PT phase transition in a Non-Hermitian gauge theory

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Outlines

1. Toy model of non-Hermitian complex scalars
2. Review-SU(N) QCD in the new Quadratic gauge
3. PT phase transition in the non-Hermitian gauge theory
4. C-symmetry and its explicit representation in this non Abelian model
5. Concluding discussions

- The model is useful to set the mathematical preliminaries for the later non Abelian non-Hermitian model.
- This model is described by the following Lagrangian

$$L = \partial_\mu \phi_1^* \partial_\mu \phi_1 + \partial_\mu \phi_2^* \partial_\mu \phi_2 + [\phi_1^* \quad \phi_2^*] M^2 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{1}$$

where

$$M^2 = \begin{bmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2. \end{bmatrix} \tag{2}$$

with $m_1^2, m_2^2, \mu^2 \geq 0$. We see that the mass matrix M^2 is not Hermitian.
- Defining the doublet of two fields as

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{3}$$

- The parity and time reversal respectively are defined on the doublet as follows

$$\Phi \xrightarrow{\text{P}} P\Phi \tag{4}$$

$$\Phi \xrightarrow{\text{T}} T\Phi^* \tag{5}$$

where complex conjugation in time reversal is due to anti-linearity.
- The parity in $\mathbb{R}^2$ suggests that

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1. \end{bmatrix} \tag{6}$$

- Then, the only choice for the time reversal T for which Eq. (1) remains PT-invariant is $\mathrm{T} = \mathbf{1}_2$ [1].
- The eigenvalues of the mass matrix M^2 are

$$M_\pm^2 = \frac{1}{2}(m_1^2 + m_2^2) \pm \sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4}. \tag{7}$$

- The theory remains in the unbroken PT- symmetric state as long as $M_\pm^2$ remain real.
So, for $|m_1^2 - m_2^2| \geq 2\mu^2 \Rightarrow$ phase of unbroken PT symmetry.

- When $|m_1^2 - m_2^2| < 2\mu^2$ happens, we step into the region of broken PT-symmetry as eigenvalues turn complex and $\text{PT}\psi_\pm \neq \pm\psi_\pm$, where $\psi_\pm$ are eigenfunctions of the mass matrix M^2 corresponding to eigenvalues $M_\pm^2$.
- The charge conjugation of this system is defined as follows

$$\Phi \quad \xrightarrow{\text{C}} \quad C\Phi^* \tag{8}$$

  with C=P [1].
- The theory is CPT invariant in both PT broken and unbroken phases. In broken PT phase, the theory violates CP also but preserves CT symmetry.

The quadratic gauge : $A^a_\mu(x)A^{\mu a}(x) = f^a(x);$ for each a [2] (9)

where $f^a(x)$ is an arbitrary function of x.

- USP: (1) This gauge is Lorentz invariant. In general, algebraic gauges are not. (2) It is not Abelian Projection. (3) It has substantial non-perturbative implications such as color confinement signatures and absence of the Gribov ambiguity in this gauge.
- **The effective theory** is given by

$$\mathcal{L}_{eff} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\xi}(A^a_\mu A^{\mu a})^2 - \overline{c^a}A^{\mu a}(D_\mu c)^a \quad (10)$$

where second, third terms are gauge fixing and ghost Lagrangian respectively and $(D_\mu c)^a = \partial_\mu c^a - g f^{abc} A^b_\mu c^c$.

Two phases of QCD in the quadratic gauge

- This theory has two different phases [2]: the normal or deconfined phase and the ghost condensed phase showing the confinement. The Lagrangian in normal phase is given by Eq. (10) itself. Ghosts are auxiliary fields in the normal phase, but play an important role in the IR regime as we discuss now.
- **Ghost condensation**
  The ghost Lagrangian contains a term $gf^{abc}\overline{c^a}c^c A^{\mu a} A^b_\mu$. In this expression, ghost bilinears multiply the terms quadratic in gauge fields. Hence if the ghosts freeze they amount to a non-zero mass matrix for the gluons as follows

$$(M^2)^{ab}_{\text{dyn}} = 2g \sum_{c=1}^{N^2-1} f^{abc} \langle \overline{c^a} c^c \rangle \tag{11}$$

  whereas diagonal components of M^2_{dyn} are zero since $f^{aac} = 0$.
- To obtain masses of gluons, we must diagonalize the matrix and find eigenvalues.

- In the $SU(N)$ symmetric state, where all ghost-anti-ghost condensates are identical i.e.,

$$\langle \overline{c^1} c^1 \rangle = ... = \langle \overline{c^1} c^{N^2-1} \rangle = ... = \langle \overline{c^{N^2-1}} c^1 \rangle = ... = \langle \overline{c^{N^2-1}} c^{N^2-1} \rangle \equiv K \tag{12}$$

the mass matrix becomes

$$(M^2)^{ab}_{\text{dyn}} = 2gK \sum_{c=1}^{N^2-1} f^{abc} \tag{13}$$

- The resulting mass matrix for the gluons has $N(N-1)$ non-zero eigenvalues only and thus has nullity $N-1$. Because of the antisymmetry, eigenvalues occur are purely imaginary and in conjugate pairs.
- Thus, the $N(N-1)$ off-diagonal gluons acquire masses and the rest $N-1$ diagonal gluons remain massless. Thus, pole of the propagator for the off-diagonal gluon is on imaginary p^2 axis, which a signals the quark confinement [3]. The massive off-diagonal gluons are presumed to provide evidence of Abelian dominance, which is one of signature of quark confinement.

- Therefore, in the ghost condensed phase the Lagrangian can effectively be given as follows

$$\mathcal{L}_{GC} = - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\zeta}(A^a_\mu A^{\mu a})^2 + M_a^2 A^a_\mu A^{\mu a} \qquad (14)$$

Here $M_a^2 = 0$ when a indexes the diagonal gluons, e.g, for $SU(3)$, $M_3^2 = M_8^2 = 0$. While for the off-diagonal gluons, $M_1^2 = +im_1^2, M_2^2 = -im_1^2$, $M_4^2 = +im_2^2, M_5^2 = -im_2^2$, $M_6^2 = +im_3^2, M_7^2 = -im_3^2$ (m_1^2, m_2^2, m_3^2 are positive real). So the gluons 1 and 2 can be considered as conjugate of each other. The same is true for other pairs.

- Hence for $SU(3)$, the last term of the effective Lagrangian in Eq. (14) would be

$$\begin{aligned} M_a^2 A^a_\mu A^{\mu a} = & + im_1^2 A^1_\mu A^{\mu 1} - im_1^2 A^2_\mu A^{\mu 2} + im_2^2 A^4_\mu A^{\mu 4} - im_2^2 A^5_\mu A^{\mu 5} \\ & + im_3^2 A^6_\mu A^{\mu 6} - im_3^2 A^7_\mu A^{\mu 7} \qquad (15) \end{aligned}$$

- PT phase transition has not been explored in gauge theories. Here we show that the non Abelian non Hermitian gauge theory of interest exhibits the PT phase transition.
- **Hermiticity of the two phases**
  The effective theory in the normal phase is given in Eq. (10)

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\zeta}(A^a_\mu A^{\mu a})^2 - \overline{c^a}A^{\mu a}(D_\mu c)^a \tag{16}$$

- Gluons must be Hermitian i.e.,

$$A^{a\dagger}_\mu = A^a_\mu \tag{17}$$

- As the operation of conjugation in principle transforms particle to its anti particle, the following is the natural choice of hermiticity property for ghosts under which the present theory can be cast as non-Hermitian model [4]

$$\begin{aligned} c^{a\dagger} &= \overline{c^a} \\ \overline{c^a}^\dagger &= c^a \end{aligned} \tag{18}$$

- Under Eqs. (17),(18), the normal phase in Eq. (10) is not invariant since the following ghost term is not Hermitian
$$(\overline{c^a}\partial_\mu c^a)^\dagger = (\partial_\mu \overline{c^a})c^a \neq \overline{c^a}\partial_\mu c^a.$$
Rest of the terms are Hermitian.
- The effective Lagrangian in the ghost condensed (confinement) phase (14) is also not Hermitian as the mass term for gluons is purely imaginary as explained.
- Important point : non hermiticity of the theory in this ghost condensed phase is free of the hermiticity convention for ghosts and thus the non hermiticity of the ghost condensed phase is profound.
- The Lagrangian (14) obeys the extended hermiticity [2] i.e., when the following inner automorphisms is applied hermiticity gets restored viz. $\mathfrak{T}\mathcal{L}_{GC}^\dagger\mathfrak{T}^\dagger = \mathcal{L}_{GC}$,
$$\begin{array}{llll} \mathfrak{T}L_1\mathfrak{T}^\dagger = L_2 & \mathfrak{T}L_4\mathfrak{T}^\dagger = L_5 & \mathfrak{T}L_6\mathfrak{T}^\dagger = L_7 & \mathfrak{T}L_3\mathfrak{T}^\dagger = L_8 \\ \mathfrak{T}L_2\mathfrak{T}^\dagger = L_1 & \mathfrak{T}L_5\mathfrak{T}^\dagger = L_4 & \mathfrak{T}L_7\mathfrak{T}^\dagger = L_6 & \mathfrak{T}L_8\mathfrak{T}^\dagger = L_3 \end{array} \quad (19$$

with the property

$$\mathfrak{T}^2 = \mathfrak{T}^{\dagger 2} = 1 \tag{20}$$

where L_i are the individual Lagrangian terms such as $-\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i}$, $-\frac{1}{2\zeta}(A^i_\mu A^{\mu i})^2$, $im^2 A^i_\mu A^{\mu i}$.

- The inner automorphism is essentially exchanging group indices between conjugate gluons i.e., $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$, $3 \leftrightarrow 8$.
- In the adjoint representation it is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- We have thus shown that both the normal and confined phases are non-Hermitian, later being profoundly. Hence, it becomes interesting to discuss state of PT symmetry in this theory.

- **PT symmetry of the theory**
  As in the case of the hermiticity, parity and time reversal properties of the gluons are well defined but not for ghosts. For gluons, parity is given as

$$\begin{aligned} A_i^a(\mathrm{x},t) &\xrightarrow{\mathrm{P}} -A_i^a(-\mathrm{x},t) \\ A_0^a(\mathrm{x},t) &\xrightarrow{\mathrm{P}} A_0^a(-\mathrm{x},t). \end{aligned} \tag{21}$$

  The rule for parity is same for all gluons as it is a linear operator.
- It is easy to see that Lagrangian in the normal phase (10) is invariant under parity if we choose ghosts to be pseudo scalars,

$$\begin{aligned} c^a(\mathrm{x},t) &\xrightarrow{\mathrm{P}} -c^a(-\mathrm{x},t) \\ \overline{c^a}(\mathrm{x},t) &\xrightarrow{\mathrm{P}} -\overline{c^a}(-\mathrm{x},t). \end{aligned} \tag{22}$$

- The time reversal is an anti-linear operation. Since some of the generators of $SU(N)$ are purely imaginary, the time reversal property is not same for all gluons.
- We shall explain it using $SU(3)$ group, further generalization to $SU(N)$ is obvious.

- In $SU(3)$, three generators namely, 2nd,5th and 7th are purely imaginary. Therefore, time reversal for gluons is given by

$$\begin{aligned} A_i^p(\mathrm{x},t) &\xrightarrow{\mathrm{T}} -A_i^p(\mathrm{x},-t) \\ A_0^p(\mathrm{x},t) &\xrightarrow{\mathrm{T}} A_0^p(\mathrm{x},-t), \end{aligned} \tag{23}$$

where index p is $1,3,4,6,8$ and,

$$\begin{aligned} A_i^q(\mathrm{x},t) &\xrightarrow{\mathrm{T}} A_i^q(\mathrm{x},-t) \\ A_0^q(\mathrm{x},t) &\xrightarrow{\mathrm{T}} -A_0^q(\mathrm{x},-t), \end{aligned} \tag{24}$$

where index q is $2,5,7$.

- Therefore, the field strength with any spacetime and group indices can utmost change up to overall negative sign i.e.,

$$F^a_{\mu\nu} \xrightarrow{\mathrm{T}} \pm F^a_{\mu\nu}. \tag{25}$$

- Thus, the action in the normal phase (10) is invariant under time reversal given that the time reversal property for ghosts is defined in the following manner,

$$c^p(\mathrm{x},t) \xrightarrow{\mathrm{T}} ic^p(\mathrm{x},-t)$$
$$\overline{c^p}(\mathrm{x},t) \xrightarrow{\mathrm{T}} i\overline{c^p}(\mathrm{x},-t) \tag{26}$$

and,

$$c^q(\mathrm{x},t) \xrightarrow{\mathrm{T}} c^q(\mathrm{x},-t)$$
$$\overline{c^q}(\mathrm{x},t) \xrightarrow{\mathrm{T}} \overline{c^q}(\mathrm{x},-t) \tag{27}$$

where the description of indices p and q are as above.

- Anti-linearity makes two sets of ghosts transform in a completely different manner.
- Thus, the theory in normal phase is individually both parity and time reversal invariant. This PT symmetry breaks down spontaneously in the confined phase as we explain now.

★ **The theory in the confined phase** is given by Eq. (14),

$$\mathcal{L}_{GC} = \; - \; \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\zeta}(A^a_\mu A^{\mu a})^2 + M^2_a A^a_\mu A^{\mu a}$$

- It is easy to check that parity (21) is still a symmetry. However, the time reversal is broken due to pure complex nature of the mass term,

$$
\begin{aligned}
M_a^2 A_\mu^a A^{\mu a} = \quad & + \quad i m_1^2 A_\mu^1 A^{\mu 1} - i m_1^2 A_\mu^2 A^{\mu 2} + i m_2^2 A_\mu^4 A^{\mu 4} - i m_2^2 A_\mu^5 A^{\mu 5} \\
& + \quad i m_3^2 A_\mu^6 A^{\mu 6} - i m_3^2 A_\mu^7 A^{\mu 7} \quad \xrightarrow{\mathrm{T}} \\
& - \quad i m_1^2 A_\mu^1 A^{\mu 1} + i m_1^2 A_\mu^2 A^{\mu 2} - i m_2^2 A_\mu^4 A^{\mu 4} + i m_2^2 A_\mu^5 A^{\mu 5} \\
& - \quad i m_3^2 A_\mu^6 A^{\mu 6} + i m_3^2 A_\mu^7 A^{\mu 7} \\
& = \quad -M_a^2 A_\mu^a A^{\mu a} \qquad (28)
\end{aligned}
$$

  and also PT$\psi \neq \pm\psi$, where ψs are eigenfunctions of the mass matrix (13). The first two terms of $\mathcal{L}_{GC}$ remain unaffected by the time-reversal.
- Thus, PT symmetry is violated in this phase. We see that the anti symmetric nature of structure constant has led to this breaking.
- Important point : the PT symmetry violation in the confined phase is profound as it is independent of the convention for ghosts.

- Thus, the transition from the normal phase to the confinement phase with $SU(N)$ symmetric ghost condensates can be identified as PT phase transition from unbroken to broken PT phase.
- There is a crucial difference between the non Abelian model and the toy model of complex scalars. Complex scalar theory has the parameter $\eta \equiv \frac{2\mu^2}{|m_1^2 - m_2^2|}$ whose value separates two phases of the PT symmetry in the theory.
- There is no such single order parameter in the non Abelian theory which governs the phase transition. Different ghost bilinears $\overline{c^a}c^c$ (a and c runs over 1 to $N^2 - 1$ independently) gradually condensing to the stated $SU(N)$ symmetric vacuum give rise to the PT phase transition in this non Abelian model.
- Thus, we have presented a gauge theory in which PT phase transition is explicitly shown for the first time.

- In the PT symmetric non Hermitian QM, a C-symmetry is defined to improve the probabilistic interpretation of the PT-inner product and is inherent in all PT symmetric systems hence it becomes essential to find C-symmetry in the given model.
- The inner automorphism provides the representation of this C-symmetry in this model. So far, no explicit representation of the C-symmetry is known in the framework of gauge theories. This symmetry in QM must satisfy the following three conditions

$$[H, C]\psi = 0, \ [PT, C]\psi = 0, \ C^2 = \mathbf{1} \tag{29}$$

- The inner automorphism satisfies QFT analogue of the conditions (29) as we explain now.
  (1) The inner automorphism exchanges group indices i.e., $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$, $3 \leftrightarrow 8$ and the Lagrangian of the initial unbroken PT theory in the normal phase contains sum over group index a. Hence, QFT analogue of the first of conditions (29) is obeyed.

- (2) PT is a space-time symmetry and the inner automorphism is the operation in the group space. Therefore, it is easy to check that changing the order of inner automorphism and PT operations on Lagrangians of both the phases in Eqs. (10) and (14) does not alter the final result. In other words, they commute. This proves the QFT analogue of the second condition in Eq. (29).
- (3) The third of Eq. (29) has already been shown. Therefore, we see that the inner automorphism forms an explicit representation of the C-symmetry, which in adjoint representation is given by the matrix (12).
- It is clear that the theory in both the phases is invariant under CPT. In the broken PT phase, the theory also violates CP symmetry but preserves the CT, in complete analogy with the scalar model described in sec. II.

Conclusions

- The first and novel example of non-Hermitian gauge theory exhibiting PT phase transition.
- Transition between two QCD phases is identified as PT phase transition
- C-symmetry and its explicit representation is identified. Hence, the present theory is consistent non-Hermitian gauge theory.

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Thank you