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Integrable Systems with balanced loss and gain

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References: arXiv:1804.02366; arXiv:1709.09648;
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Plan of the seminar

- Prologue
- Bateman Oscillator
- System-bath coupling: Equilibrium state
- Hamiltonian formulation for generic systems
  1. General Construction
  2. Generic features
  3. An interpretation in terms of Magnetic Field
- Quantization
- Integrability: Classical & Quantum
- Examples: Solved classical models
- Calogero-type models: Classical & Quantum
- Epilogue

Prologue

Several advantages of having a Hamiltonian system

- Application of canonical perturbation theory & canonical quantization scheme
- Application of KAM theory
- Application of Geometric Mechanics & tools for studying phase-transitions in configuration space
- ....

Necessitates Hamiltonian(H) for dissipative systems

System + Bath $\Rightarrow$ H with balanced loss & gain

Objectives

- Systematic method for constructing H
- Finding H admitting equilibrium state
- Integrability of H

Bateman Oscillator

$$\text{System:} \quad \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \Rightarrow \text{Dissipative Oscillator}$$
$$\text{Bath:} \quad \ddot{y} - 2\gamma\dot{y} + \omega_0^2 y = 0 \Rightarrow \text{Auxiliary Oscillator}$$

DO & AO together form a Hamiltonian system:

$$H_B = P_x P_y + \gamma(yP_y - xP_x) + (\omega_0^2 - \gamma^2)xy$$
$$P_x = \dot{y} - \gamma y, \quad P_y = \dot{x} + \gamma x$$

- Gain and loss are equally balanced
- DO & AO are time-reversed version of each other
- H_B is $\mathcal{PT}$-symmetric:

$$\mathcal{T} : t \to -t, \ \mathcal{P} : x \to y, y \to x$$
$$\mathcal{PT} : x \to y, y \to x, \quad P_x \to -P_y, \ P_y \to -P_x$$

- No equilibrium state

System-bath coupling

Hamiltonian formulation of dissipative nonlinear system necessarily introduces Unidirectional Coupling between System & Bath

$$H_U = H_B + yf(x)$$
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x + f(x) = 0$$
$$\ddot{y} - 2\gamma\dot{y} + \left[\omega_0^2 + f'(x)\right] y = 0$$

- $f(x)$ introduces nonlinearity. $f'(x)$ depends on x.
- Time-evolution of system is independent of bath
- $y(t) \sim$ solution of time-dependent linear equation
- Bi-directional coupling $\Rightarrow$ Equilibrium state

$$H = H_B + V(x,y), \; V_x \equiv \frac{\partial V}{\partial x}, \quad V_y \equiv \frac{\partial V}{\partial y}$$

  V_x and V_y depend on both x and y

- $V(x,y)$ and H are not necessarily $\mathcal{PT}$ symmetric

Example: Equilibrium state

$$V(x,y) = \frac{1}{2}\left(\epsilon_1 x^2 + \epsilon_2 y^2\right) + \frac{g}{2(x-y)^2}$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x + \epsilon_2 y + \frac{g}{(x-y)^3} = 0$$

$$\ddot{y} - 2\gamma\dot{y} + \omega_0^2 y + \epsilon_1 x - \frac{g}{(x-y)^3} = 0$$

- H is $\mathcal{PT}$ symmetric for $\epsilon_1 = \epsilon_2$
- Condition for equilibrium state

$$-\frac{\omega_0}{2} < \gamma < \frac{\omega_0}{2}, \quad \epsilon_1\epsilon_2 > 0$$

$$4\gamma\sqrt{\omega_0^2 - 4\gamma^2} < \sqrt{\epsilon_1\epsilon_2} < \omega_0^2$$

- $\epsilon_1 = \epsilon_2 = -\omega_0^2$: A_2-type Rational Calogero-Model
- $\epsilon_1 = \epsilon_2 = 0$: Sutherland-variant of the model
- $\epsilon_1 = \epsilon_2, g = 0$:Coupled SHO of Bender et. al.
- RCM & coupled oscillator model: exactly solvable

Hamiltonian formulation for generic systems

$$\ddot{x}_i + \eta_i(x_1, x_2, \ldots, x_N)\dot{x}_i + G_i(x_1, x_2, \ldots, x_N) = 0$$

- System with space-dependent gain-loss term
- Wide areas of applicability including photonics
- Conservative system:

$$\sum_{i=1}^{N} \eta_i(x_1, x_2, \ldots, x_N) = 0$$

- Hamiltonian system for $\eta_i = 0 \; \forall \; i$ and $G_i = \frac{\partial V}{\partial x_i}$
- For all $\eta_i \neq 0$, is it a Hamiltonian system?
  Any systematic procedure for constructing it?

General Construction

- Definitions, Notations etc.

$$X^T = (x_1, x_2, \ldots, x_N), \;\; P^T = (p_1, p_2, \ldots, p_N),$$
$$F^T = (F_1, F_2, \ldots, F_N), \;\; F_i \equiv F_i(x_1, x_2, \ldots, x_N)$$

- Generalized Momenta: $\Pi = P + AF$
  A is $N \times N$ constant matrix.
  $\mathcal{A} = AF$ may be interpreted as gauge potential

- Hamiltonian

$$H = \Pi^T M \Pi + V(x_1, x_2, \ldots, x_N), \;\; M^T = M$$

M is $N \times N$ non-singular, constant matrix

- Equations of Motion

$$\ddot{X} - 2D\dot{X} + 2M\frac{\partial V}{\partial X} = 0$$
$$[J]_{ij} \equiv \frac{\partial F_i}{\partial x_j}, \quad R \equiv AJ - (AJ)^T, \;\; D := MR$$
$$\frac{\partial V}{\partial X} \equiv \left(\frac{\partial V}{\partial x_i}, \frac{\partial V}{\partial x_2}, \ldots \frac{\partial V}{\partial x_N}\right)^T.$$

Generic features

- Hamiltonian ⇒ Balanced loss-gain [$Tr(D) = 0$]

$$M^T = M, \; R^T = -R, \; D^T = D$$
$$\{M, R\} = 0, \; \{M, D\} = 0, \; \{R, D\} = 0$$

- Pair-wise balancing for $N = 2m, m \in \mathbb{Z}^+$

$$\det(D) \left[1 - (-1)^N\right] = 0$$

$N = 2m + 1$: At least one eigenvalue of D is zero

- H in the background of a Pseudo-Euclidean metric

$$\begin{aligned}
M_d &= \hat{O} M \hat{O}^T \qquad (O^T O = I_{2m}) \\
&= diagonal(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_m, -\lambda_m) \\
\tilde{X} &= \hat{O} X, \; \tilde{P} = \hat{O} P, \; \tilde{\Pi} = \hat{O} \Pi \\
H &= \tilde{\Pi}^T M_d \tilde{\Pi} + V(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)
\end{aligned}$$

Representation of Matrices

- A particular choice for $N = 2m$

$$M = I_m \otimes \sigma_x, A = \frac{-i\gamma}{2} I_m \otimes \sigma_y, \;\; D = \gamma \chi_m \otimes \sigma_z$$

$$[\chi_m]_{ij} = \frac{1}{2} \delta_{ij} Q_i(x_1, x_2, \ldots, x_N)$$

- Condition for determining Q_i: $D = MR$

$$\chi_m \otimes I_2 = \frac{1}{2} J + \frac{1}{2} \{I_m \otimes \sigma_y\} J^T \{I_m \otimes \sigma_y\}$$

- Assumption based on Pairwise Balancing:

$$F_{2i-1} \equiv F_{2i-1}(x_{2i-1}, x_{2i}), \;\; F_{2i} \equiv F_{2i}(x_{2i-1}, x_{2i})$$

- J has the expression: $J = \sum_{i=1}^{m} U_i^{(m)} \otimes V_i^{(2)}$

$$\left[U_a^{(m)}\right]_{ij} \equiv \delta_{ia}\delta_{ja}, \quad V_a^{(2)} \equiv \begin{pmatrix} \frac{\partial F_{2a-1}}{\partial x_{2a-1}} & \frac{\partial F_{2a-1}}{\partial x_{2a}} \\ \frac{\partial F_{2a}}{\partial x_{2a-1}} & \frac{\partial F_{2a}}{\partial x_{2a}} \end{pmatrix}$$

- $Q_a(x_{2a-1}, x_{2a}) = Trace(V_a^{(2)})$

- $N = 2m + 1$: Add one extra column & one row to M, R, D such that all $(i, 2m+1)$th and $(2m+1, i)$th elements are zero except $M_{2m+1,2m+1}$

An interpretation

- $\hat{O} = \frac{1}{\sqrt{2}}\left[I_m \otimes (\sigma_x + \sigma_z)\right]$ diagonalizes M and generates the Co-ordinate transformation

$$z_i^{\pm} = \frac{1}{\sqrt{2}}(x_{2i-1} \pm x_{2i}), \;\; P_{z_i^{\pm}} = \pm\frac{1}{2}\left(\dot{z}_i^{\pm} - \gamma F_i^{\mp}\right)$$

$$F_i^{\pm} = \frac{1}{\sqrt{2}}\left(F_{2i-1} \pm F_{2i}\right), \;\; F_i^{\pm} \equiv F_i^{\pm}(z_i^{+}, z_i^{-})$$

- H describes a system of m particles on a Pseudo-Euclidean plane interacting with each other through $V \equiv V(z_1^-, \dots, z_m^-, z_1^+, \dots, z_m^+)$

$$H = \sum_{i=1}^{m}\left[\left(P_{z_i^+} + \frac{\gamma}{2}F_i^-\right)^2 - \left(P_{z_i^-} - \frac{\gamma}{2}F_i^+\right)^2\right] + V$$

- The i'th particle is subjected to magnetic field Q_i

$$Q_i = \frac{\partial F_i^+}{\partial z_i^+} + \frac{\partial F_i^-}{\partial z_i^-}$$

Gauge Transformations & Lagrangian

- Different choices of (F_i^+, F_i^-) leading to same Q_i are related through gauge transformation

- Explicit form of H depends on choice of the gauge

- Lagrangians differ by total time-derivative term under gauge transformations

$$L = \sum_{i=1}^{m} \left[\frac{1}{4} \left\{ (\dot{z}_i^+)^2 - (\dot{z}_i^-)^2 \right\} + \frac{\gamma}{2} \left(\dot{z}_i^- F_i^+ - \dot{z}_i^+ F_i^- \right) \right] - V$$

- Equations of motion remain the same

$$\ddot{z}_i^{\pm} - \gamma Q_i(z_i^+, z_i^-) \dot{z}_i^{\mp} \pm 2 \frac{\partial V}{\partial z_i^{\pm}} = 0,$$

- Space-dependent loss/gain co-efficients Q_i are identified with analogous magnetic field

Quantization

- $z_i^{\pm}$ and $P_{z_i}^{\pm} := -i\partial_{z_i^{\pm}}$ are treated as operators with the non-vanishing commutation relations($\hbar = 1$):

$$\left[z_j^+, P_{z_j^+}\right] = i, \left[z_j^-, P_{z_j^-}\right] = i$$

- Generalized momenta $\hat{\Pi}_{z_i^{\pm}} := -i\partial_{z_i^{\pm}} \pm \frac{\gamma}{2} F_i^{\mp}$

$$\left[\hat{\Pi}_{z_i^{\pm}}, \hat{\Pi}_{z_j^{\pm}}\right] = 0, \quad \left[\hat{\Pi}_{z_i^-}, \hat{\Pi}_{z_j^+}\right] = -\delta_{ij} \frac{i\gamma}{2} Q_i(z_i^-, z_i^+)$$

- In general, $\hat{H}$ is non-hermitian for standard B.C.

$$\hat{H} = \sum_{i=1}^{m} \left[\left(\hat{\Pi}_{z_i^+}\right)^2 - \left(\hat{\Pi}_{z_i^-}\right)^2 \right] + V(z_1^{\pm}, \ldots, z_m^{\pm})$$

Normalizable wf only in appropriate Stoke wedges

- $\hat{H}$ for different gauges are related through unitary transformation

$$S_{\pm} := exp\left[\frac{i\gamma}{2} \sum_{j=1}^{m} \int F_j^{\pm}(z_j^-, z_j^+) dz_i^{\mp}\right]$$

Integrability

- Translational invariant system(TIS)

  $$V \equiv V(z_1^-, z_2^-, \ldots, z_m^-), \quad Q_i \equiv Q_i(z_1^-, z_2^-, \ldots, z_m^-)$$

- Symmetry transformation:

  $$x_{2i-1} \rightarrow x_{2i-1} + \eta_i, \; x_{2i} \rightarrow x_{2i} + \eta_i$$

  η_i are m independent parameters

- Integrals of motion

  $$\Pi_i = 2P_{z_i^+} + \gamma F_i^- - \gamma \int Q_i(z_1^-, z_2^-, \ldots, z_m^-) dz_i^-$$

- Partial(complete) integrability for $m > 1$(m=1)

  $$\{H, \Pi_i\}_{PB} = 0 \quad \{\Pi_i, \Pi_j\}_{PB} = 0$$

- Similar results for $V \equiv V(z_1^+, z_2^+, \ldots, z_m^+), Q_i \equiv Q_i(z_1^+, z_2^+, \ldots, z_m^+)$

- Quantum Integrability: $\{., .\}_{PB} \rightarrow [., .]$

Rotational Invariant System(RIS)

- Parametrization of co-ordinates

$$z_i^+ = r_i \cosh\theta_i, \quad z_i^- = r_i \sinh\theta_i$$

- Symmetry transformation

  Hyperbolic rotation in each '$z_i^- - z_i^+$' plane

- Condition for invariance of action

$$V \equiv V(r_1, \ldots, r_m)$$

$$F_i^+ = z_i^+ g(r_1, \ldots, r_m), \; F_i^- = z_i^- g(r_1, \ldots, r_m)$$

- Integrals of motion

$$L_i = -r_i^2 \dot{\theta}_i + \gamma r_i^2 g(r_1, \ldots, r_m)$$

- Partial(complete) integrability for $m > 1$(m=1)

$$\{H, L_i\}_{PB} = 0 \quad \{L_i, L_j\}_{PB} = 0$$

- Quantum Integrability: $\{., .\}_{PB} \to [., .]$

Solved classical models:TIS

- Cubic oscillator with constant gain/loss term:

$$V(z_1^-) = -2\omega_0^2(z_1^-)^2 - \frac{\alpha}{4}(z_1^-)^4, \ \omega, \alpha \in \Re,$$

$$\ddot{z}_1^- + \omega^2 z_1^- + \alpha(z_1^-)^3, \ \omega^2 \equiv 4(\omega_0^2 - \gamma^2)$$

$$z_1^+(t) = 2\gamma \int z_i^-(t)dt$$

- Non-singular solutions for (a) $-\omega_0 < \gamma < \omega_0$, $\omega^2 > 0$, (b) $-\gamma < \omega_0 < \gamma$, $\omega^2 < 0, \alpha > 0$

- γ has an unbounded upper-range for case (b)

$$z_1^-(t) = A \ cn(\Omega t, k), \ \sqrt{\frac{2|\ \omega^2 \ |}{\alpha}} \leq A < \infty$$

$$z_1^+(t) = \frac{2\gamma}{\Omega} \frac{\cos^{-1}\{dn(\Omega t, k)\} sn(\Omega t, k)}{\sqrt{1 - dn^2(\Omega t, k)}}$$

$$\Omega = \sqrt{-|\ \omega^2 \ | + \alpha A^2}, \ k^2 = \frac{\alpha A^2}{2\Omega^2}.$$

More systems with equilibrium state

- Coupled chain of nonlinear oscillator with constant gain/loss term and embedded rotational symmetry

$$r^2 = \sum_{i=1}^{m} z_i^2$$

$$V(r) = -2\omega_0^2 r^2 - \frac{\alpha}{4} r^4 - \frac{\delta}{2r^2}, \ \alpha, \delta \in \Re$$

- Exactly solvable and admits non-singular solutions

- RIS with space-dependent gain/loss term

$$g = cr_1, V = \frac{1}{4}\omega_0^2 r_1^2 + \frac{1}{8}\alpha_0 r_1^4$$

- Exact non-singular solutions for $\alpha_0 > 2\gamma^2 c^2$

$$\begin{aligned} z_1^+ &= Acn[\Omega t, k] \cosh\theta, \qquad z_1^- = Acn[\Omega t, k] \sinh\theta, \\ \theta &= \frac{cA\gamma}{\Omega} \frac{\cos^{-1}\{dn(\Omega t, k)\} sn(\Omega t, k)}{\sqrt{1 - dn^2(\Omega t, k)}} + B \end{aligned}$$

$$\Omega = \sqrt{\omega_0^2 + \alpha A^2}, \ k^2 = \frac{\alpha A^2}{2\Omega^2},$$

Calogero-type models

Many-body system with $V_C(\omega_0, g) = V_I(\omega_0) + V_{II}(g)$

$$V_I(\omega_0) = -\sum_{i}^{m} \omega_0^2 (x_{2i-1} - x_{2i})^2$$

$$V_{II}(g) = -\sum_{\substack{i,j=1 \\ i<j}}^{m} \frac{g^2}{(x_{2i-1} - x_{2i} - x_{2j-1} + x_{2j})^2}$$

- Like RCM, V_{II} scales inverse-squarely & V_I describes pair-wise harmonic interaction

- V_C is translational invariant involving m independent parameters. RCM is translational invariant involving a single parameter

- Unlike RCM, V_{II} involves four-body interaction terms and harmonic interaction does not include all possible pairs

- Unlike RCM, V_{II} is not invariant under permutation symmetry S_{2m}. If each pair (x_{2i-1}, x_{2i}) is considered as an element, then, V_{II} is invariant under S_m

Equations of motion

$$\begin{aligned}
\ddot{x}_{2l-1} - 2\gamma\dot{x}_{2l-1} &= -2\omega_0^2(x_{2l-1} - x_{2l}) \\
&+ \sum_{\substack{i=1 \\ i\neq l}}^{m} \frac{2g^2}{(x_{2i-1} - x_{2i} - x_{2l-1} + x_{2l})^3}, \\
\ddot{x}_{2l} + 2\gamma\dot{x}_{2l} &= 2\omega_0^2(x_{2l-1} - x_{2l}) \\
&- \sum_{\substack{i=1 \\ i\neq l}}^{m} \frac{2g^2}{(x_{2i-1} - x_{2i} - x_{2l-1} + x_{2l})^3}
\end{aligned}$$

Mapped to Calogero Model in terms of $z_i^{\pm}$ co-ordinates:

$$V_C(z_i^-) = -\sum_{i=1}^{m} 2\omega_0^2(z_i^-)^2 - \sum_{\substack{i,j=1 \\ i<j}}^{m} \frac{g^2}{2(z_i^- - z_j^-)^2},$$

$$\ddot{z}_i^- + \omega^2 z_i^- - \sum_{j,(j\neq i)}^{m} \frac{g^2}{(z_i^- - z_j^-)^3} = 0$$

$$z_i^+(t) = 2\gamma \int z_i^-(t)dt + C_i, \quad i = 1, 2, \ldots m.$$

Exactly solvable with periodic solutions

Quantum Calogero-type models

- Quantum Hamiltonian $\hat{H}_L = S^{-1}\hat{H}S$ is suitable for box normalization. $V_L = V_C(\frac{\omega}{2}, \sqrt{2g})$

$$S \quad := \quad exp\left[\frac{i\gamma}{2}\sum_{j=1}^{m} z_j^+ z_j^-\right],$$

$$\hat{H}_L \quad = \quad \sum_{i=1}^{m}\left[\left(-i\partial_{z_i^+} + \gamma z_i^-\right)^2 - P_{z_i^-}^2\right] + V_L$$

- $P_{z_i^+}$ are quantum integrals of motion

$$[\hat{H}_L, P_{z_i^+}] = 0, \;\; [P_{z_i^+}, P_{z_j^+}] = 0. \;\; \forall\ i, j,$$

- Separation of variables: $\hat{H}_L\chi = E\chi$

$$\chi(z_1^\pm, \ldots, z_m^\pm) = \psi(z_1^-, \ldots, z_m^-)\exp\left[i\sum_{j=1}^{m} z_j^+ k_j\right]$$

$$\sum_{j=1}^{m}\left[\partial_{z_j^-}^2 + \gamma^2\left(z_j^- + \frac{k_j}{\gamma}\right)^2\right]\psi + V_L\psi = E\psi$$

Eigenvalues

Eigenvalue equation of standard Calogero model is obtained in the sector $k_j = k \; \forall \; j$ with energy $-\tilde{E}$

$$\sum_{j=1}^{m} \left(-\partial_{z_j}^2 + \Omega^2 z_j^2\right) \psi + \sum_{\substack{i,j=1 \\ i<j}}^{m} \frac{g}{(z_i - z_j)^2} \psi = -\tilde{E} \psi,$$

$$\Omega^2 = \frac{1}{2}(\omega^2 - 2\gamma^2), \;\; \tilde{E} = E - \frac{m\omega^2 k^2}{2\Omega^2}, \;\; z_j = z_j^- - \frac{\gamma k}{\Omega^2}$$

Exactly solvable for $\lambda = \frac{1}{2}[1 + (1+4g)^{\frac{1}{2}}]$ and a real Ω demands $-\frac{\omega}{\sqrt{2}} \leq \gamma \leq \frac{\omega}{\sqrt{2}}$

$$E = -2\Omega[2n + l + \frac{1}{2}m + \frac{\lambda}{2}m(m-1)] + \frac{mk^2\omega^2}{2\Omega^2}$$

- $k = 0$: E is bounded from below for $\Omega < 0$
- E consists of discrete as well as continuous spectra
- Box normalization: $0 \leq z_i^+ \leq L, \;\; \forall i$

$$E = 2|\;\Omega\;|[2n + l + \frac{1}{2}m + \frac{\lambda}{2}m(m-1)] + \frac{2m\pi^2\omega^2\hat{k}^2}{L^2\Omega^2}$$

Normalization of wave-functions

- Asymptotic form of the wave function χ

$$\chi \sim \exp[\frac{\mid \Omega \mid}{2} \sum_{i=1}^{m} z_i^2 + i\frac{2\pi \hat{k} m}{L} z_j^+]$$

- Eigenfunctions are not normalizable along real z_i lines. Normalizable solutions in complex z_i-planes

$$z_i = r_i exp[i\theta_i], \quad \sum_{i=1}^{m} \cos(2\theta_i) < 0$$

- Possible solution: $\theta_i = \theta \ \forall \ i$, a pair of Stoke wedges with opening angle $\frac{\pi}{2}$ and centered about the positive and negative imaginary axes

Correlation functions

$$R_n(x_1, x_2, ...x_n) \;=\; \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=n+1}^{N} dx_i$$
$$\times \;\; |\, \chi(x_1, x_2, ..., x_N)\, |^2, \;\; n < N$$

Define $y_i = \sqrt{\frac{\Omega}{\lambda}} z_i$. Standard result from RMT & RCM may be used with the following modifications:

- Integrations over z_i^- variables should be carried out in proper Stoke Wedges
- Contribution coming from the measure $\int_0^L \ldots \int_0^L \prod_{i=1}^m dz_i^+$ should be taken into account
- Normalization constant C for χ: $C \propto L^{-\frac{m}{2}}$

Mapping to integrals of RCM only for even n $(y = y_1)$

$$R_2 = \begin{cases} \frac{N(N-1)}{m\pi L}(2m - y^2)^{\frac{1}{2}}, & y^2 < 2m \\ 0, & y^2 > 2m. \end{cases}$$

Differs from RCM by a constant multiplicative factor

Epilogue

- Hamiltonian formulation of generic many-particle systems with space-dependent balanced loss & gain is presented along with general features

- Constructed partial & completely integrable systems related to underlying translation and rotational symmetry

- Presented examples of systems with balanced loss/gain admitting equilibrium state and solved

- A nonlinear oscillator system allows equilibrium state even if the constant gain/loss parameter is unbounded from above

- A Calogero-type model with balanced loss/gain is introduced and solved at the classical as well as quantum level including exact $2n$-particle correlation functions for the ground-state