Class of nonlocal nonlinear Schrödinger equations: Symmetry, integrability and solvability

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Introduction

$\mathcal{PT}$-symmetric NLSE:

1. Nonlinear optics with a $\mathcal{PT}$-symmetric optical potential.
2. Wave guide systems with balanced loss and gain.

A new type of $\mathcal{PT}$-symmetric NLSE:

- $\mathcal{PT}$-symmetric nonlocal NLSE, arising due to a nonlocal reduction in the Lax-pair formulation.

- The $\mathcal{PT}$-symmetric nonlocal vector nonlinear Schrödinger equation (NLSE).
- A generalized version of the nonlocal NLSE in an external potential with a space-time modulated coefficient of the nonlinear interaction term and/or gain-loss terms.
- Symmetries of a generalized nonlocal NLSE in (d+1) dimension.

Nonlocal nonlinear Schrödinger equation

Non-local NLSE in 1+1 dimensions:

$$i\psi_t(x,t) = -\frac{1}{2}\psi_{xx}(x,t) + G\ \psi^*(-x,t)\psi(x,t)\psi(x,t), \ \ G \in R.$$

- The self-induced potential $V(x) = \psi^*(-x)\psi(x)$, in the corresponding stationary problem, is $\mathcal{PT}$ symmetric.
- The equation is non-local.
- This equation possess a Lax pair and is integrable.
- Admits dark as well as bright soliton solutions for $G < 0$.

The nonlocal vector nonlinear Schrödinger equation (VNLSE)

A possible generalization of Eq.(1) is the following VNLSE:

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} + 2\mathbf{Q}\mathbf{Q}^{\mathbf{P}}\mathbf{Q}, \tag{1}$$

where

$$\mathbf{Q}^{\mathbf{P}} = \begin{pmatrix} q_1^*(-x,t) \\ q_2^*(-x,t) \end{pmatrix}. \tag{2}$$

and $\mathbf{Q}_t = \frac{d\mathbf{Q}}{dt}$, $\mathbf{Q}_{xx} = \frac{d^2\mathbf{Q}}{dx^2}$, $\mathbf{Q} = (q_1(x,t), q_2(x,t))$ with $q_1(x,t), q_2(x,t)$ being two complex fields.

- This system is integrable.

The Lax pair formulation for the nonlocal VNLSE:

$$v_x = \begin{pmatrix} -ik\mathbf{I}_{1\times 1} & \mathbf{Q} \\ \mathbf{R} & ik\mathbf{I}_{2\times 2} \end{pmatrix} v, \tag{3}$$

$$v_t = \begin{pmatrix} 2ik^2 + i\mathbf{QR} & -2k\mathbf{Q} - i\mathbf{Q}_x \\ -2k\mathbf{R} + i\mathbf{R}_x & -2ik^2\mathbf{I}_{\mathbf{2\times 2}} - i\mathbf{RQ} \end{pmatrix} v. \tag{4}$$

- $\mathbf{R} = (r_1(x,t), r_2(x,t))^T$, $\mathbf{I}_{n\times n}$:$n \times n$ square matrix.
- All the fields vanish rapidly as $|x| \to \infty$.
- The compatibility condition $v_{xt} = v_{tx}$ give rise to the nonlocal VNLSE under symmetry reduction, $\mathbf{R} = -\mathbf{Q}^{\mathbf{P}}$.
- Eq. (3) is the AKNS spectral problem. Eq. (4) gives the time evolution of the scattering data.

One soliton solution for two component nonlocal VNLSE

$$q_1(x) = -\sqrt{2}\frac{(\eta+\tilde{\eta})e^{i\tilde{\theta}_1}e^{-4i\tilde{\eta}^2 t}e^{-2\tilde{\eta}x}}{1+e^{i(\theta_1+\tilde{\theta}_1)}e^{-4i(\tilde{\eta}^2-\eta^2)t}e^{-2x(\eta+\tilde{\eta})}},$$
$$q_2(x) = -\sqrt{2}\frac{(\eta+\tilde{\eta})e^{i\tilde{\theta}_2}e^{-4i\tilde{\eta}^2 t}e^{-2\tilde{\eta}x}}{1+e^{i(\theta_1+\tilde{\theta}_1)}e^{-4i(\tilde{\eta}^2-\eta^2)t}e^{-2x(\eta+\tilde{\eta})}}, \tag{5}$$

$$r_1(x) = \sqrt{2}\frac{(\eta+\tilde{\eta})e^{i\theta_1}e^{4i\eta^2 t}e^{-2\eta x}}{1+e^{i(\theta_1+\tilde{\theta}_1)}e^{-4i(\tilde{\eta}^2-\eta^2)t}e^{-2x(\eta+\tilde{\eta})}},$$
$$r_2(x) = \sqrt{2}\frac{(\eta+\tilde{\eta})e^{i\theta_2}e^{4i\eta^2 t}e^{-2\eta x}}{1+e^{i(\theta_1+\tilde{\theta}_1)}e^{-4i(\tilde{\eta}^2-\eta^2)t}e^{-2x(\eta+\tilde{\eta})}}, \tag{6}$$

with $k = i\eta$ and $\tilde{k} = -i\tilde{\eta}$, $\eta \neq \tilde{\eta}$, $\eta, \tilde{\eta} > 0$.

Inhomogeneous non-autonomous non-local NLSE

We investigate the possible exact solutions of the following non-autonomous NLSE:

$$\begin{aligned} i\psi_t &= -\frac{1}{2}\psi_{xx} + [V(x,t) + iW(x,t)]\psi + g(x,t) \\ &\quad \psi^{*p}(-x,t)\psi^p(x,t)\psi(x,t), \;\; p \in N, \end{aligned} \tag{7}$$

- The external potential $v(x,t) = V(x,t) + iW(x,t)$ is chosen to be complex.
- The external potential $v(x,t)$ becomes $\mathcal{PT}$ symmetric for $V(x,t) = V(-x,-t)$ and $W(x,t) = -W(-x,-t)$.

For $V(x,t) = W(x,t) = 0$, $g(x,t) = G$, we get back the homogeneous non-local NLSE.

$$i\psi_t = -\frac{1}{2}\psi_{xx} + G\psi^{*p}(-x,t)\psi^p(x,t)\psi(x,t). \tag{8}$$

- For $p = 1$, many exact solutions of this Eq. are known.
- For arbitrary p, a possible solution is

$$\psi(x,t) = \Phi_0 \; e^{i\frac{A^2}{2p^2}t} \; sech^{\frac{1}{p}}(Ax), \tag{9}$$

where

$$G = -\frac{A^2(1+p)}{2p^2\Phi_0^{2p}}, \tag{10}$$

is necessarily negative.

Similarity transformation method for non-autonomous non-local NLSE

We use the similarity transformation

$$\psi(x,t) = \rho(x,t)e^{i\phi(x,t)}\Phi(X),\ \ X \equiv X(x,t) \tag{11}$$

to map eq.

$$\begin{aligned} i\psi_t &= -\frac{1}{2}\psi_{xx} + [V(x,t) + iW(x,t)]\psi + g(x,t) \\ & \psi^{*p}(-x,t)\psi^p(x,t)\psi(x,t),\ \ p \in N, \end{aligned} \tag{12}$$

to the following equation:

$$\mu\Phi(X) = -\frac{1}{2}\Phi_{XX}(X) + G\Phi^{*p}(-X)\Phi^p(X)\Phi(X). \tag{13}$$

This mapping is valid only when $X(x,t)$ is an odd function of x, i.e.,

$$X(-x,t) = -X(x,t), \tag{14}$$

The following additional consistency conditions hold simultaneously:

$$\begin{aligned}
2\rho\rho_t + (\rho^2\phi_x)_x &= 2\rho^2 W(x,t) && (15)\\
(\rho^2 X_x)_x &= 0 && (16)\\
X_t + \phi_x X_x &= 0 && (17)\\
V(x,t) &= \frac{\rho_{xx}}{2\rho} - \phi_t - \frac{\phi_x^2}{2} - \mu X_x^2 && (18)\\
g &= \frac{G}{\rho^p(-x,t)\rho^p(x,t)e^{ip(\phi(x,t)-\phi(-x,t))}} X_x^2 && (19)
\end{aligned}$$

Solution of eqs. (8) and (9) gives ρ and ϕ:

$$\begin{aligned} \rho(x,t) &= \sqrt{\frac{\delta(t)}{X_x}} \\ \phi(x,t) &= -\int dx \frac{X_t}{X_x} + \phi_0(t), \end{aligned} \tag{20}$$

where $\delta(t)$ and $\phi_0(t)$ are two integration constants.

- It follows that $\phi(x,t)$ is even in x.
- This allows to re-write $g(x,t)$ in eq. (19) as,

$$g(x,t) = \frac{G\delta^2(t)}{\rho^{2(p+2)}}. \tag{21}$$

Inhomogeneous autonomous non-local NLSE

Consider a spacial class of similarity transformation by considering,

$$\rho(x,t) \equiv \rho(x), \phi(x,t) \equiv -Et,\ X \equiv X(x), \tag{22}$$

- In this case $W(x,t) = 0$ and g and V become independent of time.
- The consistency conditions give:

$$X(x) = \int_o^x \frac{ds}{\rho^2(s)} \tag{23}$$

$$g(x) = \frac{G}{\rho^{2(p+2)}} \tag{24}$$

$$V(x) = \frac{\rho_{xx}}{2\rho} + E - \frac{\mu}{\rho^4} \tag{25}$$

- Eq. (23) implies that ρ must have a definite parity as X is an odd function of x.
- It immediately follows from eqs. (24) and (25) that both $g(x)$ and $V(x)$ must be an even function of x.

$$\rho(-x) = \pm\rho(x),\ \ g(-x) = g(x),\ \ V(-x) = V(x). \tag{26}$$

- The similarity transformation technique is applicable to the non-local NLSE, only when both the confining potential $V(x)$ and the space-modulated nonlinear interaction term $g(x)$ are even in x.
- The expressions for $X(x)$ and $g(x)$ can be obtained, once an explicit form of $\rho(x)$ is known.

- For a given $V(x)$, $\rho(x)$ is obtained from Eq. (25) having the form,

$$\frac{1}{2}\rho_{xx} + [E - V(x)]\,\rho = \frac{\mu}{\rho^3} \tag{27}$$

which is the Ermakov-Pinney equation.

- The solution of this equation:

$$\rho = \left[a\phi_1^2(x) + 2b\phi_1(x)\phi_2(x) + c\phi_2^2(x)\right]^{\frac{1}{2}}, \tag{28}$$

where a, b, c are constants and $\phi_1(x)$, $\phi_2(x)$ are the two linearly independent solutions of the equation,

$$-\frac{1}{2}\phi_{xx} + V(x)\phi(x) = E\phi(x). \tag{29}$$

- The constant μ is determined as,
$\mu = (ac - b^2)\left[\phi_1'(x)\phi_2(x) - \phi_1(x)\phi_2'(x)\right]^2$.

Examples:

i) Vanishing External Potential:
A solution of eq. (12) for $G < 0$, $V = W = 0$ and $g(x,t) = G\,[1 + \alpha\cos(\omega x)]^{-(p+2)}$ reads

$$\psi(x,t) = e^{-iEt}\left(\frac{E(\alpha^2-1)(p+1)}{\mid G \mid}\right)^{\frac{1}{2p}}$$

$$[1+\alpha\cos(\omega x)]^{\frac{1}{2}}\;\; sech^{\frac{1}{p}}\left(p\sqrt{2E(\alpha^2-1)}X_{-}(x)\right)$$

where $\omega = 2\sqrt{2\mid E\mid}$ and $\mu = (1-\alpha^2)E$, and $E > 0$.

- Other examples: Harmonic Confinement, Reflection-less Potential.

Non-autonomous non-local NLSE

i) Non-separable $X(x,t)$: One may choose the following ansatz,

$$X(t,x) = F(\xi),\ \xi(t,x) \equiv \gamma(t)x,\ F(-\xi) = -F(\xi), \tag{30}$$

where $\gamma(t)$ is an arbitrary function of t.

- Unlike in the case of local NLSE , a purely time-dependent term can not be added to the ansatz for $X(x,t)$ due to the condition $X(-x,t) = -X(x,t)$.

- The consistency condition fixes $W(x,t) = 0$.

- For exponentially localized non-linearity $F(\xi) = \int e^{-\xi^2} d\xi$ with a combination of harmonic and dipole traps, the non-local NLSE admits all kind of soliton solutions as in the corresponding local case.

- Moving solitons are not allowed for the non-local NLSE due to the condition (14) which forbids the addition of a purely time-dependent term to the ansatz for $X(x, t)$.

ii) Separable $X(x,t)$:

$$X(x,t) \equiv \alpha(t) f(x), \quad f(-x) = -f(x). \tag{31}$$

With this choice of X, consistency conditions take the following form in terms of $\alpha(t)$ and $f(x)$:

$$\begin{aligned} \rho(x,t) &= \sqrt{\frac{\delta(t)}{\alpha(t) f'(x)}}, \\ \phi(x,t) &= -\frac{\alpha_t}{\alpha(t)} \int dx \frac{f(x)}{f'(x)}, \; g(x,t) = \frac{G\alpha^{p+2}}{\delta^p} (f')^{p+2} \end{aligned} \tag{32}$$

$$
\begin{aligned}
W(x,t) &= \frac{1}{2\alpha(t)\delta(t)}(\delta_t\alpha - 2\alpha_t\delta) + \frac{\alpha_t}{\alpha}(\frac{f''f}{f'^2}), \\
V(x,t) &= -(\frac{2f'''f' - 3f''^2}{8f'^2}) \\
&+ \frac{\alpha_{tt}\alpha - \alpha_t^2}{\alpha^2}\int \frac{f(x)}{f'(x)}dx - \frac{\alpha_t^2 f^2}{2\alpha^2 f'^2} - \mu\alpha^2 f'^2.
\end{aligned}
$$

- The external potential $v(x,t)$ is $\mathcal{PT}$ symmetric whenever both $\alpha(t)$ and $\delta(t)$ have definite parity.
- The nonlinear interaction $g(x,t)$ becomes $\mathcal{PT}$ symmetric when both $\delta(t)$ and $\alpha(t)$ have the same parity or p is even.

Example

Harmonic confinement:

$$V(x,t) = \frac{1}{2}\omega_0^2 x^2 - \mu\alpha^2, \tag{33}$$

$$g(x,t) = G\left(C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\right)^{p-2}. \tag{34}$$

A solution of eq. (12) with $p = 1$ and $G < 0$.

$$\psi_V = \left(\frac{2\mu m}{G(1+m)\left(C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\right)}\right)^{\frac{1}{2}} e^{-i\frac{\omega_0\left(C_1\sin(\omega_0 t) - C_2\cos(\omega_0 t)\right)}{2\left(C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)\right)}x^2} sn\left(\sqrt{\frac{2\mu}{1+m}}X, m\right)$$

where $\frac{1}{2} < \mu \leq 1$ and the value of m lies within the range $0 < m \leq 1$.

Schrödinger invariance of non-local NLSE

A $(d+1)$ dimensional generalization of nonlocal NLSE has the form

$$i\psi_t(\mathbf{x},t) = -\frac{1}{2}\nabla^2\psi(\mathbf{x},t) + g\left\{\psi^*(\mathcal{P}\mathbf{x},t)\psi(\mathbf{x},t)\right\}^p\psi(\mathbf{x},t),$$

where $\mathcal{P}$ denotes the parity transformation in $(d+1)$ dimension. The Lagrangian density:

$$\begin{aligned}\mathcal{L} &= i\psi^*(\mathcal{P}\mathbf{x},t)\partial_t\psi(x,t) - \frac{1}{2}\nabla\psi^*(\mathcal{P}\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) \\ &- \frac{g}{p+1}\left\{\psi^*(\mathcal{P}\mathbf{x},t)\psi(\mathbf{x},t)\right\}^{p+1}, \qquad (35)\end{aligned}$$

where $\psi(\mathbf{x},t)$ and $\psi^*(\mathcal{P}\mathbf{x},t)$ are treated as two independent fields.

- The action $\mathcal{A} = \int \mathcal{L} d^d \mathbf{x} dt$ is invariant under space-time translations, spatial rotation, Galilean transformation and a global gauge transformation. The action $\mathcal{A}$ is invariant under dilatation and special conformal transformation for the special case $pd = 2$.
- The Noether charges satisfy the $d+1$ dimensional Schrödinger algebra:

$$\begin{aligned}
&\{H, D\} = H, \ \{H, K\} = 2D, \ \{D, K\} = K, \\
&\{\mathbf{P}, D\} = \frac{1}{2}\mathbf{P}, \ \{\mathbf{P}, K\} = \mathbf{B}, \ \{P_i, L_{jk}\} = -\left(\delta_{ij} P_k - \delta_{ik} P_j\right), \\
&\{L_{ij}, L_{kl}\} = (\delta_{ik} L_{jl} - \delta_{il} L_{jk} - \delta_{jk} L_{il} + \delta_{jl} L_{ik}) \\
&\{H, \mathbf{B}\} = \mathbf{P}, \ \{D, \mathbf{B}\} = \frac{\mathbf{B}}{2}, \{P_i, B_j\} = \delta_{ij} N, \\
&\{B_i, L_{jk},\} = -\left(\delta_{ij} B_k - \delta_{ik} B_j\right).
\end{aligned} \tag{36}$$

Decomposition of the field as a sum of parity-even and parity-odd terms:

The formal expressions of these conserved Noether charges are in general complex and are not semi-positive definite.

- Example: The conserved quantity N corresponding to the global $U(1)$ transformation:

$$N = \int \rho(\mathbf{x}, t) d^d\mathbf{x}, \;\; \rho(\mathbf{x}, t) \equiv \psi^*(\mathcal{P}\mathbf{x}, t)\psi(\mathbf{x}, t). \tag{37}$$

- Decomposition of the field $\psi(\mathbf{x}, t)$ as a sum of parity-even and parity-odd terms:

$$\begin{aligned} \psi(\mathbf{x}, t) &= \psi_e(\mathbf{x}, t) + \psi_o(\mathbf{x}, t), \quad \psi_e(\mathbf{x}, t) = \frac{\psi(\mathbf{x}, t) + \psi(\mathcal{P}\mathbf{x}, t)}{2}, \\ \psi_o(\mathbf{x}, t) &= \frac{\psi(\mathbf{x}, t) - \psi(\mathcal{P}\mathbf{x}, t)}{2}. \end{aligned}$$

$$\rho(\mathbf{x}, t) = \rho_r(\mathbf{x}, t) + \rho_c(\mathbf{x}, t) \tag{38}$$

with

$$\begin{aligned} \rho_r(\mathbf{x}, t) &= \mid \psi_e(\mathbf{x}, t) \mid^2 - \mid \psi_o(\mathbf{x}, t) \mid^2, \\ \rho_c(\mathbf{x}, t) &= \psi_e^*(\mathbf{x}, t)\psi_o(\mathbf{x}, t) - \psi_o^*(\mathbf{x}, t)\psi_e(\mathbf{x}, t), \end{aligned} \tag{39}$$

where $\rho_r(\mathbf{x}, t)$ and $\rho_c(\mathbf{x}, t)$ satisfy the properties:

$$\begin{aligned} \rho_r^*(\mathbf{x}, t) = \rho_r(\mathbf{x}, t), \quad \mathcal{P}\rho_r(\mathbf{x}, t) = \rho_r(\mathbf{x}, t), \\ \rho_c^*(\mathbf{x}, t) = -\rho_c(\mathbf{x}, t), \quad \mathcal{P}\rho_c(\mathbf{x}, t) = -\rho_c(\mathbf{x}, t), \end{aligned} \tag{40}$$

- The total number N does not receive any contribution from the parity-odd purely imaginary term $\rho_c(\mathbf{x}, t)$ and is real, $N = \int d^d\mathbf{x}\rho_r(\mathbf{x}, t)$.

- Similar decomposition of the field indicates that the conserved quantities corresponding to the time translation, dilatation and special conformal transformation are real valued.

Conclusions:

- A non-local VNLE is consider which is shown to be integrable. The soliton solutions are obtained using IST methods.
- The exact soliton solutions for the inhomogeneous and/or non-autonomous non-local NLSE is obtained using ST.
- The invariance of the action of a $d + 1$ dimensional generalization of the non-local NLSE under different symmetry transformations is presented.

Conclusions:

- It is shown that some of the conserved Noether charges are real-valued, although the formal expressions of these conserved Noether charges are in general complex.

D. Sinha and P. K. Ghosh, Phys. Rev. E **91**, 042908 (2015).

D. Sinha and P. K. Ghosh, Physics Letter A, 381, 3, 124-128 (2017).

Thank You

Norming constants

$$C_1(0) = \frac{1}{\sqrt{2}}(\eta + \tilde{\eta})e^{i(\theta_1 + \frac{\pi}{2})}, \quad \tilde{C}_1(0) = \frac{1}{\sqrt{2}}(\eta + \tilde{\eta})e^{i(\tilde{\theta}_1 + \frac{\pi}{2})},$$
$$C_2(0) = \frac{1}{\sqrt{2}}(\eta + \tilde{\eta})e^{i(\theta_2 + \frac{\pi}{2})}, \quad \tilde{C}_2(0) = \frac{1}{\sqrt{2}}(\eta + \tilde{\eta})e^{i(\tilde{\theta}_2 + \frac{\pi}{2})}. \quad (41)$$

with

$$(\theta_1 + \tilde{\theta}_1) = (\theta_2 + \tilde{\theta}_2). \quad (42)$$

Conformal symmetry for $pd = 2$:

$$\begin{aligned}
\mathbf{x} \rightarrow \mathbf{x}_h &= \dot{\tau}^{-\frac{1}{2}}(t)\mathbf{x}, \quad t \rightarrow \tau = \tau(t) \\
\psi(\mathbf{x}, t) \rightarrow \psi_h(\mathbf{x}_h, \tau) &= \dot{\tau}^{\frac{d}{4}} exp(-i\frac{\ddot{\tau}}{4\dot{\tau}} x_h^2)\psi(\mathbf{x}, t) \\
\psi^*(\mathcal{P}\mathbf{x}, t) \rightarrow \psi_h^*(\mathcal{P}\mathbf{x}_h, \tau) &= \dot{\tau}^{\frac{d}{4}} exp(i\frac{\ddot{\tau}}{4\dot{\tau}} x_h^2)\psi^*(\mathcal{P}\mathbf{x}, t),
\end{aligned}$$

where

$$\tau(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \ \alpha\delta - \beta\gamma = 1. \tag{43}$$

The particular choices $\tau(t) = t + \beta$, $\tau(t) = \alpha^2 t$, and $\tau(t) = \frac{t}{1+\gamma t}$ correspond to time translation, dilatation and special conformal transformation, respectively.

The conserved charges corresponding to dilatation(D) and special conformation transformation(K) are,

$$D = tH - I_2 \tag{44}$$

$$K = -t^2 H + 2tD + I_1, \tag{45}$$

where the moments I_1 and I_2 are defined as,

$$I_1(t) = \frac{1}{2}\int d^d\mathbf{x}\ x^2\ \rho(\mathbf{x},t), \quad I_2(t) = \frac{1}{2}\int d^d\mathbf{x}\ \mathbf{x}\cdot\mathbf{J}, \tag{46}$$