# The damped wave equation with unbounded and singular damping

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Based on

[1] P. Freitas, P. Siegl, and C. Tretter (2018). "Damped wave equation with unbounded damping". In: *J. Differential Equations* 264, pp. 7023–7054

[2] A. Arifoski and P. Siegl (2018). *Pseudospectra of damped wave equation with unbounded damping*.

1. Basic concepts
2. Examples (singular and unbounded damping)
3. Results (existence & uniqueness, general spectral features, diverging damping, essential spectrum, pseudospectrum)

## Damped wave equation in $\Omega \subset \mathbb{R}^d$

$$u_{tt}(t,x) + \boxed{2a(x)}\, u_t(t,x) = (\Delta - q(x))u(t,x)$$

- $t \in \mathbb{R}_+$ time, $x \in \Omega$ spatial coordinate
- $a : \Omega \to \mathbb{R}_+$ damping, $q : \Omega \to \mathbb{R}_+$ potential
- initial conditions $u(0,\cdot) = u_0$, $u_t(0,\cdot) = u_1$
- Dirichlet boundary conditions at $\partial\Omega$ (if $\partial\Omega \neq \emptyset$)

[1] F. Gesztesy and H. Holden (2011). *J. Differential Equations* 251, pp. 1086–1127; F. Gesztesy et al. (2012). *Ann. Mat. Pura Appl.* 191, pp. 1–46.

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## Towards spectral analysis and semigroups

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}}_{G} \begin{pmatrix} u \\ v \end{pmatrix}$$

- $G$ acts in a suitable Hilbert space $\mathcal{H} = \mathcal{W}(\Omega) \times L^2(\Omega)$
- $a = 0 \implies \mathrm{i}G$ is symmetric w.r.t. $\langle \nabla \cdot, \nabla \cdot \rangle_1 + \langle q^{\frac{1}{2}} \cdot, q^{\frac{1}{2}} \rangle_1 + \langle \cdot, \cdot \rangle_2$

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## Associated quadratic pencil

- formally for eigenvalues

$$\begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

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- generalization of spectra of operators: $\lambda \in \sigma(A) \iff 0 \in \sigma(A - \lambda)$
- time-dependent solution: $\boxed{e^{\lambda t} u(x)}$

$$T(\lambda) = -\Delta + q + 2\lambda a_0 + \lambda^2, \quad a_0 \in \mathbb{R}_+$$

### Finite interval $\Omega = (-1, 1)$ and $q = 0$

- resolvent of $T(\lambda)$ is compact for every $\lambda \in \mathbb{C} \Rightarrow$ only EV's in $\sigma(T)$

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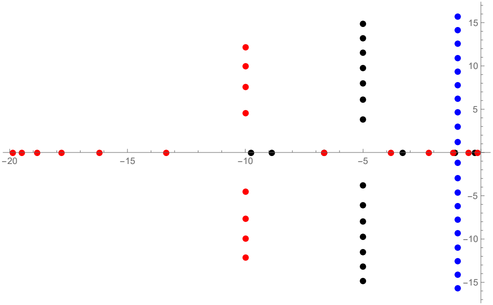

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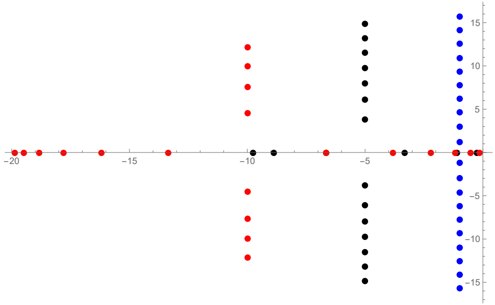


http://mathlets.org/mathlets/damped-wave-equation/

Finite interval $\Omega = (-1, 1)$, $a \in L^\infty(\Omega)$

- asymptotic behavior of eigenvalues[2]

$$\lambda_n = \pm \mathrm{i}\frac{n\pi}{2} - \frac{\int a(x)\mathrm{d}x}{2} + \mathcal{O}(n^{-1}), \quad n \to \infty$$

- eigenfunctions form a suitable (Riesz) basis[3]

$\Rightarrow$ decay rate of a general solution determined by $\min\{\operatorname{Re}\lambda_n : n \in \mathbb{N}\}$

[2] P. Freitas and E. Zuazua (1996). *J. Differential Equations* 132, pp. 338–352.

[3] S. Cox and E. Zuazua (1994). *Comm. Partial Differential Equations* 19, pp. 213–243; P. Djakov and B. Mityagin (2010). *Math. Nachr.* 283, pp. 443–462.

Finite interval $\Omega = (0, 1)$, $a = \frac{1}{x} \notin L^1(\Omega)$

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$$T(\lambda) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\lambda}{x} + \lambda^2$$

  is discrete,

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- the spectrum of
$$T(\lambda) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\lambda}{x} + \lambda^2$$
is discrete, but empty
- every solution of
$$\begin{cases} u_{tt} + \dfrac{2}{x} u_t = u_{xx} \\ u(\cdot, 0) = u(\cdot, 1) = 0 \\ (u(0,\cdot), u_t(0,\cdot)) = (u_0, u_1) \in W^{1,2}(0,1) \times L^2(0,1) \end{cases}$$
vanishes in a finite time[4]
$$\boxed{u(t,\cdot) = 0, \quad t > 2}$$

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## 1D examples on $\Omega = \mathbb{R}$

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- eigenvalues $\lambda_k = 2^{\frac{1}{2n+1}} e^{\pm \mathrm{i}\pi \frac{n+1}{2n+1}} \mu_{k;n}^{\frac{n+1}{2n+1}} - \frac{2(n+1)}{2n+1} a_0 + o_k(1), \quad k \to \infty$
- $\{\mu_{k;n}\}_k$ are eigenvalues of the self-adjoint $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2n}$ in $L^2(\mathbb{R})$
- for $n = 1$: $\mu_{k;1} = 2k + 1$, $k = 0, 1, 2, \ldots$

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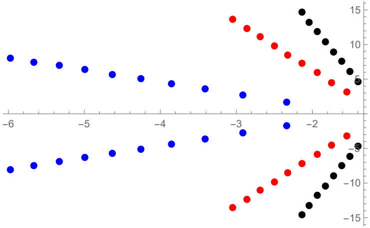

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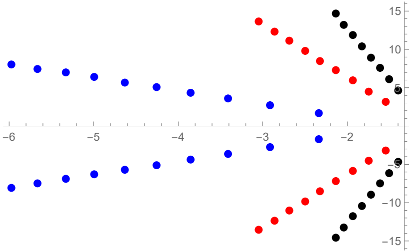


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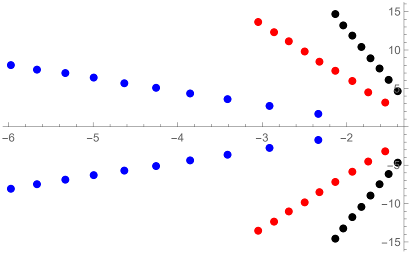


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- similar effect as for Schrödinger operators (convergence to the "square-well")

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2n} \quad \text{in} \quad L^2(\mathbb{R}) \xrightarrow{n\to\infty} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \quad \text{with Dirichlet BC in} \quad L^2(-1,1)$$

## Infinite 2D strip

- $\Omega = \mathbb{R} \times (-1, 1)$ and $a(x, y) = x^2 + a_0$
- separation of variables: algebraic equation for eigenvalues $\lambda$

$$2\lambda\mu_k = (\lambda^2 + \sigma_j^2 + 2\lambda a_0)^2$$

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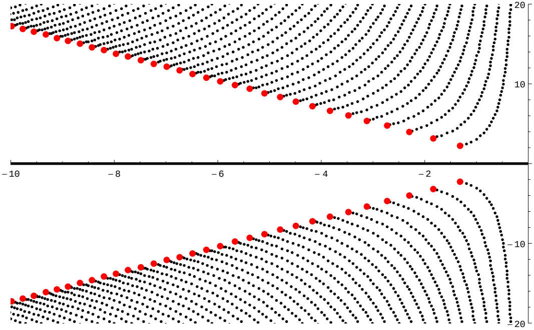

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### Theorem (semigroup)

Under suitable regularity assumptions:

- $-G$ is m-accretive

$\Rightarrow$ $G$ generates a contraction semigroup in $\mathcal{H} = \mathcal{W}(\Omega) \times L^2(\Omega)$

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Under suitable regularity assumptions: $\forall \lambda \in \mathbb{C} \setminus (-\infty, 0]$

$$\lambda \in \sigma(G) \iff 0 \in \sigma(T(\lambda))$$
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If $a$ is unbounded at infinity:

- $\sigma(G) \cap \mathbb{C} \setminus (-\infty, 0]$ = isolated EV's of finite multiplicity
- possible accumulations only to $\mathbb{R}_-$

### Assumption: limiting damping

- sequence of unbounded dampings $\{a_n\}$: $\lim_{|x|\to\infty} a_0(x) = \infty$ and

$$\forall n \in \mathbb{N}, \quad a_n \geq a_0$$

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- no pollution: if
$$\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \setminus (-\infty, 0], \quad \lambda_n \in \sigma_{\mathrm{p}}(T_n)$$
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- the corresponding eigenfunctions converge in $L^2$

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- "tricky" spectra of $T(\lambda)$ with $\lambda < 0$:

$$\sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2) = \mathbb{R} \quad \text{vs.} \quad \sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^4) = \sigma_{\mathrm{disc}}(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^4)$$

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### Theorem (negative essential spectrum)

For $d = 1$, if $\mathbb{R}_+ \subset \Omega$ and $\lambda < 0$ is such that

- $A(x) := -(q(x) + 2\lambda a(x) + \lambda^2)$ is (eventually) increasing on $\mathbb{R}_+$,
- $\lim_{x\to+\infty} A(x) = +\infty$,
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For $d > 1$, the same holds if $\Omega$ contains a (possibly shrinking) neighborhood of a semi-infinite segment where $A$ grows to $+\infty$.

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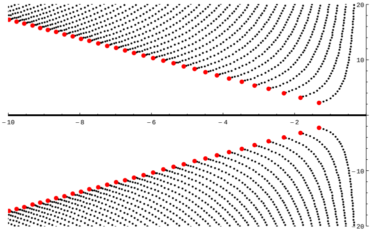
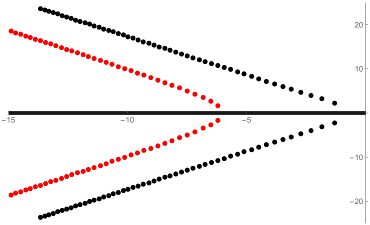
then $\boxed{\lambda \in \sigma_{\mathrm{ess}}(G)}$.

For $d > 1$, the same holds if $\Omega$ contains a (possibly shrinking) neighborhood of a semi-infinite segment where $A$ grows to $+\infty$.

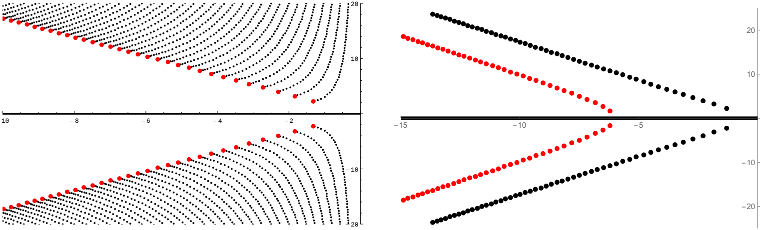
### Examples in $d = 1$: $a_n(x) = x^{2n}$ and $q(x) = q_0$

- $\boxed{\sigma_{\mathrm{ess}}(G) = (-\infty, 0]}$




[5] R. Ikehata and H. Takeda (2018). *Funkcialaj Ekvacioj*.

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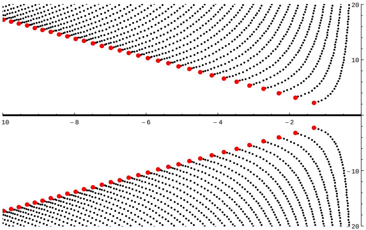


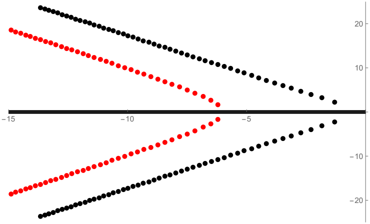


## Overdamping due to $\sigma_{\text{ess}}(G)$

- no exponential decay of the semigroup possible
- shifting $\sigma_{\text{ess}}$ of 0: adding “larger” $q$ than $a$: e.g. $q(x) = 10x^2$, $a(x) = x^2$

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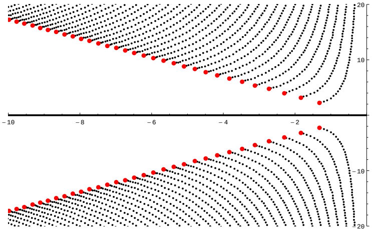


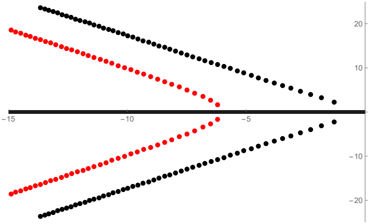


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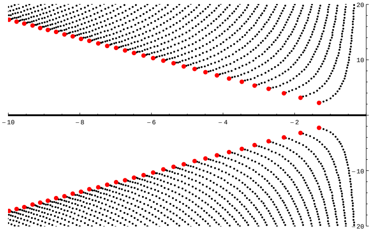


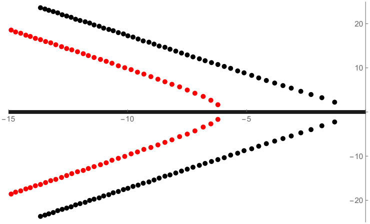


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- polynomial decay[5] for $\Omega = \mathbb{R}^d$, $d \geq 3$ and $a(x) \geq a_0 > 0$

$$\|e^{itG}\| = \mathcal{O}(t^{-1}), \qquad t \to +\infty$$

[5] R. Ikehata and H. Takeda (2018). *Funkcialaj Ekvacioj*.

- $\varepsilon$-pseudospectrum

$$\sigma_\varepsilon(G) := \sigma(G) \cup \left\{ z \in \mathbb{C} \ : \ \|(G-z)^{-1}\| > \frac{1}{\varepsilon} \right\}$$

[6] M. Embree (July 2013).

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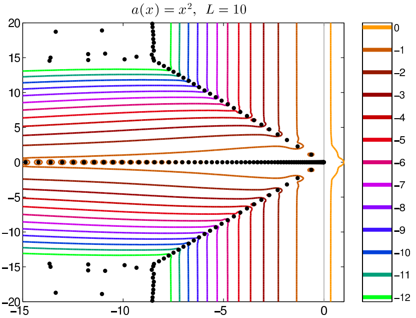


Eigenvalues inside the $\varepsilon = 10^{-12}$-pseudospectrum (top and bottom left) are not reliable.

6

[6] M. Embree (July 2013).

**Theorem** [Arifoski & PS] (Pseudospectrum)

Let

- $d = 1$, $\Omega = \mathbb{R}$
- $q(x) = 0$, $a(x) = x^{2n}$ with some (fixed $n \in \mathbb{N}$)

Then for every $\omega \in (\pi/2, \pi) \cup (\pi, 3\pi/2)$

$$\boxed{\lim_{t \to +\infty} \|(G - e^{i\omega} t)^{-1}\|_{\mathcal{H}} = \infty.}$$

## Generation of the semigroup

- accretivity of $-G$ is easy (numerical range)

[7] W. N. Everitt and M. Giertz (1978). *Proc. Roy. Soc. Edinburgh Sect. A* 79, pp. 257–265.

## Generation of the semigroup

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$$\|(-\Delta + q + \gamma a)f\| \approx \|(-\Delta + q)f\| + |\gamma|\|af\|$$

if $\gamma \in \mathbb{C} \setminus (-\infty, 0]$ and the regularity assumption **I** holds

$$|\nabla a| \leq \varepsilon a^{\frac{3}{2}} + M_{\nabla}(q^{\frac{1}{2}} + 1)$$

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- defined through the form

$$t_\lambda[\psi] = \|\nabla \psi\|^2 + \|q^{\frac{1}{2}} \psi\|^2 + 2\lambda \|a^{\frac{1}{2}} \psi\|^2 + \lambda^2 \|\psi\|^2$$

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$\{\|T_n(\lambda)\phi_n\| + \|\phi_n\|\}_{n\in\mathbb{N}}$ is bounded $\Longrightarrow$
$\{\|T_n(\lambda)\phi_n\| + \|\phi_n\|\}_{n\in\mathbb{N}}$ has a convergent subsequence in $L^2(\Omega)$

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- based on WKB approximation

$$\phi_n(x) = \varphi_n(x)\exp\left(\mathrm{i}\int_0^x \sqrt{-(q(t) + 2\lambda a(t) + \lambda^2)}\mathrm{d}t\right)$$

- $\varphi_n$: dispersing cut-off with $\operatorname{supp}\varphi_n \to \infty$

## Pseudospectrum

$$a(x) = x^2 \quad \Longrightarrow \quad \|(G-\lambda)^{-1}\|_{\mathcal{H}} \geq C_N |\operatorname{Re}\lambda|^N, \qquad \operatorname{Re}\lambda \to -\infty \ \& \ |\operatorname{Im}\lambda| \geq \delta > 0.$$

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- based on WKB approximation

$$f_\lambda := \xi_\lambda g_\lambda, \qquad g_\lambda(x) = e^{-\mathrm{i}\int_0^x \sqrt{-\lambda^2 - 2\lambda a(t) - q(t)}\, dt}.$$

- $\xi_\lambda \in C_0^\infty(\mathbb{R})$: $\lambda$-dependent cut-off