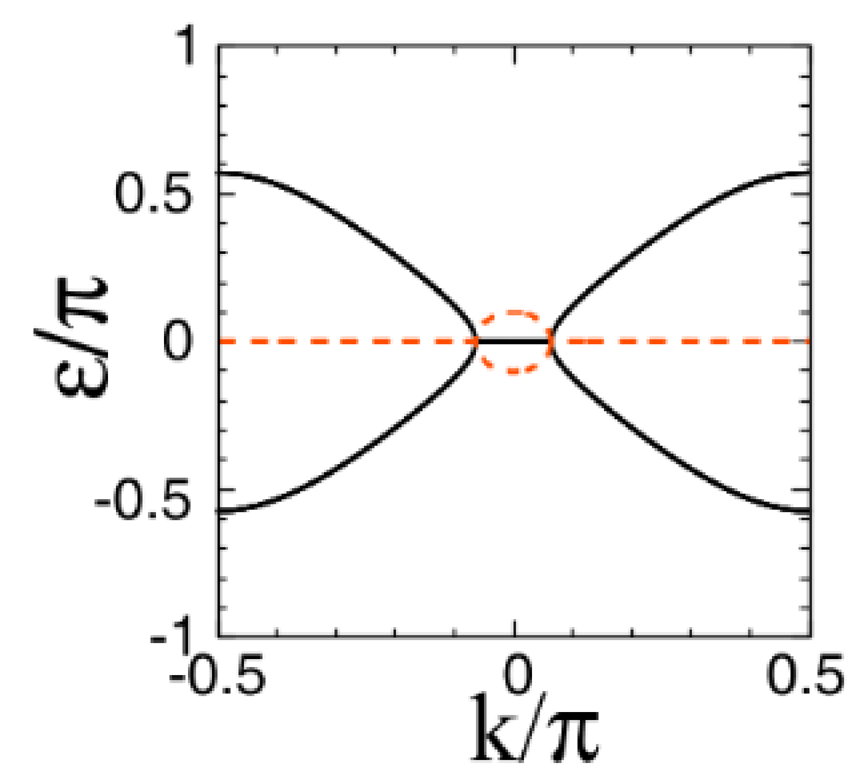
unbroken phase:

$$|\lambda| = 1 \Rightarrow \varepsilon \in \mathbb{R}$$

broken phase:

$$\cancel{|\lambda| = 1} \Rightarrow \varepsilon \in \mathbb{C}$$





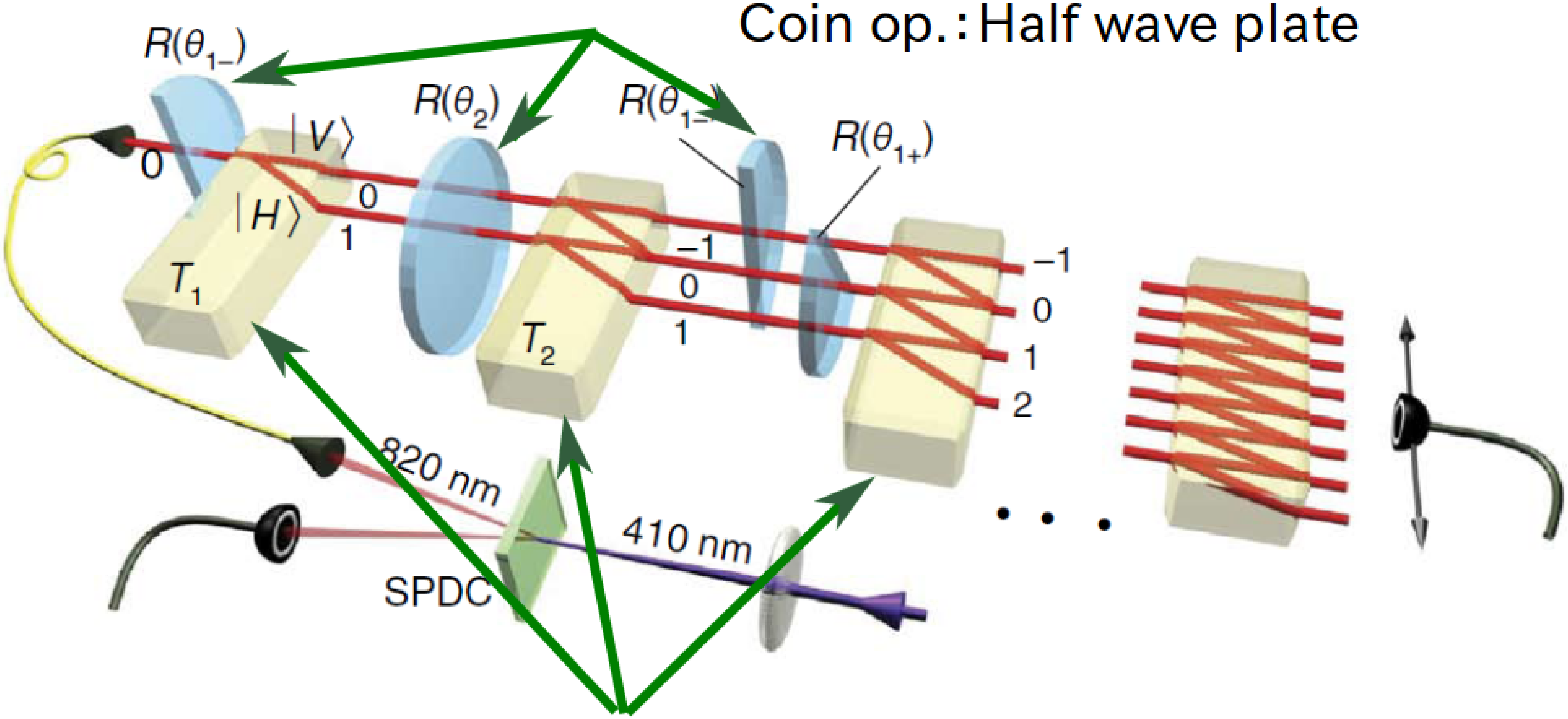
Experiment: bulk optics with photons

internal states :

vertical & horizontal polarizations

Kitagawa, Broome, Fedrizzi *et al.*, Nature Comm. **3**, 882 ('12)

Coin op.：Half wave plate

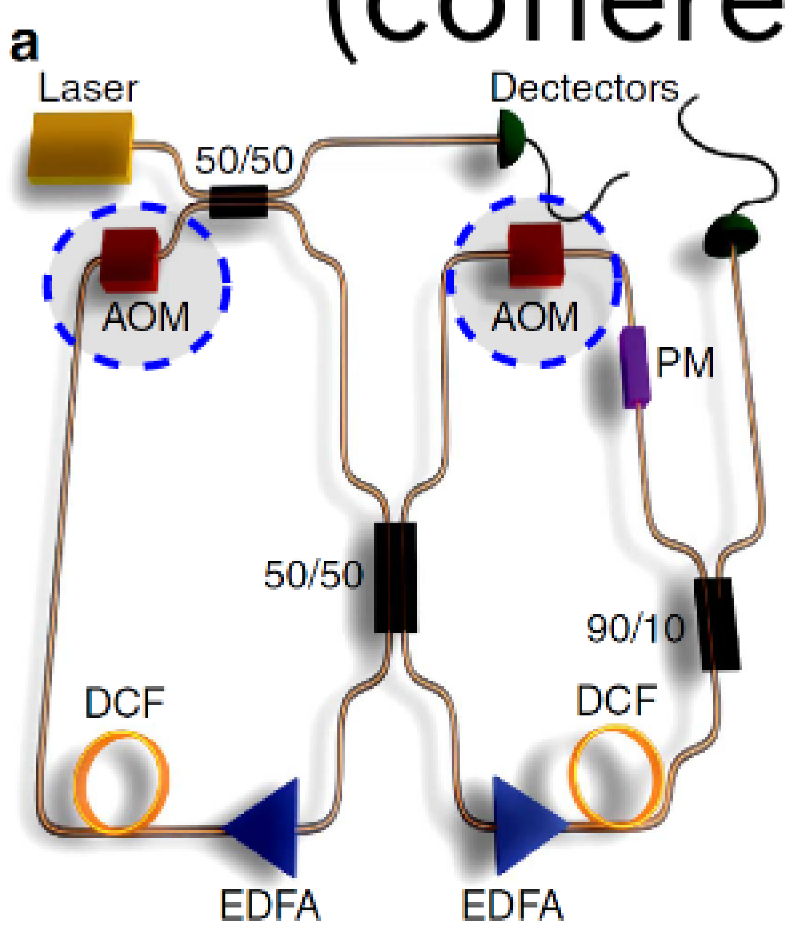


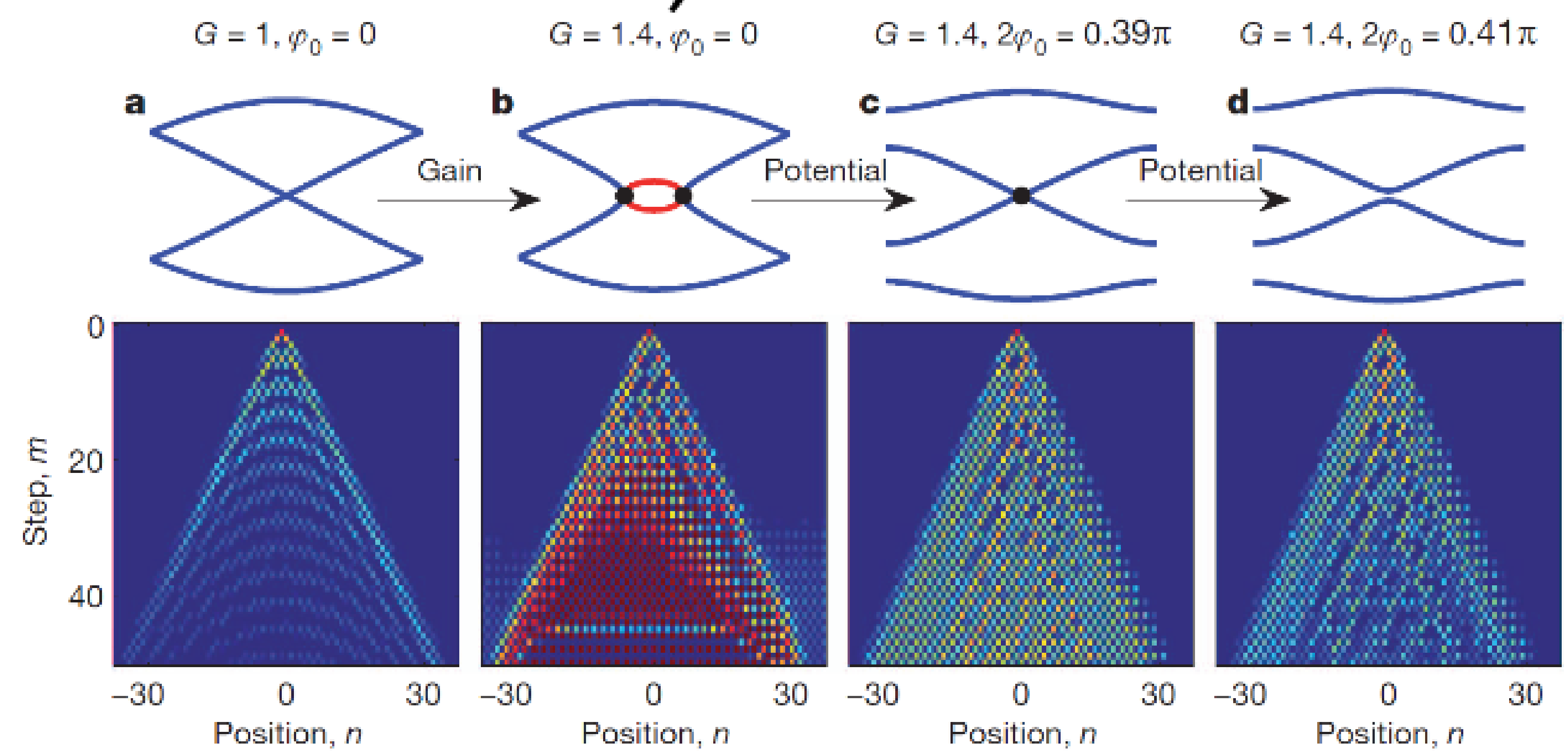
Shift op.：Beam displacer

Experiment: optical fiber loop with laser

Regensburger, Bersch, Miri et al., Nature **488**, 167 ('12)

Optical fiber loops:
(coherent classical laser)





A way to define symmetry of unitary QWs

Symmetry for topological phases

- Time-reversal symm.

$$\mathcal{T} H \mathcal{T}^{-1} = H$$

$\mathcal{T}$: anti-unitary

$$\mathcal{T}^2 = \pm 1$$

- Particle-hole symm.

$$\Xi H \Xi^{-1} = -H$$

Ξ : anti-unitary

$$\Xi^2 = \pm 1$$

- Chiral symm.

$$\Gamma H \Gamma^{-1} = -H$$

Γ : unitary

Classification table:

universality class		TRS	PHS	chiral symmetry	$d=1$	$d=2$	$d=3$
Standard	A	0	0	0	-	$\mathbb{Z}$	-
(Wigner-Dyson)	AI	+1	0	0	-	-	-
	AII	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	AIII	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
	BDI	+1	+1	1	$\mathbb{Z}$	-	-
	CII	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
	C	0	−1	0	-	$\mathbb{Z}$	-
	DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	+1	−1	1	-	-	$\mathbb{Z}$

Symmetry relations for QWs

- Time-evolution operator & Hamiltonian

$$U = \exp(-iH_F)$$
$$= 1 - iH_F - \frac{1}{2}H_F^2 + \frac{i}{3}H_F^3 + \cdots$$
$$1 + iH_F - \frac{1}{2}H_F^2 - \frac{i}{3}H_F^3 + \cdots = \exp(+iH_F) = U^{-1}$$

- Time-reversal symmetry (TRS)

$$\mathcal{T} H \mathcal{T}^{-1} = H \quad \Rightarrow \quad \mathcal{T} U \mathcal{T}^{-1} = U^{-1}$$

- Particle-hole symmetry (PHS)

$$\Xi H \Xi^{-1} = -H \quad \Rightarrow \quad \Xi U \Xi^{-1} = U$$

- Chiral symmetry

$$\Gamma H \Gamma^{-1} = -H \quad \Rightarrow \quad \Gamma U \Gamma^{-1} = U^{-1}$$

Naive approach for single step QWs

$$U = SC(\theta)$$

- Particle-hole symmetry:

$$\Xi U \Xi^{-1} = U$$

$$\Xi S \Xi^{-1} \cdot \Xi C(\theta) \Xi^{-1} = S \cdot C(\theta)$$

$$\Rightarrow \quad \Xi S \Xi^{-1} = S, \qquad \Xi C(\theta) \Xi^{-1} = C(\theta)$$

$$\Rightarrow \Xi = \sum_x |x\rangle\langle x| \otimes K$$

Global operator

K : complex conjugation

$$C[\theta(x)] = \sum_x |x\rangle\langle x| \otimes \mathcal{R}[\theta(x)]$$

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$S = \sum_x \Big(|x+1\rangle\langle x| \otimes |R\rangle\langle R| + |x-1\rangle\langle x| \otimes |L\rangle\langle L| \Big)$$

$$\Rightarrow \Xi = \sum_x |x\rangle\langle x| \otimes K$$ Global operator

K : complex conjugation

- Time-reversal symmetry / chiral symmetry:

$$\mathcal{T} U \mathcal{T}^{-1} = U^{-1}$$

$$\underline{\mathcal{T} S \mathcal{T}^{-1}} \cdot \underline{\mathcal{T} C(\theta) \mathcal{T}^{-1}} = \underline{C}^{-1}(\theta) \cdot \underline{S}^{-1}$$

$$\Rightarrow \mathcal{T} S \mathcal{T}^{-1} = C^{-1}(\theta)$$

$$\Rightarrow \mathcal{T} = \sum_{x,x'} |x'\rangle\langle x| t_{x,x'}(\theta) K$$ Local operator

Problem:
Local operator is fragile for local perturbation.

Identify chiral symmetry of QWs

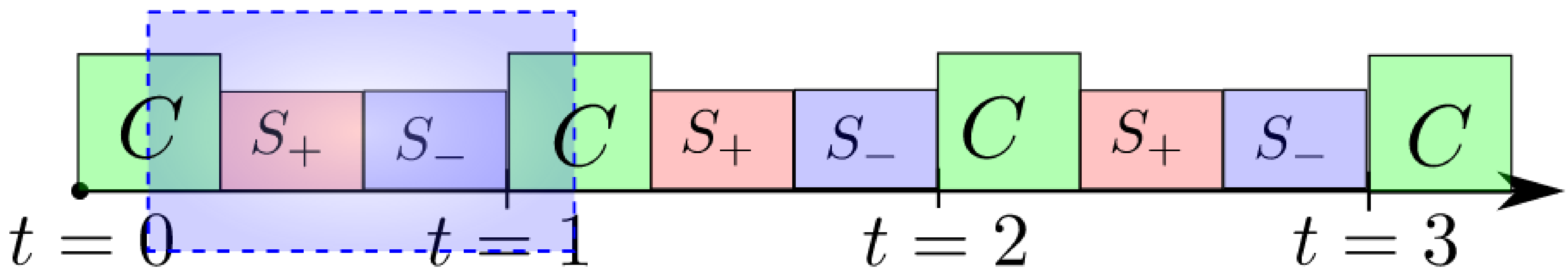
single-step: $U = SC(\theta) = S_+ S_- C(\theta)$

Asboth, HO, PRB 88 121406 ('13)

$$S = S_- S_+ \qquad \begin{cases} S_+ = \sum_x |x\rangle\langle x| \otimes |L\rangle\langle L| + |x+1\rangle\langle x| \otimes |R\rangle\langle R| \\ S_- = \sum_x |x-1\rangle\langle x| \otimes |L\rangle\langle L| + |x\rangle\langle x| \otimes |R\rangle\langle R| \end{cases}$$

- Symmetry time frame

shifting the origin of time!



$$U' = C(\theta/2) S C(\theta/2)$$

Chiral symmetry : $\Gamma U' \Gamma^{-1} = U'^{-1}$

LHS : $(\Gamma C(\theta/2)\Gamma^{-1})(\Gamma S \Gamma^{-1})(\Gamma C(\theta/2)\Gamma^{-1})$

RHS : $C^{-1}(\theta/2) \cdot S^{-1} \cdot C^{-1}(\theta/2)$

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RHS : $C^{-1}(\theta/2)\cdot S^{-1}\cdot C^{-1}(\theta/2)$

1. Condition for the coin op.

$$\Gamma C(\theta)\Gamma = C^{-1}(\theta)$$

2. Condition for the shift op.

$$\Gamma S\Gamma^{-1} = S^{-1}$$

$$\Gamma = \sum_x |x\rangle\langle x| \otimes \sigma_1$$

$$\mathcal{T} = \Gamma\Xi = \sum_x |x\rangle\langle x| \otimes \sigma_1 K$$

Global operators!

$$\begin{aligned} C(\theta) &= e^{-i\theta\sigma_2} \\ S &= e^{-ik\sigma_3} \end{aligned}$$

$$U = SC(\theta)$$

belongs to class BDI

Assuming

$$U = A \cdot B$$

A, Γ are given

Then

$B = \Gamma A^{-1} \Gamma^{-1}$ gives $U = A \cdot B$ with chiral symmetry.

$$\Gamma U \Gamma^{-1} = \Gamma A \Gamma^{-1} \cdot \Gamma B \Gamma^{-1} = B^{-1} \cdot A^{-1} = U^{-1}$$

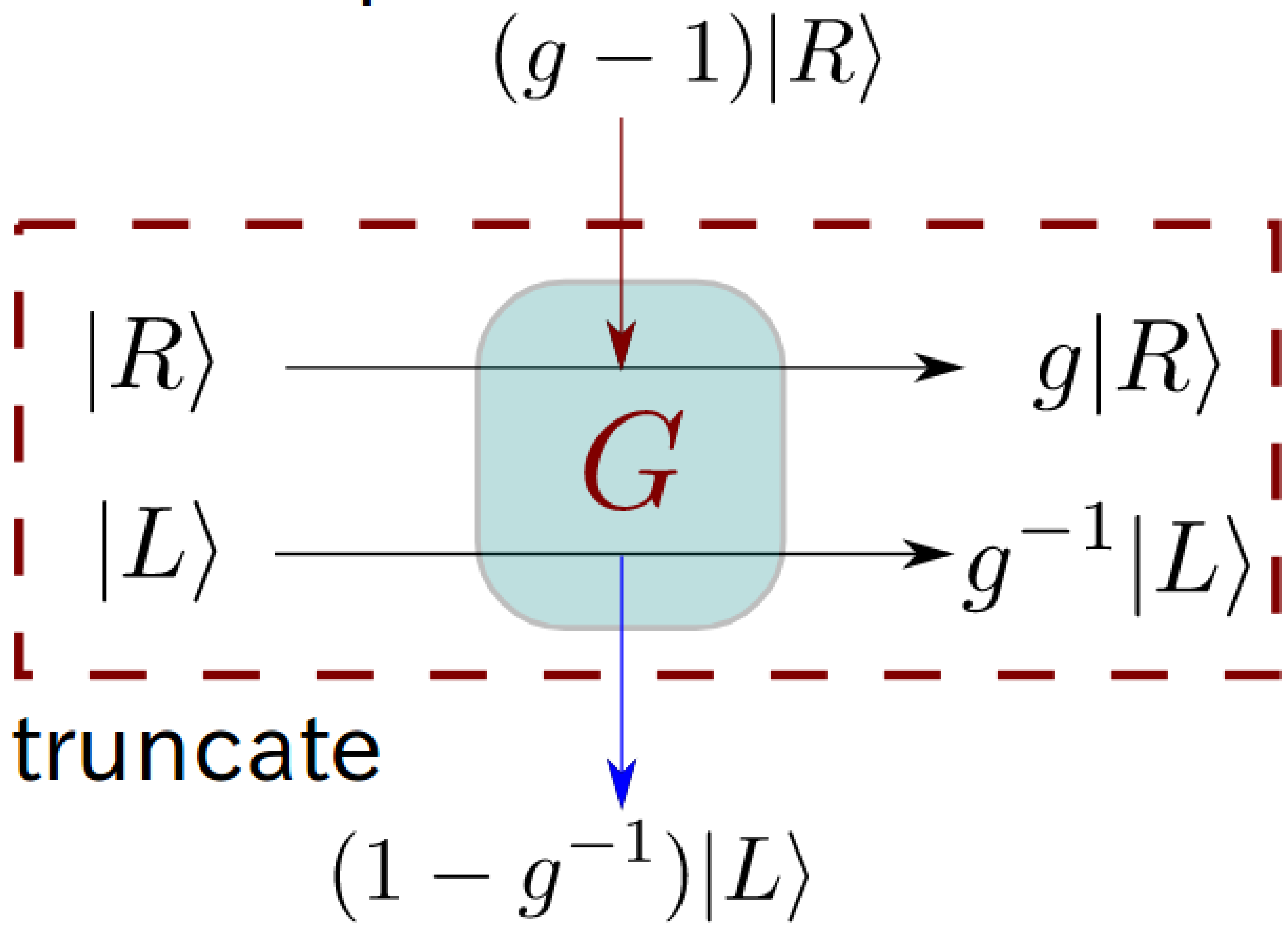
PT symmetric non-unitary QWs

Non-unitary QWs with Gain & Loss

- (phenomena logical)gain & loss operators:

$$G = \sum_x |x\rangle\langle x| \otimes \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}$$

$$(g > 1)$$



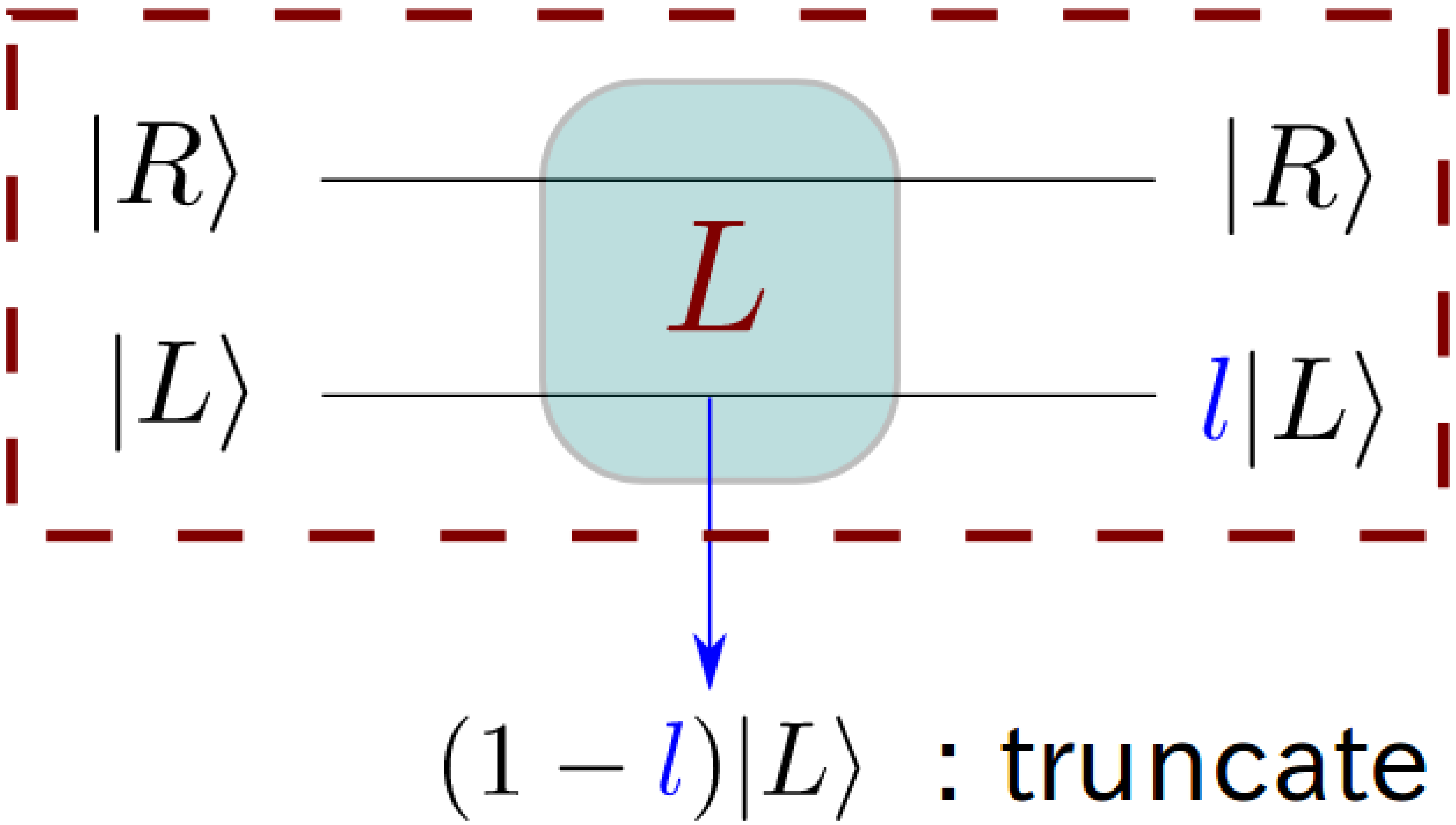
Non-unitary QWs with Loss

- Loss operators:

$$L = \sum_x |x\rangle\langle x| \otimes \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$$

$$L' = \sum_x |x\rangle\langle x| \otimes \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$$

$$(l \leq 1)$$



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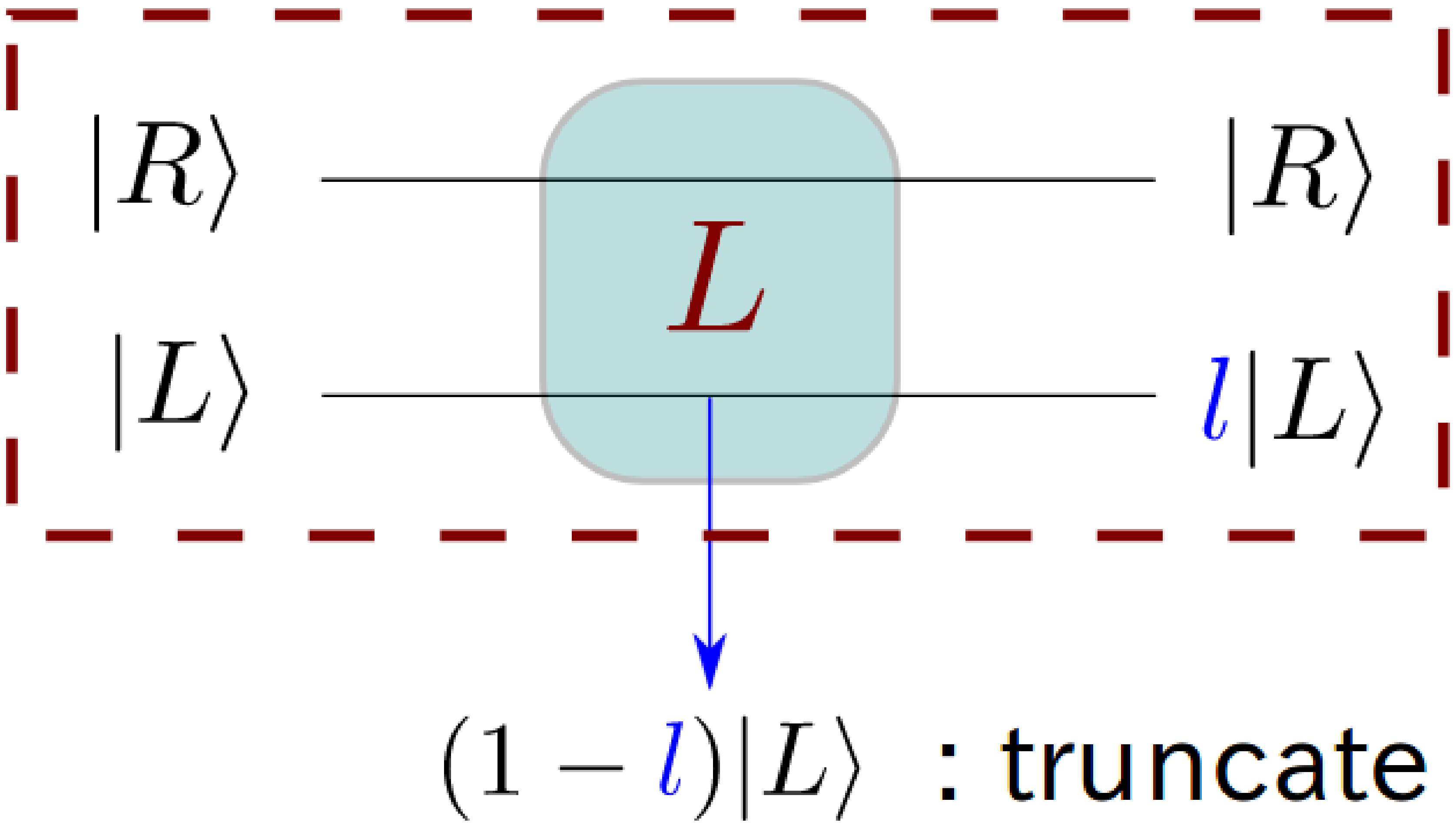

- Non-unitary time-evolution operator:

$$U_{ll} = L' S C(\theta_2) L S C(\theta_1)$$

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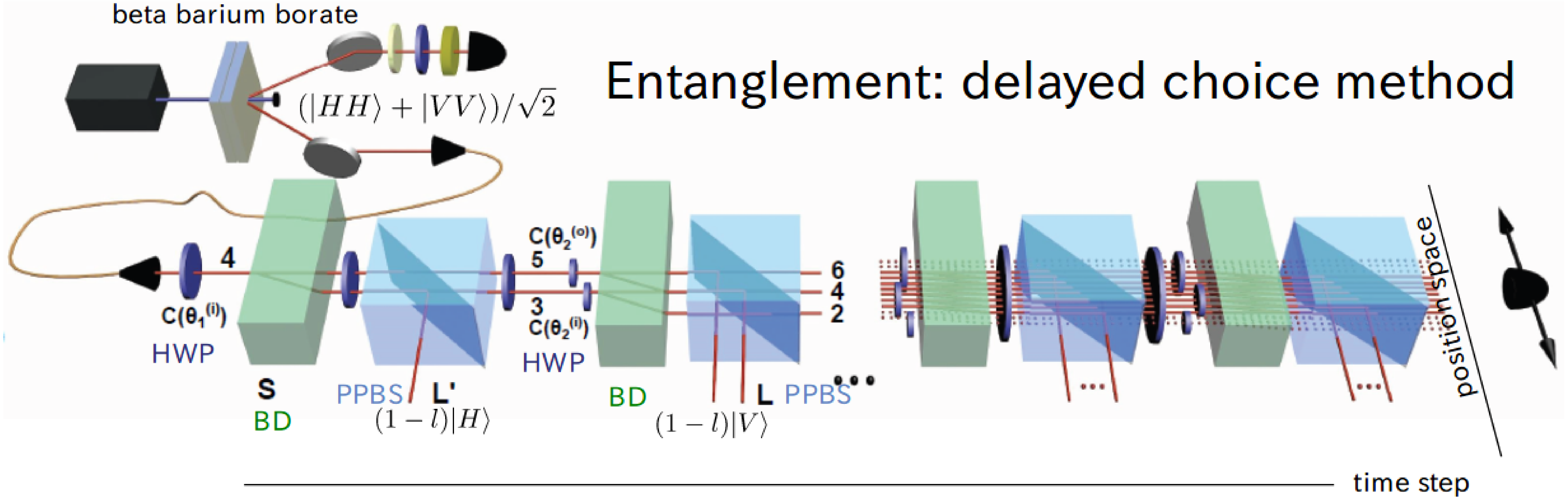
$$U_{ll} = L' S C(\theta_2) L S C(\theta_1)$$

Experiment

Nature Physics 13, 1117 (2017)

a pair of entangled photons

Peng Xue's group (Southeast U., China)



Identifying PT symmetry

$$U_{ll} = L'SC(\theta_2)LSC(\theta_1)$$

- Shifting origin of imaginary energy: $U_{ll} \xrightarrow{\times e^{\varepsilon_0}(=1/\ell)} U_{gl}$

$$U_{gl} = G^{-1}SC(\theta_2)GSC(\theta_1)$$

- Shifting origin of time: $U_{gl} \to U'_{gl}$
  (symmetry time frame)

- condition of $\mathcal{PT}$ symmetry

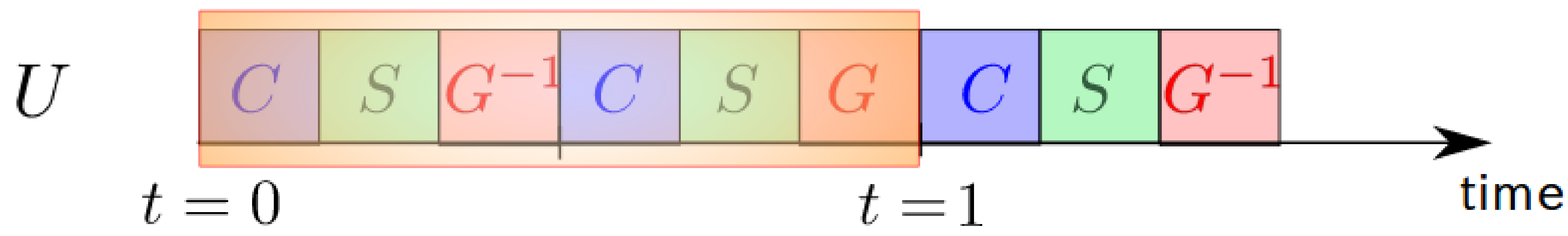
$$\mathcal{P} = \sum_x |-x\rangle\langle x| \otimes \tilde{\mathcal{P}}$$

$$(\mathcal{PT})H(\mathcal{PT})^{-1} = H$$

$$\exp(-iH) = U$$

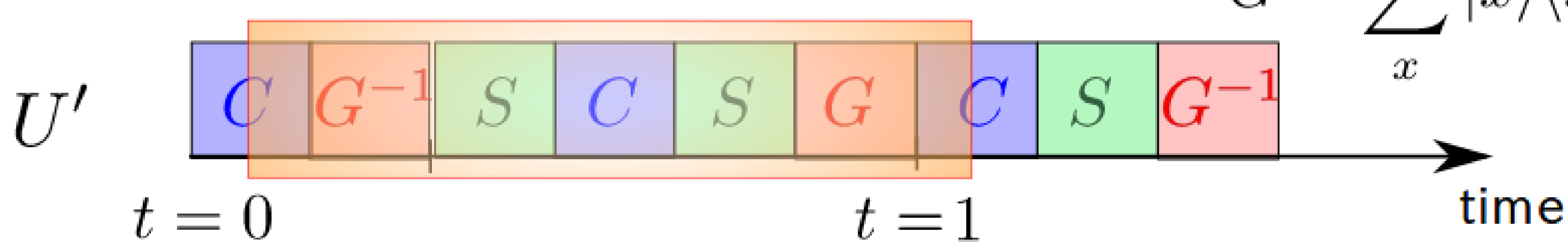
$$\Rightarrow \quad (\mathcal{PT})\,U\,(\mathcal{PT})^{-1} = U^{-1}$$

- Fitting to the symmetry time frame:



$$U^{-1} = G^{-1}S^{-1}C^{-1}GS^{-1}C^{-1}$$

$$G = \sum_x |x\rangle\langle x| \otimes \begin{pmatrix} l^{1/2} & 0 \\ 0 & l^{-1/2} \end{pmatrix}$$



$$U'^{-1} = C(\theta_1/2)^{-1}S^{-1}G^{-1}C(\theta_2)^{-1}GS^{-1}C(\theta_1/2)^{-1}$$

- In the momentum space (homogeneous system)

$$C(\theta) = e^{i\theta\sigma_2}$$
$$G = e^{\gamma\sigma_3}$$
$$S(k) = e^{ik\sigma_3}$$

$$\mathcal{PT}\,U'(\mathcal{PT})^{-1} = U'^{-1}$$

Presence of $\mathcal{PT}$ symmetry!

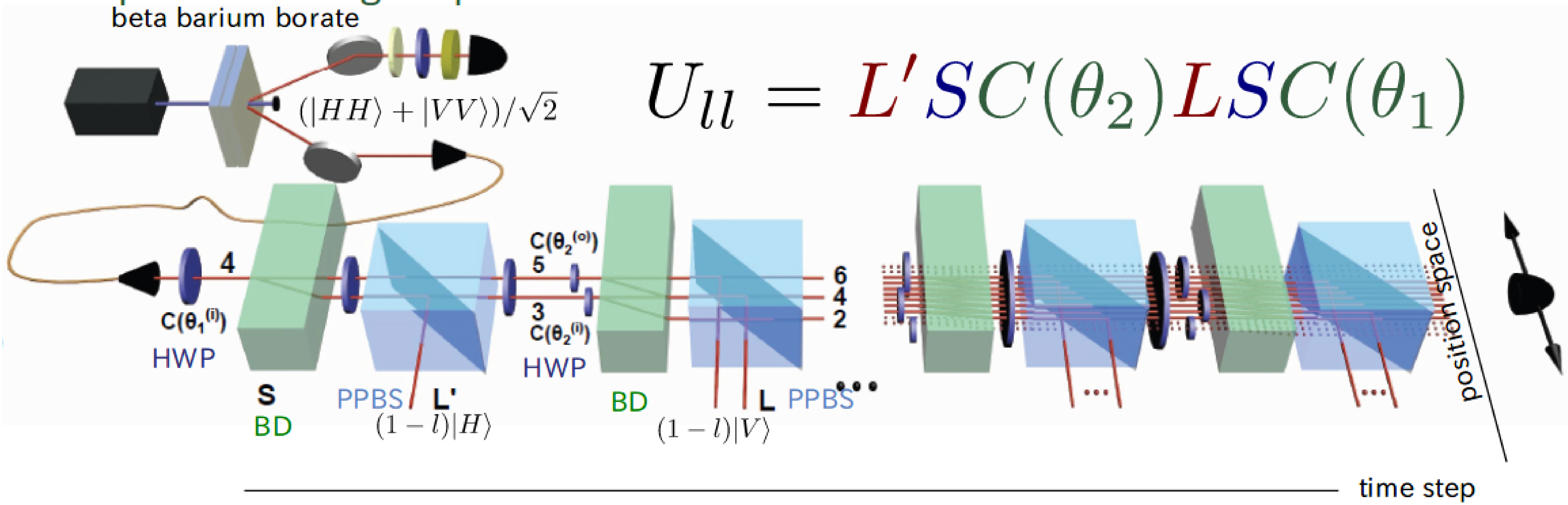
$$\mathcal{PT} = \sum_x |-x\rangle\langle x| \otimes \sigma_3 K$$

Experimental result

Experiment

Nature Physics 13, 1117 (2017)

a pair of entangled photons



Parameters:

$$\theta_1, \theta_2, l = 0.8, 0.64$$

Raw probability

$$P_{\rm raw}(x,t) = \frac{\text{\#detected photons at x and t}}{\text{\# photon pairs}}$$

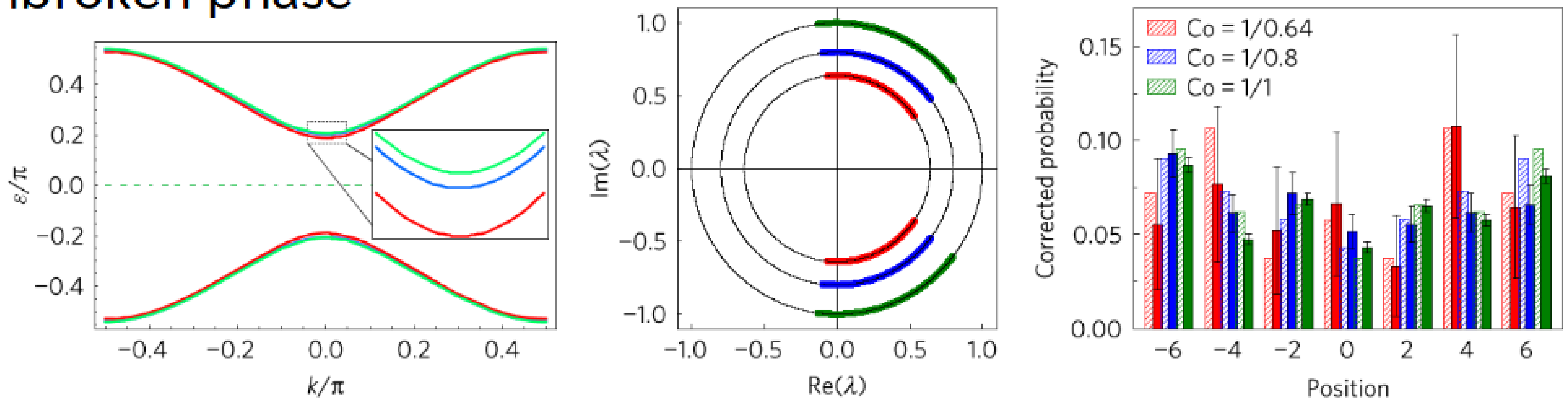
up to 6 time step

Delayed-choice of the initial state for entangled photon pairs

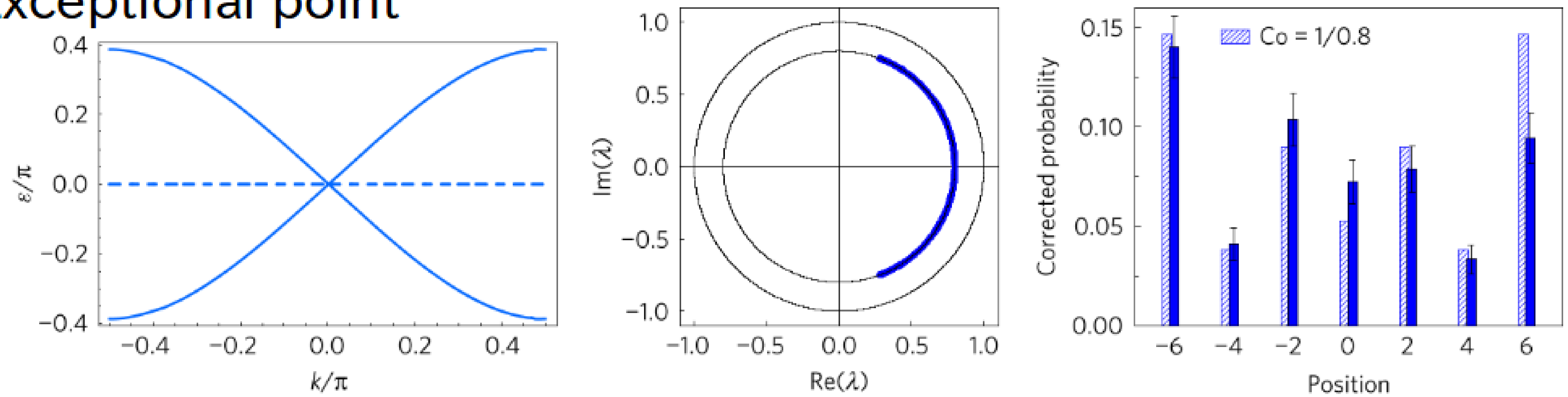
~10,000 photons

Result: homogeneous systems

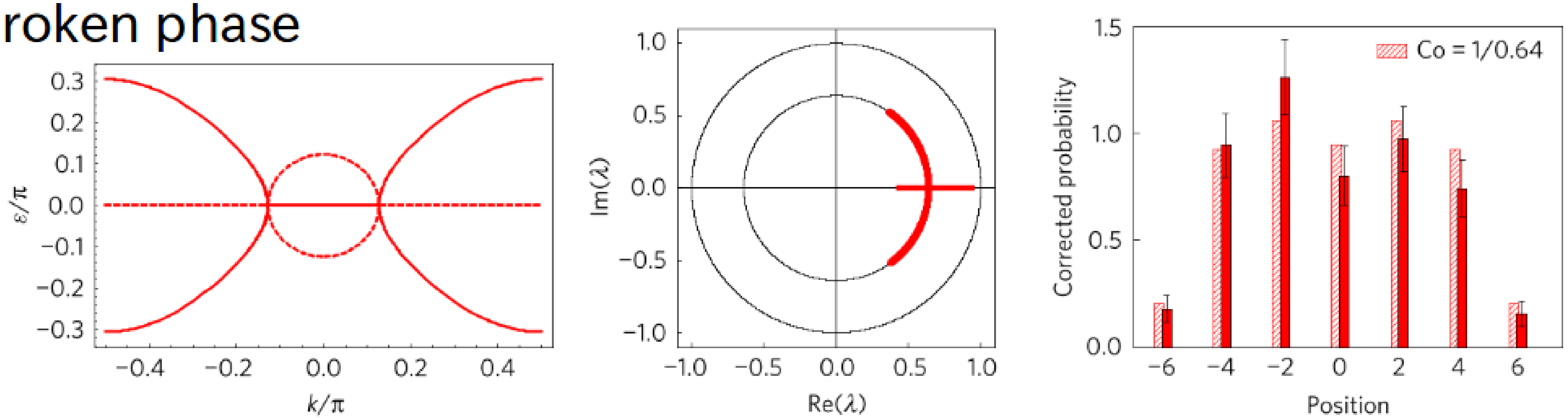
Unbroken phase



Exceptional point

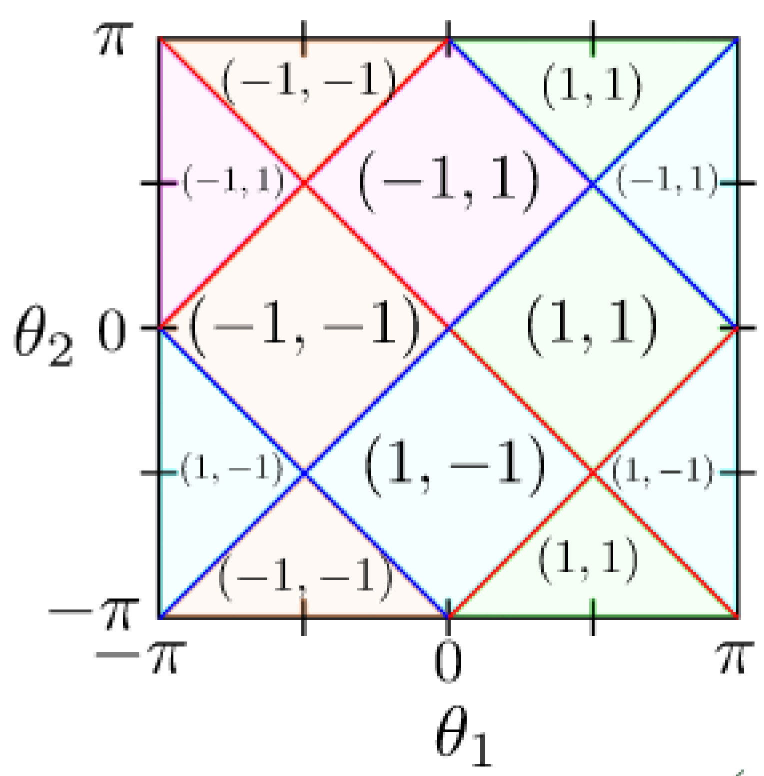


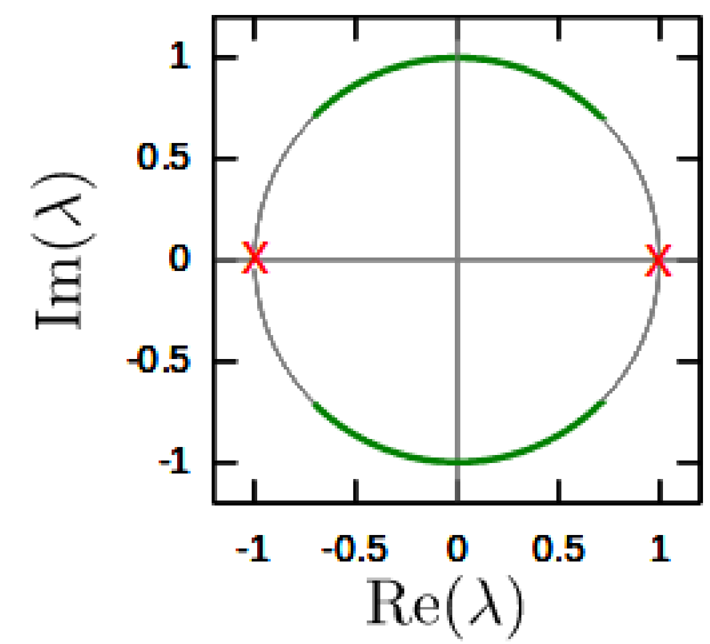
Broken phase

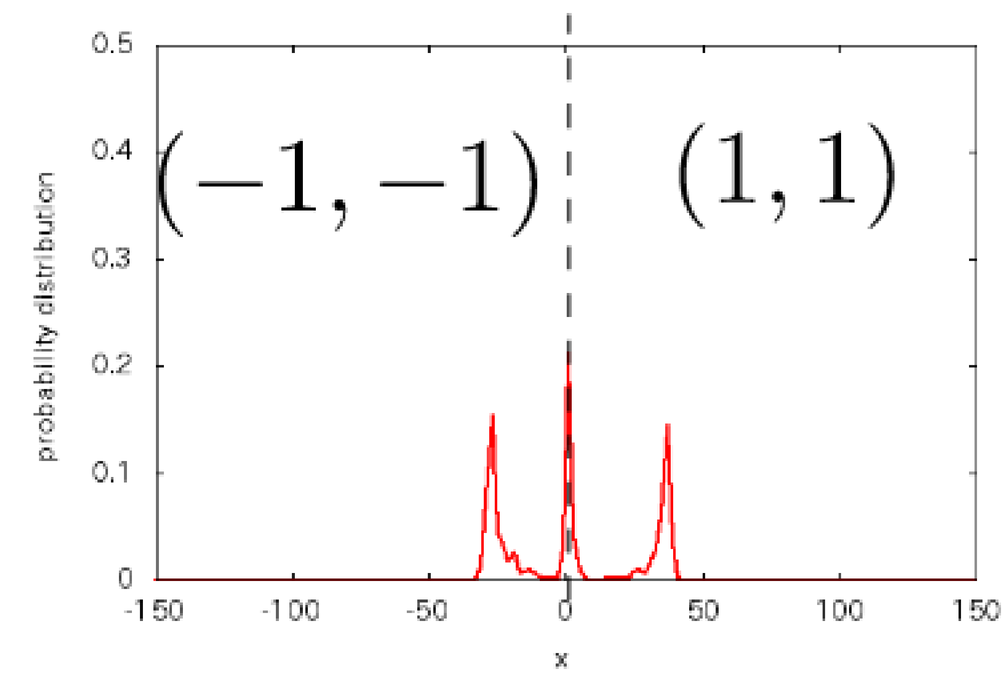


Floquet Topological Phases in QWs

- Unitary two-step QW $U = SC(\theta_2) \cdot SC(\theta_1)$

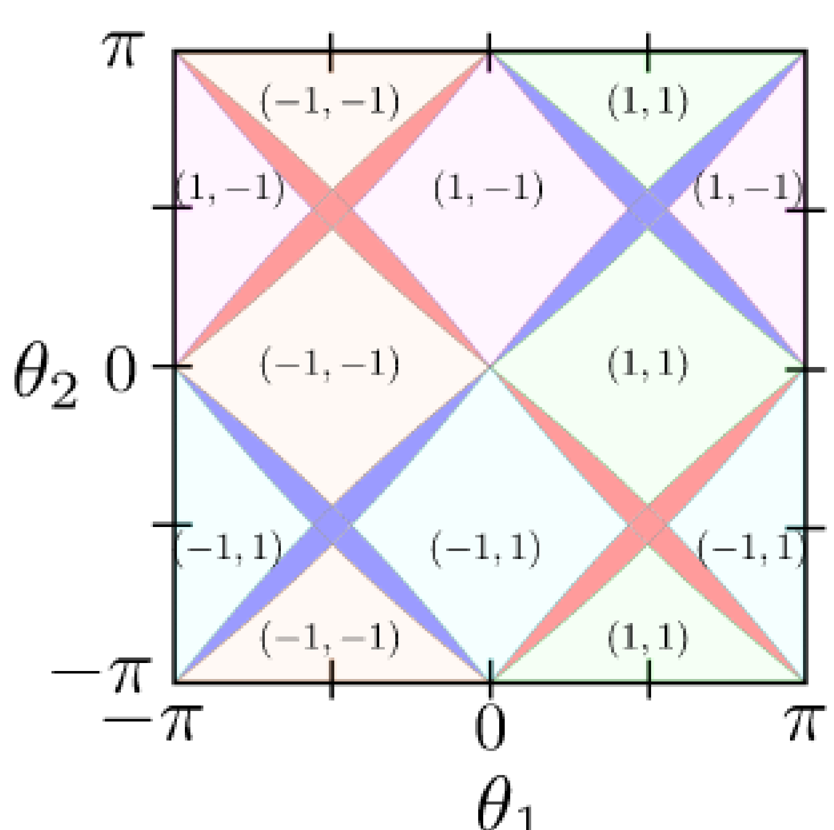




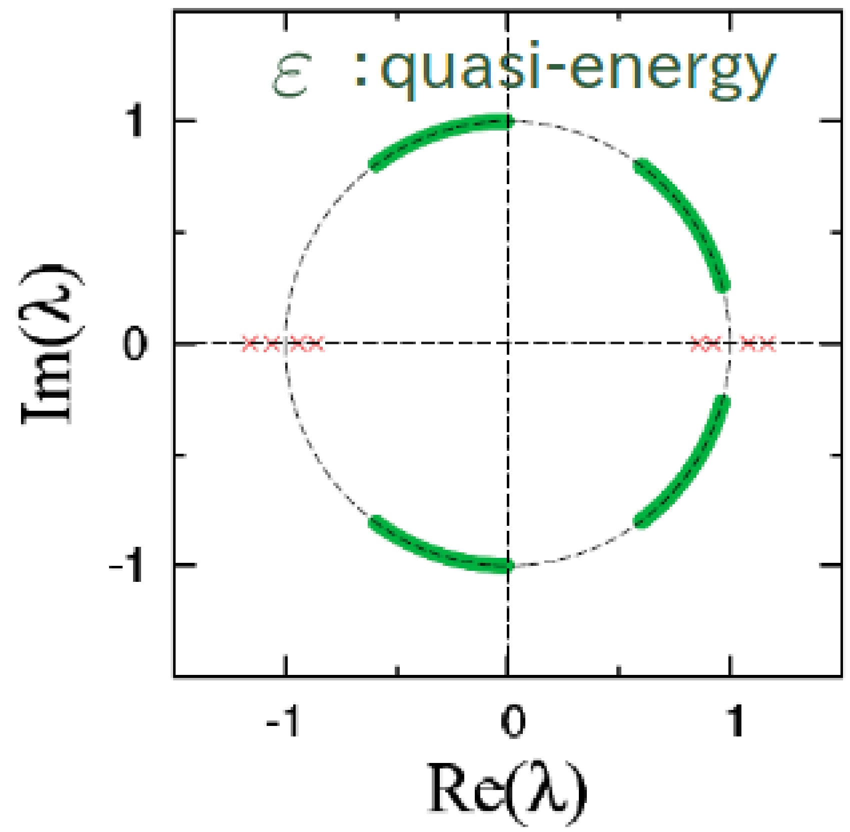


Topological numbers (ν_0, ν_π):

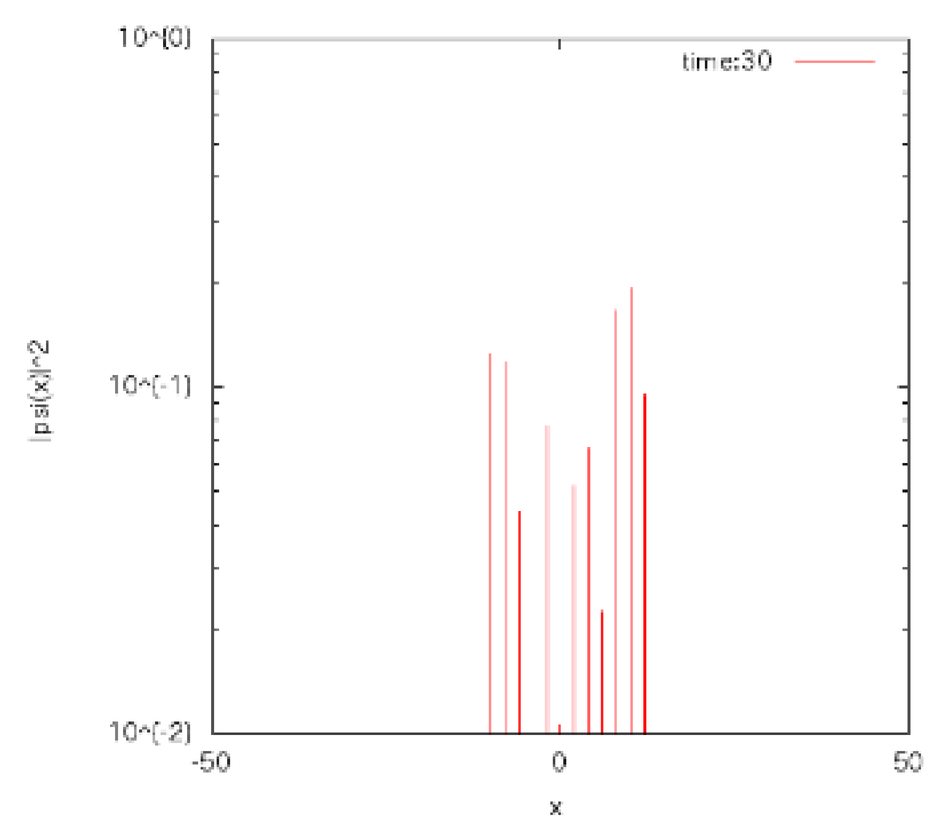
- PT symmetric non-unitary QW $U_{gl} = G^{-1}SC(\theta_2)GSC(\theta_1)$



topological numbers
(ν_0, ν_π)

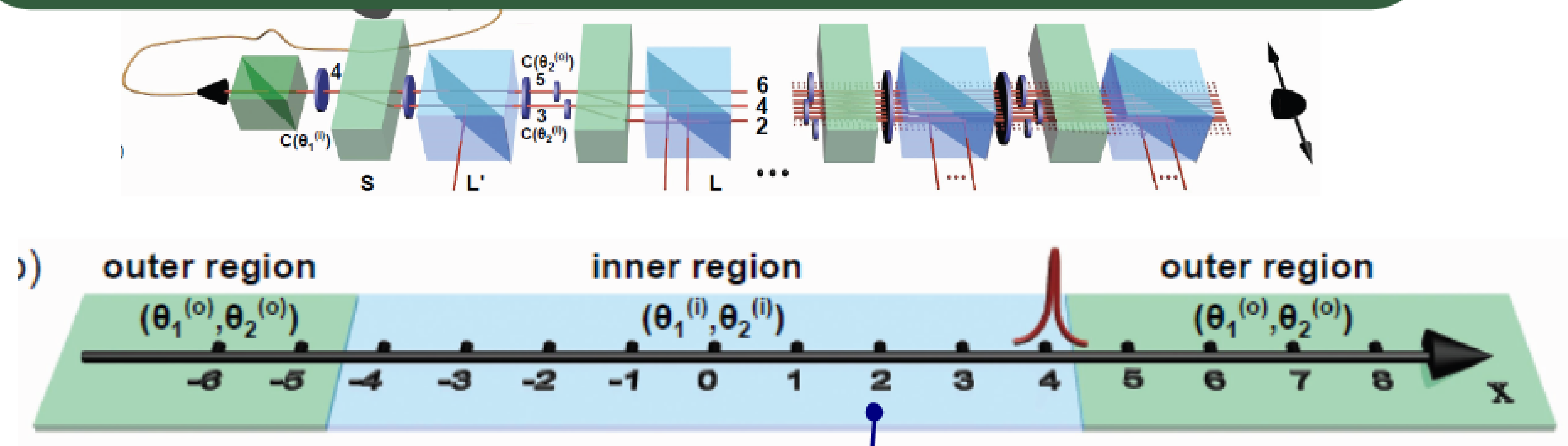


bulk-edge correspondence

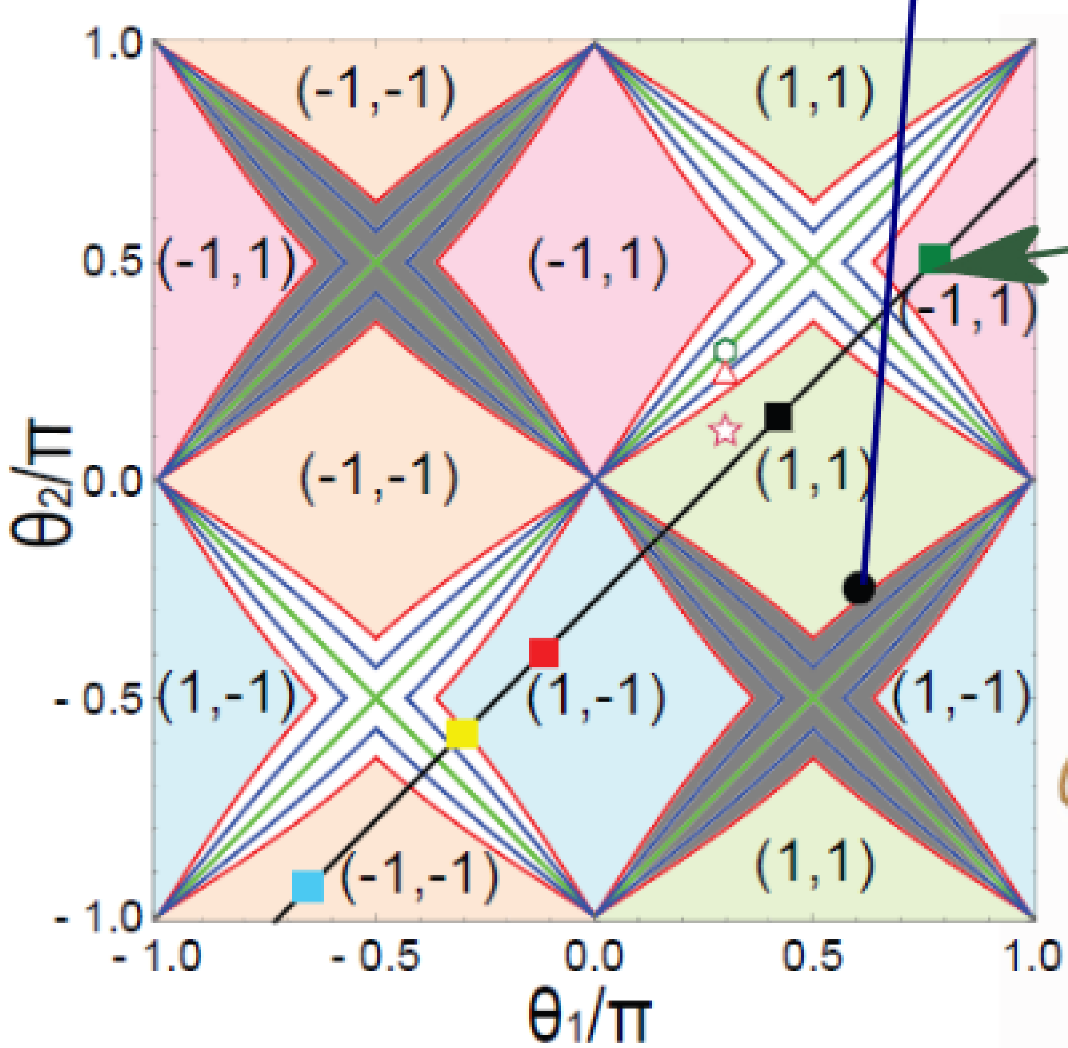


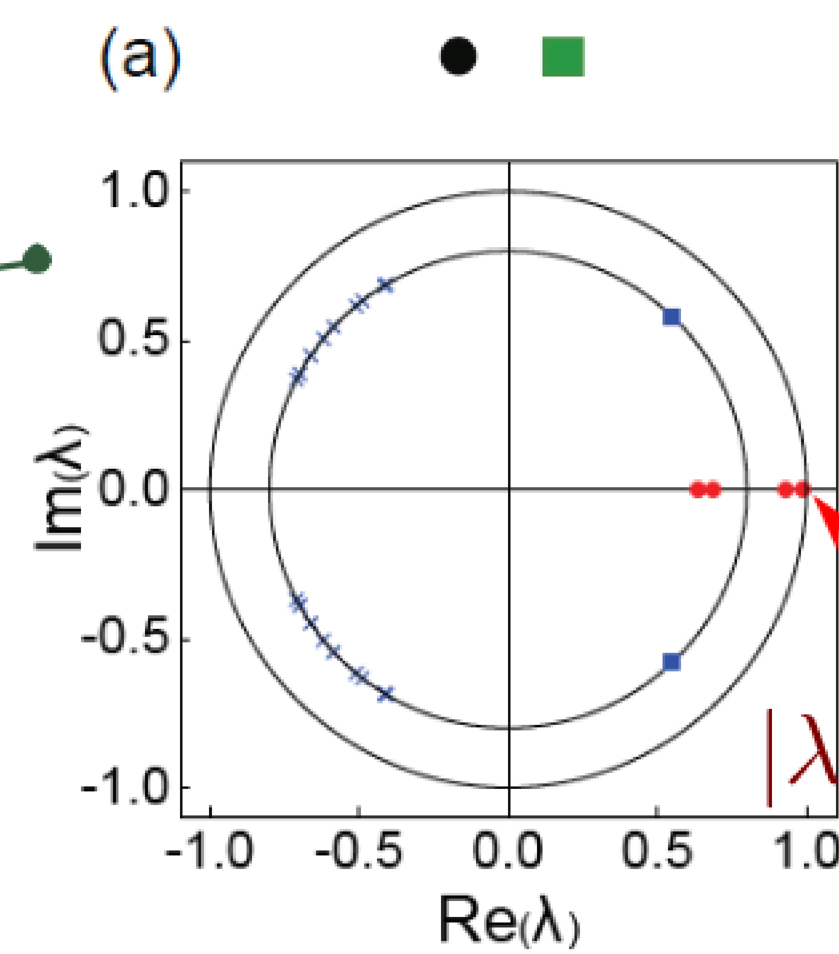
enhancement of wavefunction amplitude

Experimental result : edge states



Topological numbers (ν_0, ν_π):

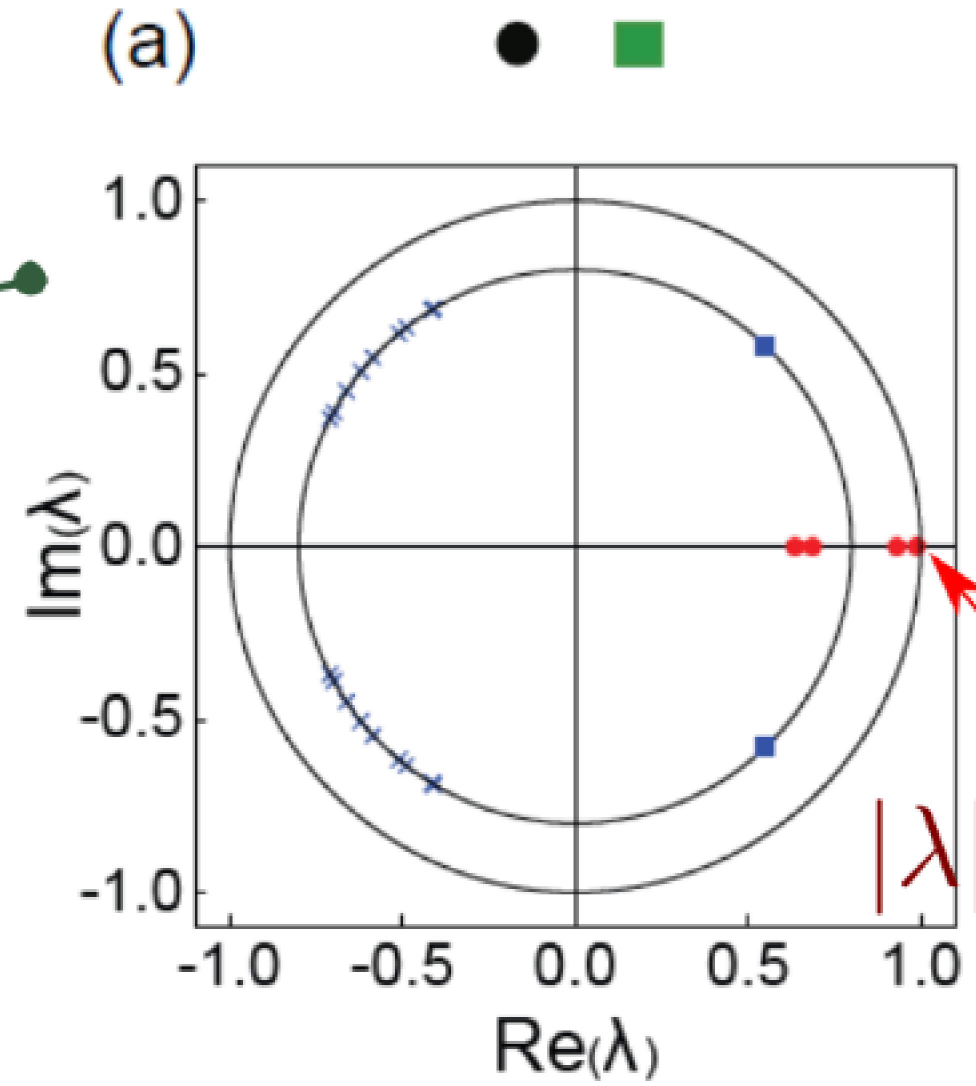




$|\lambda| = 0.9893..$

≈ 1

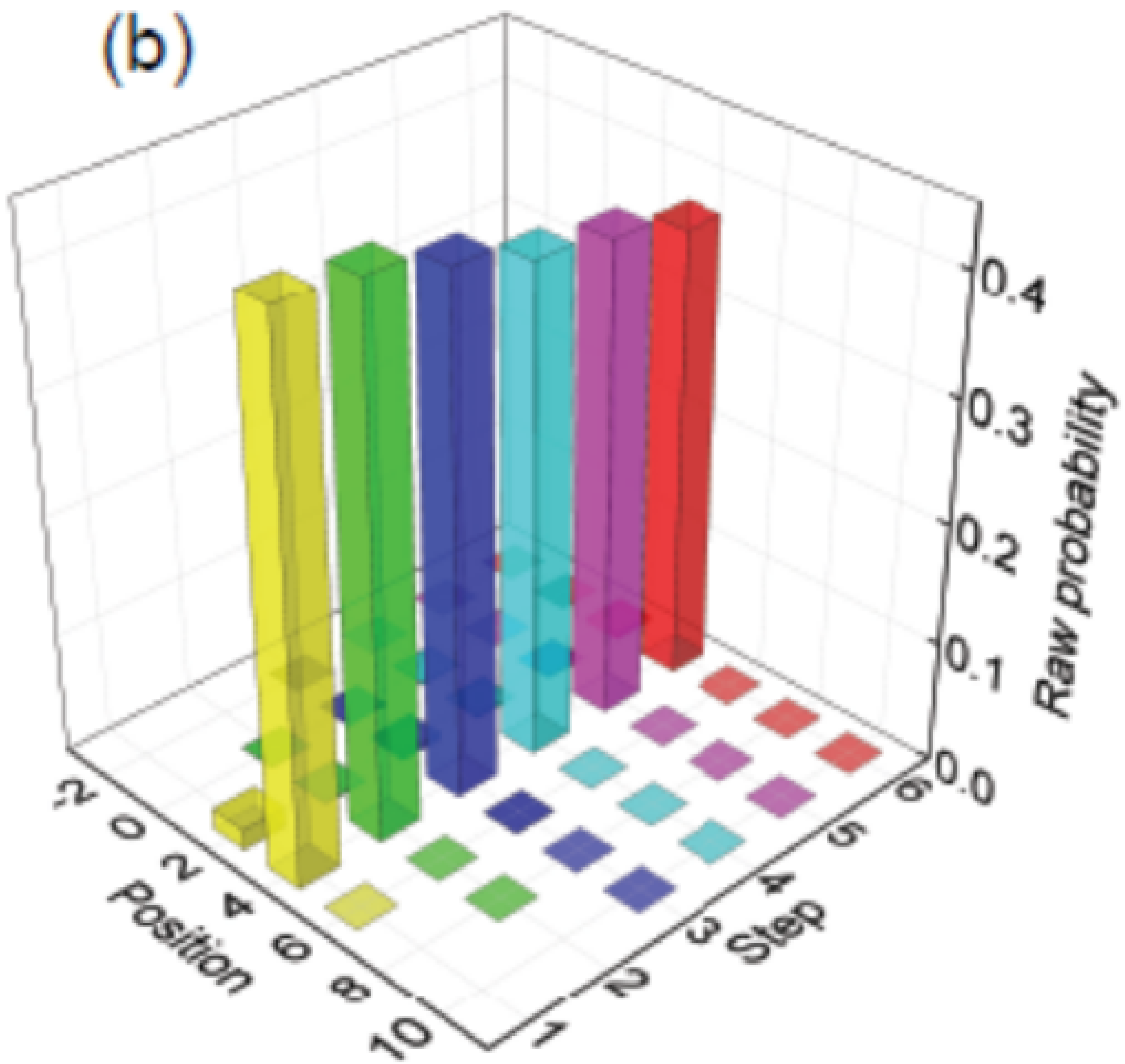
Survival prob. for bulk states :



$|\lambda| = 0.9893...$

≈ 1

$P_{\mathrm{raw}}(x,t)$



Survival prob. for bulk states : 3%

$$l^{2\cdot 6}/2 = 0.8^{2\cdot 6}/2 \approx 0.03$$

The higher prob. at x=4 originates from $\mathcal{PT}$symmetry breaking of edge states.

Summary

- Theory : Identifying symmetries & developing a theory for Floquet topological phases in unitary QW and $\mathcal{PT}$ symmetric non-unitary QWs.
- Experiment : Observation of the edge states by using $\mathcal{PT}$ symmetric/pseudo-unitary photonic QW with loss in quantum regime.
- Nice agreement between theoretical predictions and experimental results.

QWs are useful to study open quantum systems

unitary QW:

PRB 84, 195139 (2011).
PRB 88, 121406(R) (2013).
PRB 92, 045424 (2015).

$\mathcal{PT}$ symmetric non-unitary QW:

PRA 93, 062116 (2016).
IIS 23, 95 (2017) [arXiv:1608.00719]
arXiv:1609.09650.
Nature Physics 13, 1117 (2017)