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Time-dependent quasi-hermiticity: Revisited

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Abstract :
The treatment for time-dependent non-Hermitian Hamiltonians with time-independent metric operators has been studied. The generalization to time-dependent metric operators have advanced the ground for treating time dependent systems and opened new venues for further studies. Nevertheless, the existence of invariants (constants of the motion or first integral) introduced by Lewis- Riesenfeld is an important factor in the study of time-dependent systems.
We revisite the quasi hermiticity relation though the invariant operator associated to non-Hermitian Hamiltonian with a time-dependent metric. The pseudo-Hermitian invariant operator is constructed in the same manner as for both the SU(1,1) and SU(2) systems. Interesting physical applications are suggested and discussed.

Quantum systems described by non-Hermitian stationary Hamiltonians $H \neq H^+$ with real spectra where their Hamiltonians are connected to its Hermitian conjugate through a linear, Hermitian, invertible and bounded metric operator $\eta=\rho^+\rho$ with a bounded inverse

$$H^+ = \eta H \eta^{-1}$$

i.e. H is Hermitian with respect to a positive definite inner product defined by $\langle \cdot, \cdot \rangle_\eta = \langle \cdot |\eta| \cdot \rangle$ and called as η -pseudo-Hermitian. It is also established by Mostafazadeh [2002] that the non Hermitian Hamiltonian H can be transformed to an equivalent Hermitian one given by

$$h = \rho H \rho^{-1}$$

where h is the equivalent Hermitian analog of H with respect to the standard inner product $\langle \cdot, \cdot \rangle$

Treatment of time-dependent non-Hermitian Hamiltonians H(t) with time-dependent metric operators $\eta(t)$ has been introduced by Znojil [2008-09] and opened new venues for further studies.

In this case, different relations between H and H^+ are established by several authors:

i. M. Znojil's point of view

$$H_{gen}^{+}(t) = \underbrace{\eta(t)H(t)\eta^{-1}(t)}_{H^{+}(t)} - i\dot{\eta}^{-1}(t)\eta(t)$$

$H_{gen}(t)$ is the nonobservable generator of evolution.

ii. A. Mostafazadeh's point of view

$$H^{+}(t) = \eta(t)H(t)\eta(t)^{-1} - i\hbar\eta(t)\dot{\eta}(t)^{-1}$$

$H^{+} = \eta H \eta^{-1}$ coincides with H if and only if the metric operator is time-independent.

iii. Point of view of A. Fring and M. Moussa

$$H^{\dagger}(t) = \eta(t)\,H(t)\,\eta^{-1}(t) + i\hbar\dot{\eta}(t)\,\eta^{-1}(t)$$

This relation is considered as the time-dependent quasi Hermiticity relation replacing the standard quasi-Hermiticity relation for a time-independent η(t) .

Recently, the work of M. Znojil [AnnPhys2017] has shed more light on the debated questions raised on this subject.

How to preserve the notion of quasi-Hermiticity for time-dependent systems ?

We will resort to the pseudo-invariant theory:

Given a non-Hermitian time-dependent Hamiltonian operator H(t), it is possible to build a pseudo-invariant operator $I^{PH}(t)$ verifying

$$\frac{dI^{PH}(t)}{dt} = \frac{\partial I^{PH}(t)}{\partial t} - i\left[I^{PH}(t), H(t)\right] = 0, \qquad \text{(I)}$$

and obeys the eigenvalue equation

$$I^{PH}(t)\left|\phi_n^H(t)\right\rangle = \lambda_n \left|\phi_n^H(t)\right\rangle,$$

where the eigenvalues are time-independent and the eigenstates are orthonormal

$$\left\langle \phi_m^H(t)\right| \eta(t) \left|\phi_n^H(t)\right\rangle = \delta_{m,n}.$$

The solutions of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\Phi^H(t)\right\rangle = H(t)\left|\Phi^H(t)\right\rangle$$

can be written in terms of the eigenfunctions $\left|\phi_n^H(t)\right\rangle$ as

$$\left|\Phi_n^H(t)\right\rangle = e^{i\varphi_n(t)}\left|\phi_n^H(t)\right\rangle,$$

where the phase functions $\varphi_n(t)$ are found from the equation

$$\frac{d\varphi_n(t)}{dt} = \langle \phi_n^H(t) | \, \eta(t) \left[i\frac{\partial}{\partial t} - \frac{H(t)}{\hbar} \right] \; | \phi_n^H(t) \rangle \, .$$

Application:

The su(1,1) and su(2) Lie algebra time-dependent systems that we consider are described by the non-Hermitian Hamiltonian

$$H(t) = 2\omega(t) K_0 + 2\alpha(t) K_- + 2\beta(t) K_+,$$

where the parameters are arbitrary complex functions of time. $K_0 = K_0^+$ and $K_+ = (K_-)^+$
The commutation relations between these operators are

$$\begin{cases} [K_0, K_+] = K_+ \\ [K_0, K_-] = -K_- \\ [K_+, K_-] = DK_0 \end{cases} .$$

D=-2 and 2 corresponds to su(1, 1) and su(2) algebras respectively.
We consider the most general invariant in the form

$$I^{PH}(t) = 2\delta_1(t) K_0 + 2\delta_2(t) K_- + 2\delta_3(t) K_+,$$

where the parameters are complex and time-dependent.

The key point of our method is to solve the standard quasi-Hermiticity relation

$$I^{PH\dagger}(t) = \eta(t) I^{PH}(t)\, \eta^{-1}(t) \;\Leftrightarrow I^{h}(t) = \rho(t) I^{PH}(t) \rho^{-1}(t) = I^{h\dagger}(t),$$

Using the Baker--Hausdorff--Campbell formula that allows us to express all elements of SU(1, 1) or of SU(2) obtained by exponentiation of an hermitic element of SU(1, 1) or of SU(2) as

$$\begin{aligned} \rho(t) &= \exp\left\{2\left[\epsilon(t) K_0 + \mu(t) K_- + \mu^*(t) K_+\right]\right\}, \\ &= \exp\left[\vartheta_+(t) K_+\right] \exp\left[\ln \vartheta_0(t) K_0\right] \exp\left[\vartheta_-(t) K_-\right], \end{aligned}$$

where

$$\begin{aligned} \vartheta_+(t) &= \frac{2\mu^* \sinh\theta}{\theta\cosh\theta - \epsilon\sinh\theta} = -\zeta(t) e^{-i\varphi(t)}, \\ \vartheta_0(t) &= \left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)^{-2} = -\frac{D}{2}\zeta^2(t) - \chi(t), \\ \vartheta_-(t) &= \frac{2\mu \sinh\theta}{\theta\cosh\theta - \epsilon\sinh\theta} = -\zeta(t) e^{i\varphi(t)}, \\ \chi(t) &= -\frac{\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta}{\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta} \qquad , \quad \theta = \sqrt{\epsilon^2 + 2D\left|\mu\right|^2}. \end{aligned}$$

We obtain the transformed invariant operator

$$I^h(t) = \frac{2}{\vartheta_0}\left[\left(\frac{D}{2}\vartheta_+\vartheta_- - \chi\right)\delta_1 + D\left(\vartheta_+\delta_2 + \chi\vartheta_-\delta_3\right)\right]K_0 + \\ + \frac{2}{\vartheta_0}\left[\vartheta_-\delta_1 + \delta_2 - \frac{D}{2}\vartheta_-^2\delta_3\right]K_- + \\ + \frac{2}{\vartheta_0}\left[\chi\vartheta_+\delta_1 - \frac{D}{2}\vartheta_+^2\delta_2 + \chi^2\delta_3\right]K_+$$

The hermiticity of $I^h(t)$ leads to

$$\left(\chi - \frac{D}{2}\phi^2\right)|\delta_1|\sin\varphi_{\delta_1} - D\phi\left[|\delta_2|\sin\left(\varphi - \varphi_{\delta_2}\right) - \chi|\delta_3|\sin\left(\varphi + \varphi_{\delta_3}\right)\right] = 0$$

$$\begin{cases} \phi\left(1-\chi\right)|\delta_1|\cos\varphi_{\delta_1} - \left(1 + \frac{D}{2}\phi^2\right)|\delta_2|\cos\left(\varphi - \varphi_{\delta_2}\right) + \left(\chi^2 + \frac{D}{2}\phi^2\right)|\delta_3|\cos\left(\varphi + \varphi_{\delta_3}\right) = 0 \\ \phi\left(1+\chi\right)|\delta_1|\sin\varphi_{\delta_1} + \left(1 - \frac{D}{2}\phi^2\right)|\delta_2|\sin\left(\varphi - \varphi_{\delta_2}\right) - \left(\chi^2 - \frac{D}{2}\phi^2\right)|\delta_3|\sin\left(\varphi + \varphi_{\delta_3}\right) = 0 \end{cases}$$

By imposing

$$I^{\dagger PH}(t) = \rho(t)I^h(t)\rho^{-1}(t)$$

and with some algebra, we deduce

$$
\begin{aligned}
I^{PH}(t) = 2 & \left[|\delta_1| \cos\varphi_{\delta_1} + i\frac{(1-\chi)}{\vartheta_0} \left[|\delta_1| \sin\varphi_{\delta_1} + D\phi\, |\delta_3| \sin\left(\varphi + \varphi_{\delta_3}\right) \right] \right] K_0 + \\
& + \frac{2e^{i\varphi}}{\vartheta_0^2} \left[\vartheta_0^2\, |\delta_2| \cos\left(\varphi - \varphi_{\delta_2}\right) - i\left(\frac{D}{2}\phi^2 + \chi^2 \right) \left[\phi\, |\delta_1| \sin\varphi_{\delta_1} + + \right.\right. \\
& \left.\left. + |\delta_2| \sin\left(\varphi - \varphi_{\delta_2}\right) + \frac{D}{2}\phi^2\, |\delta_3| \sin\left(\varphi + \varphi_{\delta_3}\right) \right] \right] K_- + \\
& + \frac{2e^{-i\varphi}}{\vartheta_0^2} \left[\vartheta_0^2\, |\delta_3| \cos\left(\varphi + \varphi_{\delta_3}\right) + i\left(\frac{D}{2}\phi^2 + 1 \right) \left[\phi\, |\delta_1| \sin\varphi_{\delta_1} + \right.\right. \\
& \left.\left. + |\delta_2| \sin\left(\varphi - \varphi_{\delta_2}\right) + \frac{D}{2}\phi^2\, |\delta_3| \sin\left(\varphi + \varphi_{\delta_3}\right) \right] \right] K_+
\end{aligned}
$$

Now we impose the invariance condition (I) to obtain

$$
\begin{aligned}
\dot{\delta}_1 &= 2iD\left(\alpha\delta_3 - \beta\delta_2\right) \\
\dot{\delta}_2 &= 2i\left(\omega\delta_2 - \alpha\delta_1\right) \\
\dot{\delta}_3 &= 2i\left(\beta\delta_1 - \omega\delta_3\right)
\end{aligned}
$$

For simplicity, we examine the case where the time dependent coefficients of the pseudo-invariant are real.

Pseudo-invariant with real parameters

In this case equations for parameters δ_1 , δ_2 and δ_3 are simplified to

$$\delta_2 = \delta_3 \chi, \qquad \delta_1 = \frac{\left(\frac{D}{2}\vartheta_-^2 - \chi\right)}{\vartheta_-}\delta_3, \qquad \delta_1 = \frac{\left(\frac{D}{2}\vartheta_+^2 - \chi\right)}{\vartheta_+}\delta_3,$$

and lead to the relation $\vartheta_+(t) = \vartheta_-(t) \equiv -\zeta(t)$ implying that the time dependent parameter $\mu(t)$ must be real, and therefore the Hemitian invariant is given by

$$I^h(t) = \frac{2}{\vartheta_0}\left[\left(\frac{D}{2}\zeta^2 - \chi\right)\delta_1 - 2D\chi\zeta\delta_3\right]K_0.$$

Let $\left|\psi_n^h\right\rangle$ be the eigenstate of K_0 with eigenvalue k_n

$$K_0\left|\psi_n^h\right\rangle = k_n\left|\psi_n^h\right\rangle.$$

The eigenstates of $I^h(t)$ are obviously given by

$$I^h(t)\left|\psi_n^h(t)\right\rangle = \frac{2}{\vartheta_0}\left[\left(\frac{D}{2}\zeta^2 - \chi\right)\delta_1 - 2D\chi\zeta\delta_3\right]k_n\left|\psi_n^h\right\rangle,$$

Whereas the pseudo Hermitian invariant operator is written in the following form

$$I^{PH}(t) = \frac{2}{\vartheta_0}\left[\left(\frac{D}{2}\zeta^2 - \chi\right)K_0 - \chi\zeta K_- - \zeta K_+\right].$$

and the invariance condition (I) implies

$$\dot{\vartheta}_0 = \frac{2\vartheta_0}{\zeta}\left[-2\zeta\,|\omega|\sin\varphi_\omega + |\alpha|\sin\varphi_\alpha + \left(\chi - D\zeta^2\right)|\beta|\sin\varphi_\beta\right],$$

$$\dot{\zeta} = -2\zeta\,|\omega|\sin\varphi_\omega + 2\,|\alpha|\sin\varphi_\alpha - D\zeta^2\,|\beta|\sin\varphi_\beta,$$

$$\begin{array}{c} \chi\,|\beta|\cos\varphi_\beta = |\alpha|\cos\varphi_\alpha \\ \left(\chi - \frac{D}{2}\zeta^2\right)|\alpha|\cos\varphi_\alpha = \chi\zeta\,|\omega|\cos\varphi_\omega \\ \zeta\,|\omega|\cos\varphi_\omega = \left(\chi - \frac{D}{2}\zeta^2\right)|\beta|\cos\varphi_\beta \end{array} ,$$

The general solution of the Schrödinger equation, which is an eigenstate of the pseudo Hermitian invariant multiplied by a time-dependent factor, is given by

$$\left|\Phi^H(t)\right\rangle = \sum\nolimits_n C_n(0)\exp\left(-ik_n\int_0^t \frac{2}{\vartheta_0}\left[|\omega|\left(\frac{D}{2}\zeta^2 - \chi\right)\cos\varphi_\omega - 2D\zeta\,|\alpha|\cos\varphi_\alpha\right]dt'\right)\left|\phi_n^H(t)\right\rangle .$$

Few special examples

- **Generalized time dependent non-Hermitian Swanson Hamiltonian**

The su(1,1) Lie algebra has a realization in terms of boson creation and annihilation operators a^+ and a such that

$$K_0 = \frac{1}{2}\left(a^+a + \frac{1}{2}\right), \quad K_- = \frac{1}{2}a^2, \qquad K_+ = \frac{1}{2}a^{+2}.$$

We construct the solutions for the generalized version of the following non-Hermitian Swanson Hamiltonian with time-dependent coefficients

$$H(t) = \omega(t)\left(a^+a + \frac{1}{2}\right) + \alpha(t)a^2 + \beta(t)a^{+2},$$

where $(\omega(t), \alpha(t), \beta(t)) \in C$ are time-dependent parameters. The form for $I^{PH}(t)$ is

$$I^{PH}(t) = \exp\left[\frac{\zeta}{2}a^2\right]\exp\left[-\frac{\ln\vartheta_0}{2}\left(a^+a + \frac{1}{2}\right)\right]\exp\left[\frac{\zeta}{2}a^{+2}\right]\ \left(a^+a + \frac{1}{2}\right)$$
$$\times\exp\left[-\frac{\zeta}{2}a^{+2}\right]\exp\left[\frac{\ln\vartheta_0}{2}\left(a^+a + \frac{1}{2}\right)\right]\exp\left[-\frac{\zeta}{2}a^2\right],$$

As eigenstates of $I^{PH}(t)$ one can then take

$$\left|\phi_n^H(t)\right\rangle = \exp\left[\frac{\zeta}{2}a^2\right] \exp\left[-\frac{\ln \vartheta_0}{2}\left\{\left(a^+a+\frac{1}{2}\right)-\left(n+\frac{1}{2}\right)\right\}\right] \exp\left[\frac{\zeta}{2}a^{+2}\right]|n\rangle ,$$

the corresponding phase $\varphi_n(t)$ is

$$\varphi_n(t) = (n+\frac{1}{2})\int_0^t \frac{1}{\vartheta_0}\left[\left(\zeta^2+\chi\right)|\omega|\cos\varphi_\omega - 4\zeta\,|\alpha|\cos\varphi_\alpha\right] dt'.$$

- **A spinning particle in a time-varying magnetic field**

A spin in a time-varying complex magnetic field is a practical example for the case D=2 . Let

$$K_0 = J_z, \quad K_- = J_-, \qquad K_+ = J_+,$$

the Hamiltonian and the invariant be

$$H(t) = 2\left[\omega(t)J_z + \alpha(t)J_- + \beta(t)J_+\right],$$

$$I^{PH}(t) = \frac{2}{\vartheta_0}\left[\left(\zeta^2-\chi\right)J_z - \chi\zeta J_- - \zeta J_+\right],$$

The form for $I^{PH}(t)$ is

$$I^{PH}(t) = \exp\left[\zeta J_-\right]\exp\left[-\ln\vartheta_0 J_z\right]\exp\left[\zeta J_+\right]\ J_z\ \exp\left[-\zeta J_+\right]\exp\left[\ln\vartheta_0 J_z\right]\exp\left[-\zeta J_-\right].$$

The instantaneous eigenstates of $I^{PH}(t)$ can be written in terms of J_z eigenstates denoted by $|m\rangle$, as

$$\left|\phi_m^H(t)\right\rangle = \exp\left[\zeta J_-\right]\exp\left[-\ln\vartheta_0\left(J_z - m\right)\right]\exp\left[\zeta J_+\right]|m\rangle\,,$$

For this case, the phase is easy to calculate and is given by

$$\varphi_m(t) = -m\int_0^t \frac{2}{\vartheta_0}\left[\left(\zeta^2 - \chi\right)|\omega|\cos\varphi_\omega - 4\zeta\,|\alpha|\cos\varphi_\alpha\right]dt'.$$

- **The Hamiltonian H(t) with real coefficients ω(t),α(t),β(t)**

When considering the time-dependent coefficients ω(t),α(t),β(t) to be real functions instead of complex ones, we get

$$\dot{\vartheta}_0 = 0,$$

$$\dot{\zeta} = 0,$$

and

$$\begin{aligned} \chi\left|\beta\right| &= \left|\alpha\right| \\ \left(\chi-\tfrac{D}{2}\zeta^{2}\right)\left|\alpha\right| &= \chi\zeta\left|\omega\right| \\ \zeta\left|\omega\right| &= \left(\chi-\tfrac{D}{2}\zeta^{2}\right)\left|\beta\right| \end{aligned} \quad .$$

Thus, the time-dependent real coefficients ω(t),α(t),β(t) of H(t) provide a time-independent metric and consequently the gaugelike term $i\hbar\dot{\eta}\left(t\right)\eta^{-1}\left(t\right)$ is equal to zero and the standard quasi-Hermiticity relation $\eta H\left(t\right)=H^{\dagger}\left(t\right)\eta$ for the Hamiltonian H(t) itself is recovered in complete analogy with the time-independent scenario.

Thus H(t) is self-adjoined operator and therefore observable and can be written in the following simple form

$$H(t)=2\frac{\omega(t)}{\left(\frac{D}{2}\zeta^{2}-\chi\right)}\left\{\left(\frac{D}{2}\zeta^{2}-\chi\right)K_{0}-\chi\zeta K_{-}-\zeta K_{+}\right\}=\frac{\omega(t)\vartheta_{0}}{\left(\frac{D}{2}\zeta^{2}-\chi\right)}I^{PH}(t),$$

we derive the metric parameter ζ in terms of parameters of the Hamiltonian H(t)

$$\zeta=\frac{1}{2\left|\beta\right|}\left(-\frac{D}{2}\left|\omega\right|\pm\sqrt{\left|\omega\right|^{2}+2D\left|\alpha\right|\left|\beta\right|}\right).$$

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