Non-Hermitian Physics - PHHQP XVIII

Super Periodic Potential

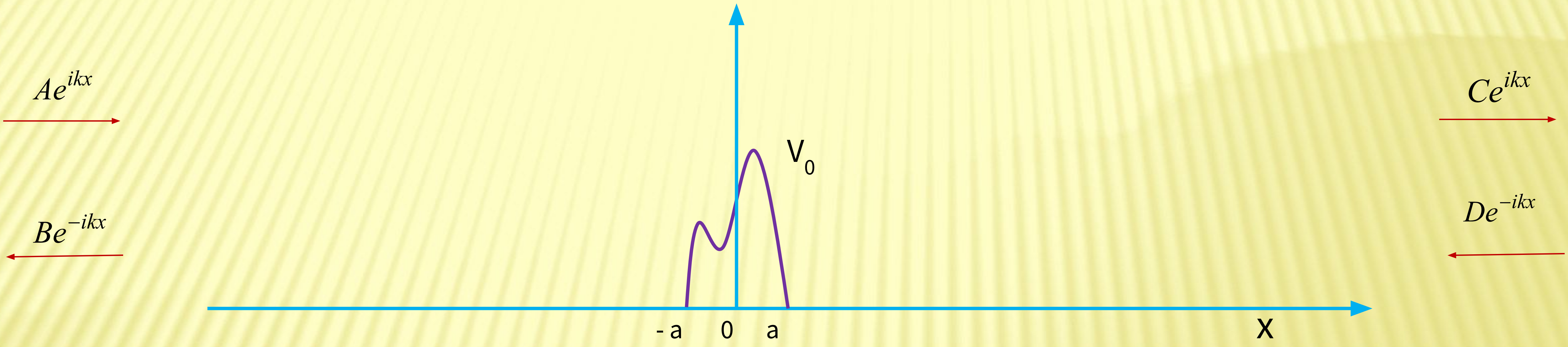
Mohammad Hasan
(Scientist)
Space Astronomy Group
U.R. Rao Satellite Centre
Indian Space Research Organisation (ISRO)
Bangalore-India

Prof. Bhabani Prasad Mandal
Department of Physics
Institute of Science
Banaras Hindu University (B.H.U.)
Varanasi-India

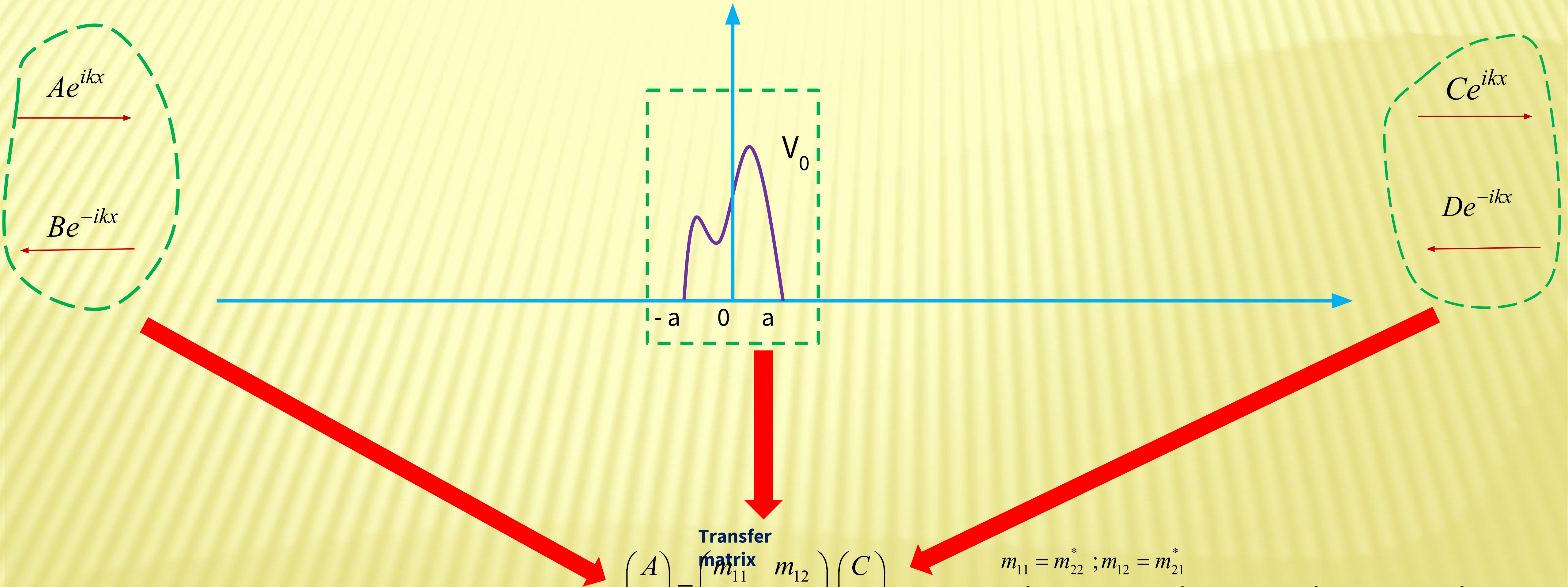
By,
Mohammad Hasan

Concepts
Next→ Transfer matrix elements
Next → Tunneling probability

Super Periodic Potential: Concepts



Super Periodic Potential: Concepts



Transfer matrix

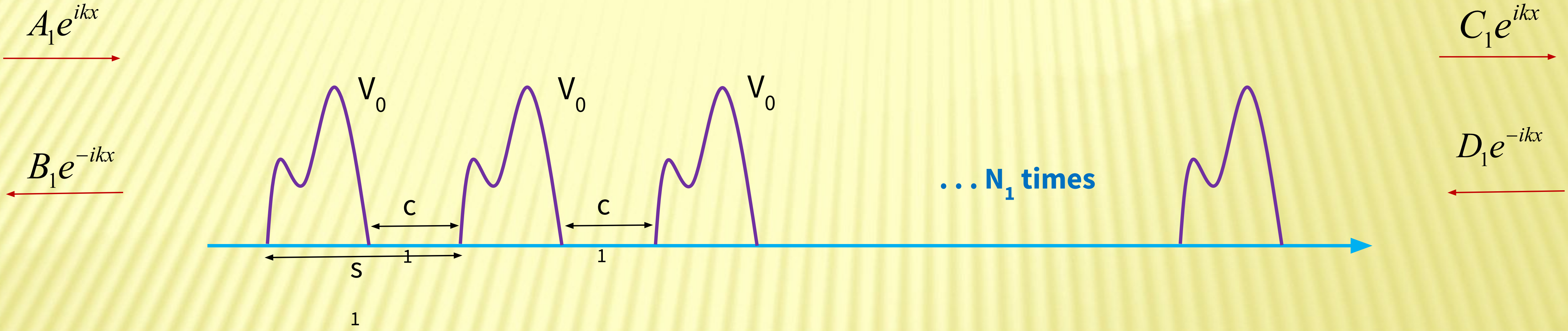
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$m_{11} = m_{22}^* \ ; m_{12} = m_{21}^*$$

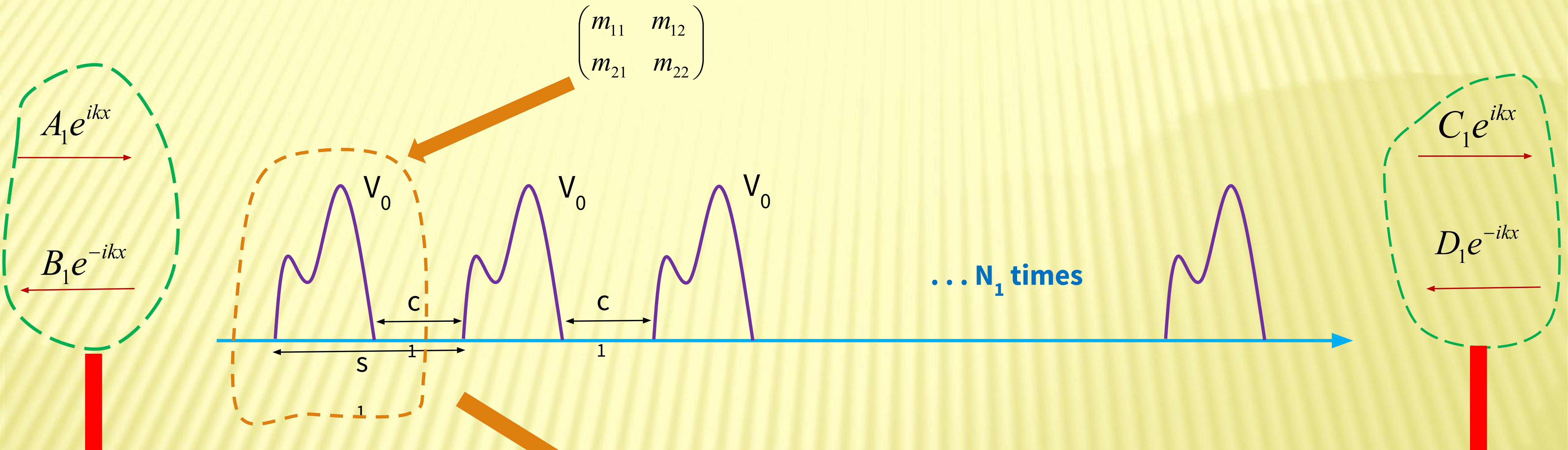
$$T_l = \left|\frac{1}{m_{22}}\right|^2, \ T_r = \left|\frac{\det M}{m_{22}}\right|^2, \ R_l = \left|\frac{m_{21}}{m_{22}}\right|^2, \ R_r = \left|\frac{m_{12}}{m_{22}}\right|^2$$

Concepts
Next→ Transfer matrix elements
Next → Tunneling probability

Super Periodic Potential: Concepts



Super Periodic Potential: Concepts



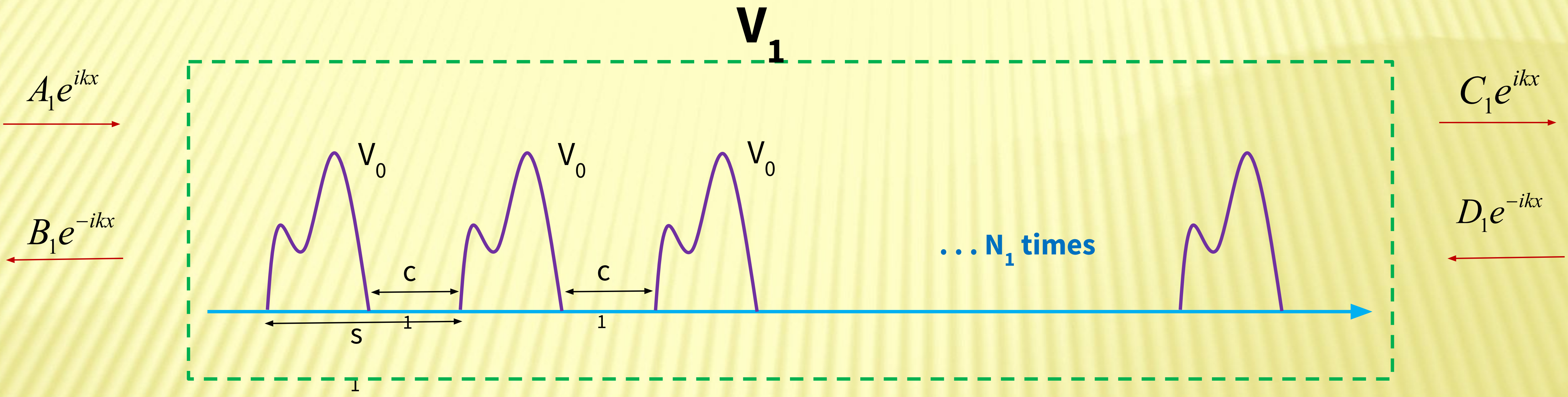
Transfer matrix

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{bmatrix} \left\{ m_{11}\ e^{-iks_1} U_{N_1-1}(\xi_1) - U_{N_1-2}(\xi_1) \right\} e^{ikN_1 s_1} & m_{12}\, U_{N_1-1}(\xi_1)\, e^{-ik(N_1-1)s_1} \\ m_{12}^{*}\, U_{N_1-1}(\xi_1)\, e^{ik(N_1-1)s_1} & \left\{ m_{11}^{*}\ e^{iks_1} U_{N_1-1}(\xi_1) - U_{N_1-2}(\xi_1) \right\} e^{-ikN_1 s_1} \end{bmatrix} \begin{pmatrix} C_1 \\ D_1 \end{pmatrix}$$

$$\xi_1 = |\, m_{22} \,| \cos(\ \alpha + k\, s_1) \ \ ; \ \ \alpha = Arg\,(m_{22})$$

David J. Griffiths and Carl A. Steinke, 'Waves in locally periodic media' , Am. J. Phys. 69 (2), 2001

Super Periodic Potential: Concepts

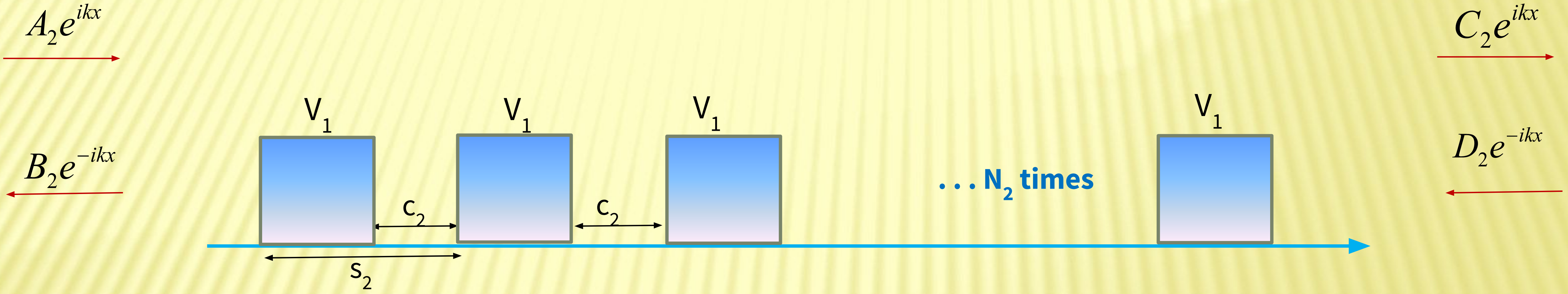


Transfer matrix

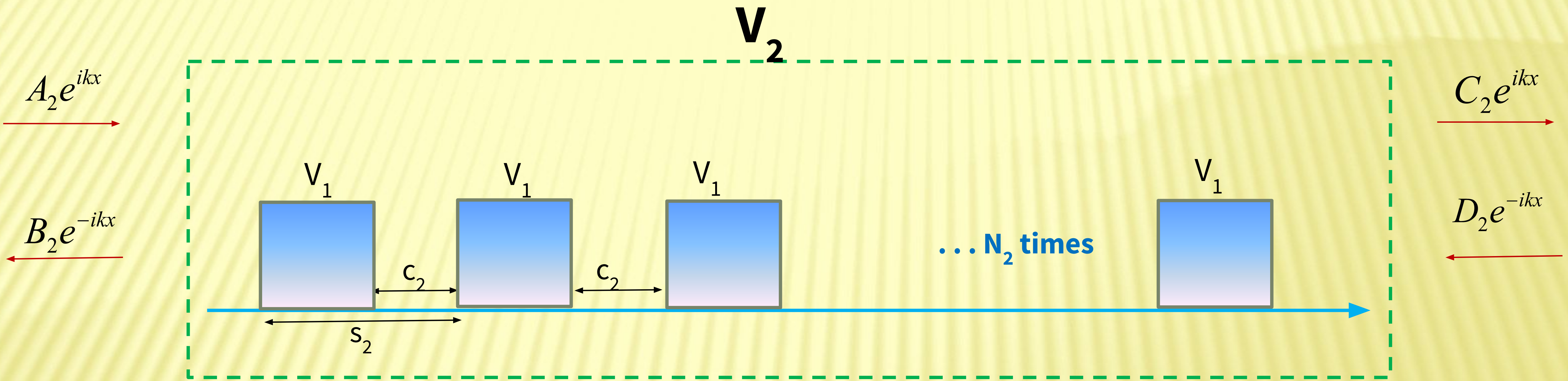
$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} (m_{11})_1 & (m_{12})_1 \\ (m_{21})_1 & (m_{22})_1 \end{bmatrix} \begin{bmatrix} C_1 \\ D_1 \end{bmatrix}$$

David J. Griffiths and Carl A. Steinke, ‘Waves in locally periodic media’ , Am. J. Phys. 69 (2), 2001

Super Periodic Potential: Concepts



Super Periodic Potential: Concepts

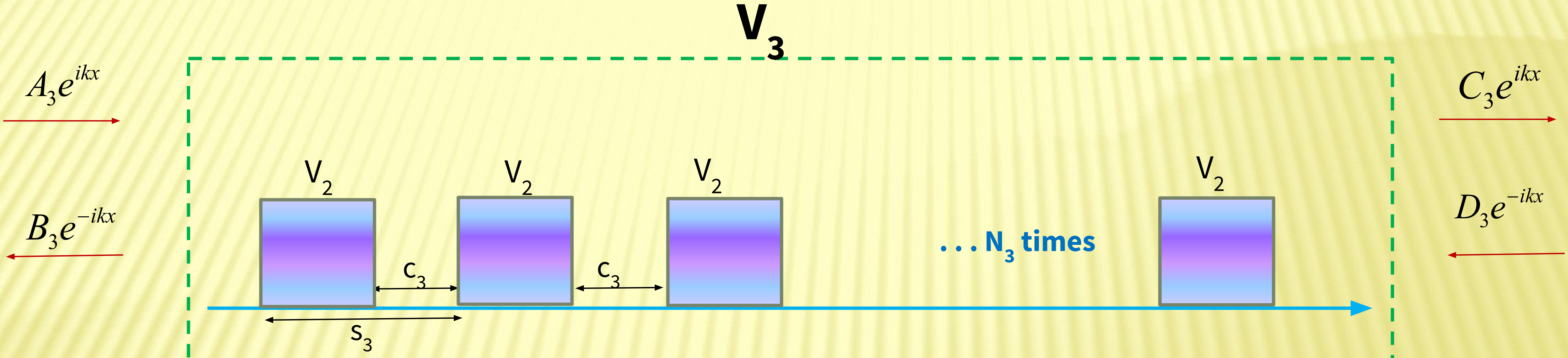


Transfer matrix

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} (m_{11})_2 & (m_{12})_2 \\ (m_{21})_2 & (m_{22})_2 \end{bmatrix} \begin{bmatrix} C_2 \\ D_2 \end{bmatrix}$$

Concepts
Next→ Transfer matrix elements
Next → Tunneling probability

Super Periodic Potential: Concepts

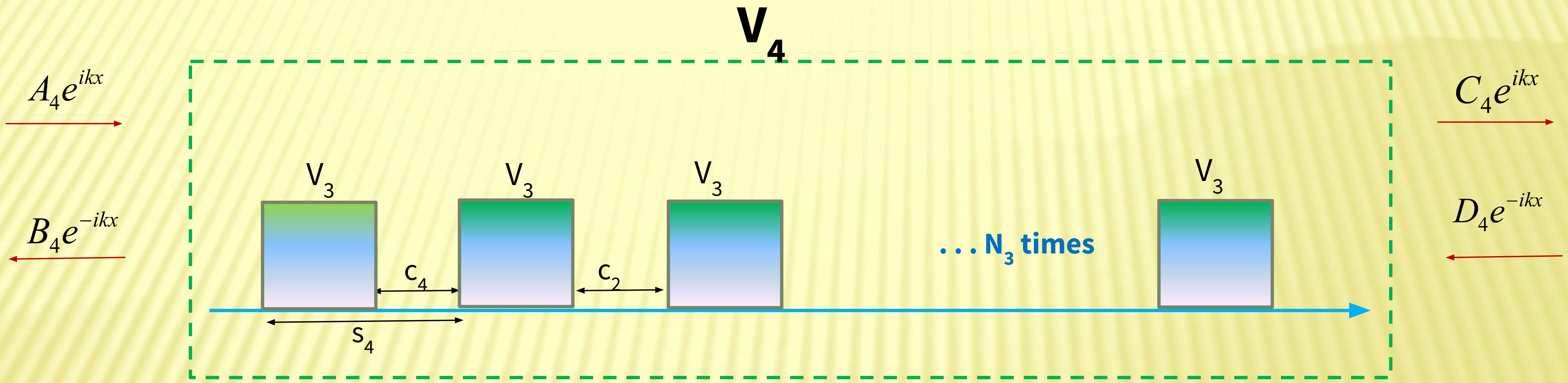


Transfer matrix

$$\begin{bmatrix} A_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} (m_{11})_3 & (m_{12})_3 \\ (m_{21})_3 & (m_{22})_3 \end{bmatrix} \begin{bmatrix} C_3 \\ D_3 \end{bmatrix}$$

Super Periodic Potential: Concepts

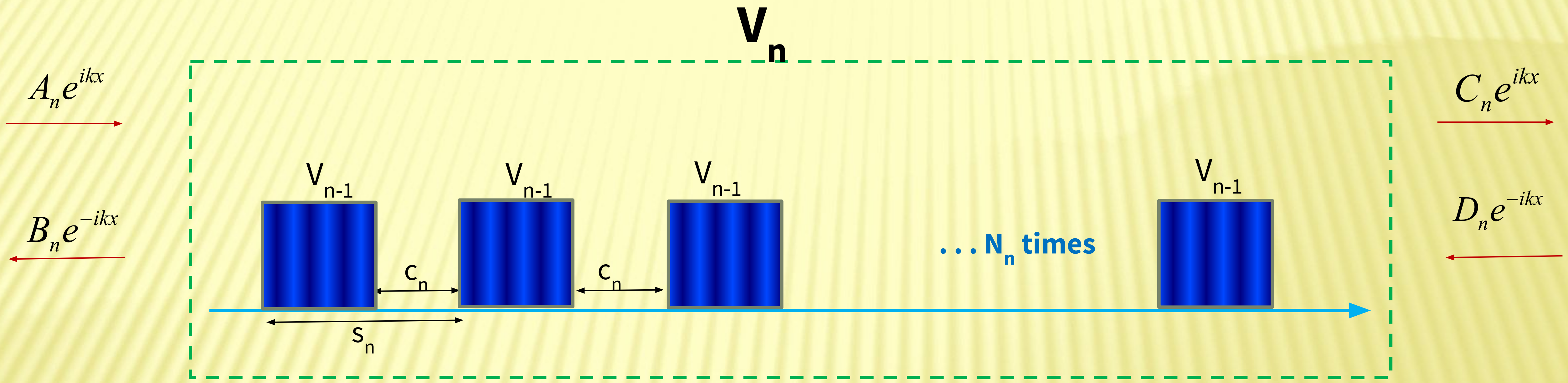
Concepts
Next→ Transfer matrix elements
Next → Tunneling probability



Transfer matrix

$$\begin{bmatrix} A_4 \\ B_4 \end{bmatrix} = \begin{bmatrix} (m_{11})_4 & (m_{12})_4 \\ (m_{21})_4 & (m_{22})_4 \end{bmatrix} \begin{bmatrix} C_4 \\ D_4 \end{bmatrix}$$

Super Periodic Potential: Concepts



Transfer matrix

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} (m_{11})_n & (m_{12})_n \\ (m_{21})_n & (m_{22})_n \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix}$$

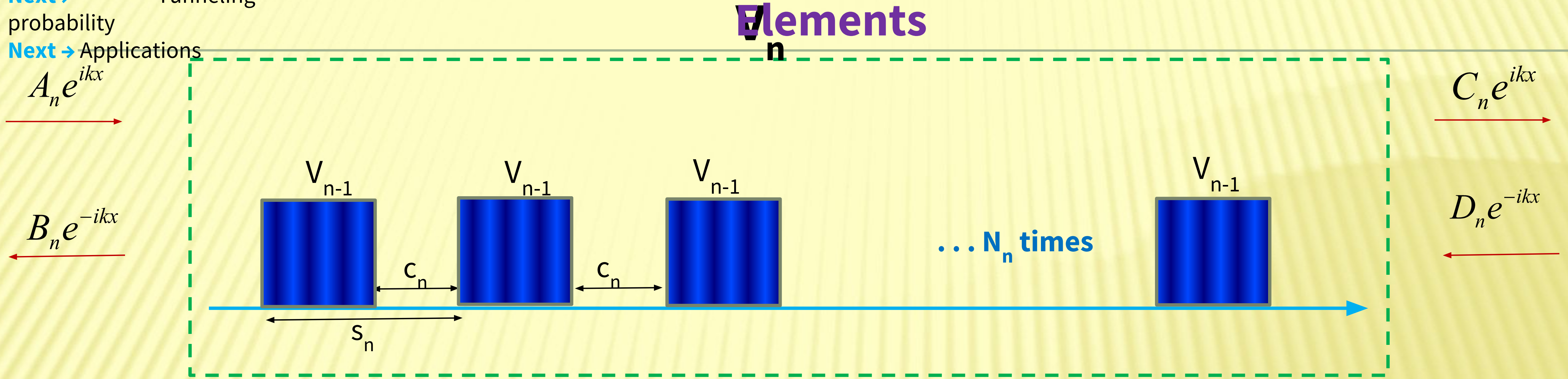
The generated ***potential V_n*** is the **super periodic potential of order 'n'**

Transfer matrix elements

Next→ Tunneling probability

Next → Applications

Super Periodic Potential: Transfer Matrix Elements



Transfer matrix

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} (m_{11})_n & (m_{12})_n \\ (m_{21})_n & (m_{22})_n \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix}$$

$$(m_{22})_n = (m_{11})_n^* \quad , \quad (m_{21})_n = (m_{12})_n^*$$

$$(m_{11})_n = m_{11} e^{ik\sum_{p=1}^{n}(N_p-1)s_p} \prod_{p=1}^{n} U_{N_p-1}(\xi_p) - U_{N_n-2}(\xi_n) e^{ikN_n s_n} - \sum_{r=1}^{n-1} G_r$$

$$G_r = e^{ik\left(\sum_{p=1}^{n} N_p s_p - \sum_{p=1}^{r} N_{p-1} s_{p-1} - \sum_{p=r+1}^{n} s_p\right)} U_{N_r-2}(\xi_r) \prod_{p=r+1}^{n} U_{N_p-1}(\xi_p) \; ; \qquad N_0 = 0 \, , s_0 = 0$$

$$(m_{12})_n = m_{12} e^{-ik\sum_{p=1}^{n}(N_p-1)s_p} \prod_{p=1}^{n} U_{N_p-1}(\xi_p)$$

M. Hasan, B.P. Mandal, Annals of Physics, 391 (2018), 240-262.

Super Periodic Potential: Transfer Matrix Elements

$$\xi_n = |m_{22}| \cos\left[\alpha - k\{\sum_{p=1}^{n-1}(N_p - 1)s_p - s_n\}\right] \prod_{p=1}^{n-1} U_{N_p-1}(\xi_p) - \sum_{r=1}^{n-1} H_r - U_{N_{n-1}-2}(\xi_{n-1}) \cos k(N_{n-1}s_{n-1} - s_n)$$

Where,

$$H_r = \cos\left[k(\sum_{p=r}^{n-1} N_p s_p - \sum_{p=r+1}^{n} s_p)\right] U_{N_r-2}(\xi_r) \prod_{p=r+1}^{n-1} U_{N_p-1}(\xi_p)$$

& $\alpha = \text{Arg}\,(m_{22})$

M. Hasan, B.P. Mandal, Annals of Physics, 391 (2018), 240-262.

Super Periodic Potential: Tunneling Probability

Super Periodic Potential	Tunneling probability
$V_1(N_1)$	$T(N_1;k)=\dfrac{1}{1+\left(\vert m_{12}\vert U_{N_1-1}(\xi_1)\right)^2}$; $\xi_1=\vert m_{22}\vert\cos(\alpha+k.s_1)$

Super Periodic Potential: Tunneling Probability

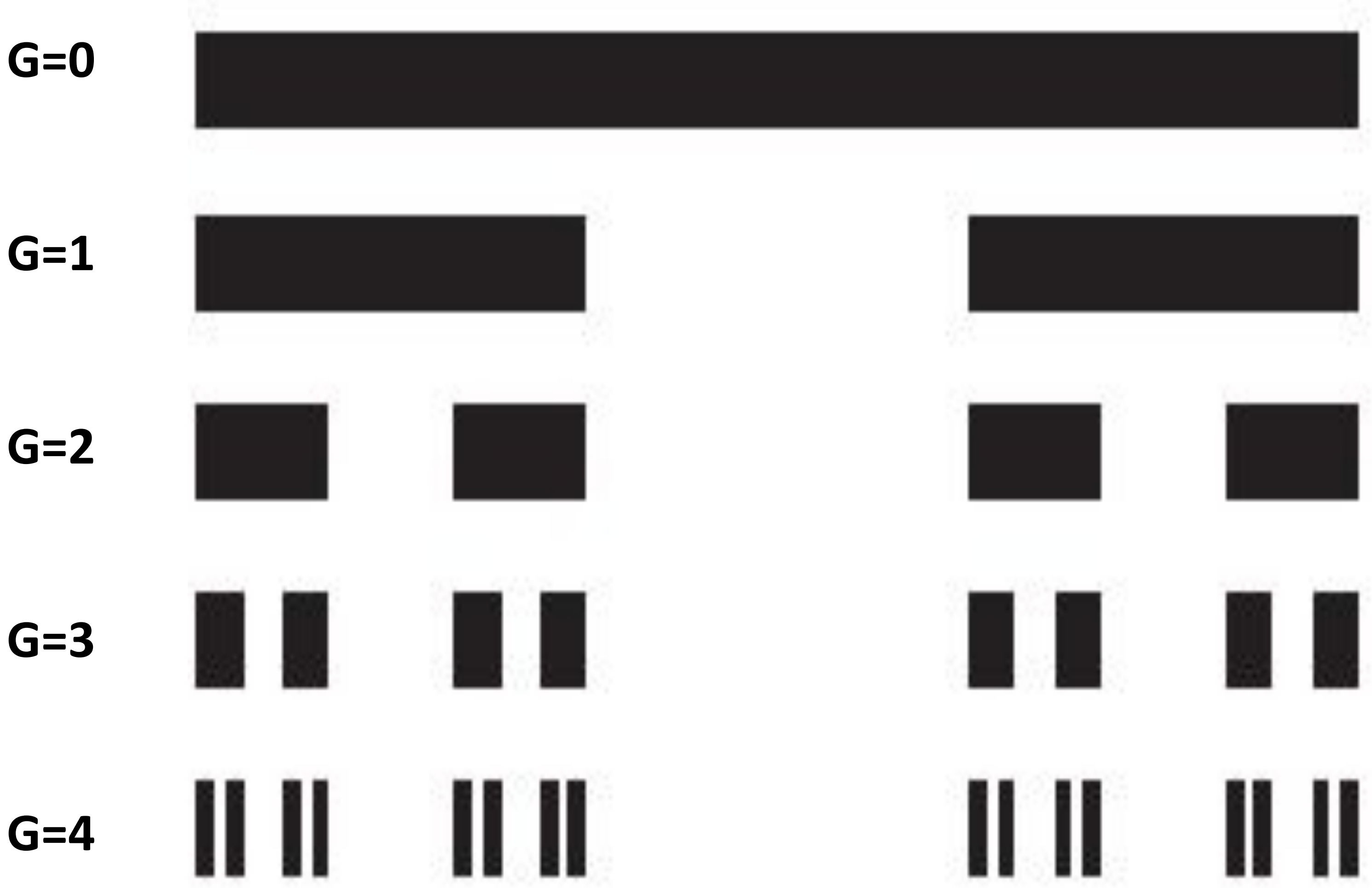
Super Periodic Potential	Tunneling probability
$V_1(N_1)$	$T(N_1;k)=\dfrac{1}{1+\left(\lvert m_{12}\rvert U_{N_1-1}(\xi_1)\right)^2}$; $\xi_1=\lvert m_{22}\rvert\cos(\alpha+k.s_1)$
$V_2(N_1,N_2)$	$T(N_1,N_2;k)=\dfrac{1}{1+\left(\lvert m_{12}\rvert U_{N_1-1}(\xi_1)U_{N_2-1}(\xi_2)\right)^2}$

Super Periodic Potential: Tunneling Probability

Super Periodic Potential	Tunneling probability
$V_1(N_1)$	$T(N_1;k)=\dfrac{1}{1+\left(\lvert m_{12}\rvert U_{N_1-1}(\xi_1)\right)^2}$; $\xi_1=\lvert m_{22}\rvert\cos(\alpha+k.s_1)$
$V_2(N_1,N_2)$	$T(N_1,N_2;k)=\dfrac{1}{1+\left(\lvert m_{12}\rvert U_{N_1-1}(\xi_1)U_{N_2-1}(\xi_2)\right)^2}$
$V_3(N_1,N_2,N_3)$	$T(N_1,N_2,N_3;k)=\dfrac{1}{1+\left(\lvert m_{12}\rvert U_{N_1-1}(\xi_1)U_{N_2-1}(\xi_2)U_{N_3-1}(\xi_3)\right)^2}$
↓	↓
$V_n(N_1,N_2,N_3,..,N_n)$	$T(N_1,N_2,N_3,....,N_n;k)=\dfrac{1}{1+\left(\lvert m_{12}\rvert\prod_{i=1}^{n}U_{N_i-1}(\xi_i)\right)^2}$

Super Periodic Potential: Applications

Cantor-1/3 Potential



Cantor 1/3 potential:

- One-third of the segment from the middle is removed at each stage G.
- The height of black region is the potential height.
- The black region is rectangular barrier potential.

Super Periodic Potential: Applications

Rectangular SPP of order 0

G=4

Super Periodic Potential: Applications

Rectangular SPP of order 1, $N_1=2$

G=4 

Super Periodic Potential: Applications

Rectangular SPP of order 2, $N_2=2$

G=4

Applications
Next⟶ General features
Next⟶ Conclusions

Super Periodic Potential: Applications

Rectangular SPP of order 3, $N_3=2$

G=4

Super Periodic Potential: Applications

Rectangular SPP of order 4, N_4=2

G=4

Super Periodic Potential: Applications

A (general) Cantor fractal potential of stage G is a rectangular super periodic potential of order G.

Rectangular SPP of order 4, N_4=2

G=4

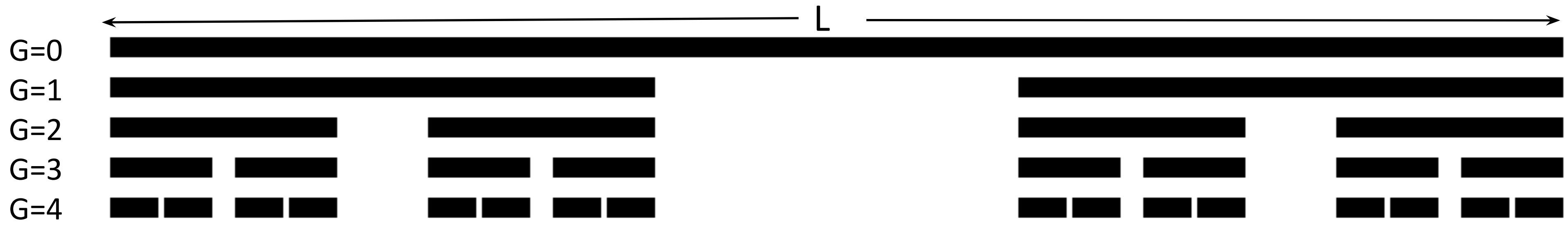
Super Periodic Potential: Applications

- ✓ Our observations suggest that in general, symmetric self-similarity in 1D is the special case of super periodicity in 1D *(yet to be proven mathematically).*

- ✓ By using SPP formalism we have derived the close form expressions of transmission amplitude for *general Cantor potential* and *standard Smith-Volterra-Cantor (SVC-4) potential* (M. Hasan, B.P. Mandal, Annals of Physics, 391 (2018), 240-262.).

Super Periodic Potential: Applications

Tunneling amplitude for standard Smith-Volterra-Cantor (SVC-4) potential of arbitrary stage G

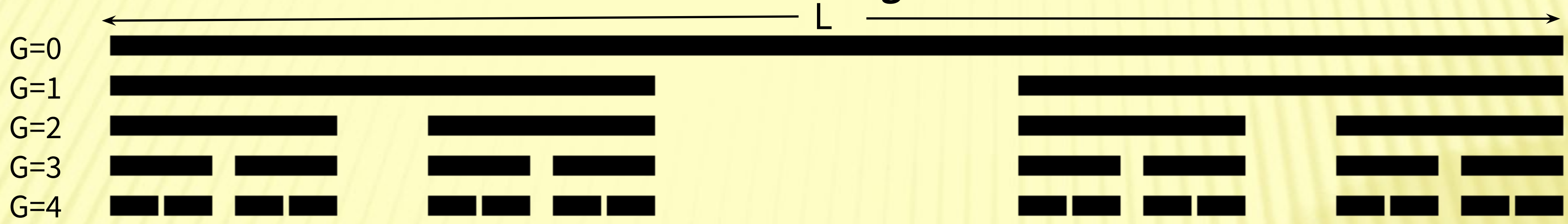


$1/4^G$ fraction from the middle is taken out at stage G in SVC fractal.

M. Hasan, B.P. Mandal, Annals of Physics, 391 (2018), 240-262.

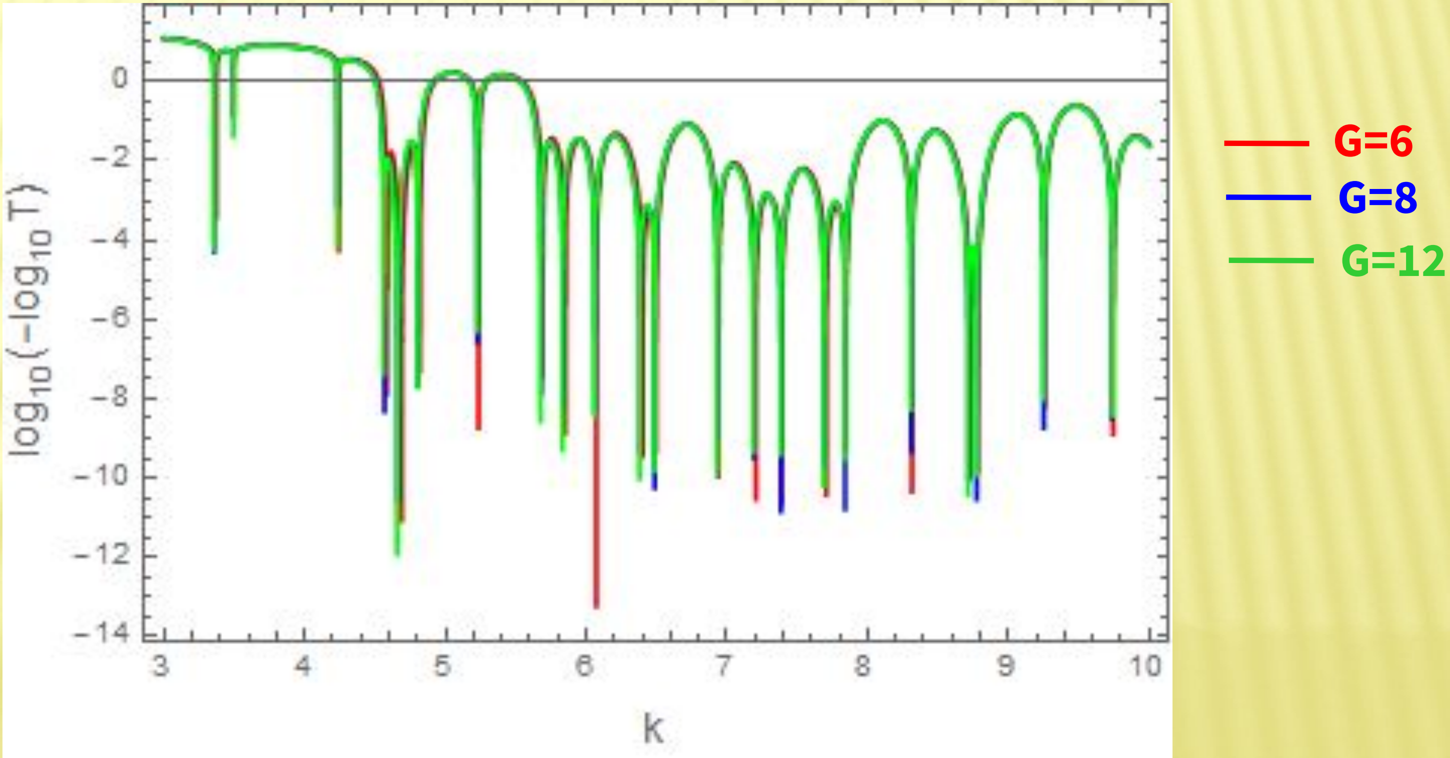
Super Periodic Potential: Applications

Tunneling amplitude for standard Smith-Volterra-Cantor (SVC-4) potential of arbitrary stage G



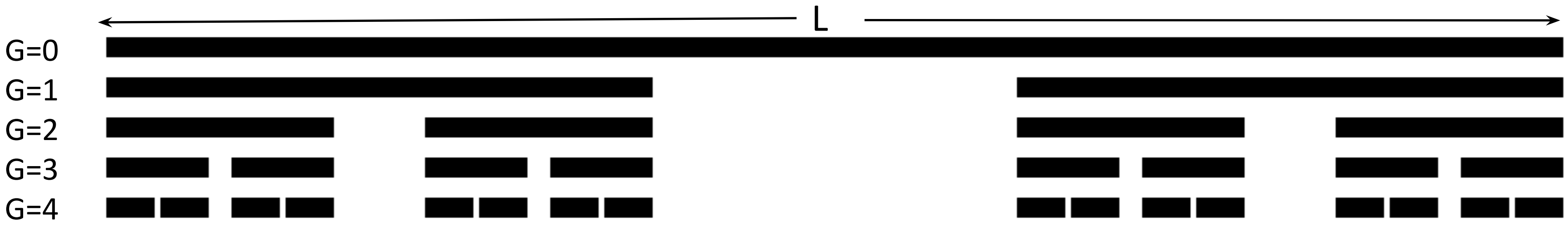
$1/4^G$ fraction from the middle is taken out at stage G in SVC fractal.

V=10, L=10

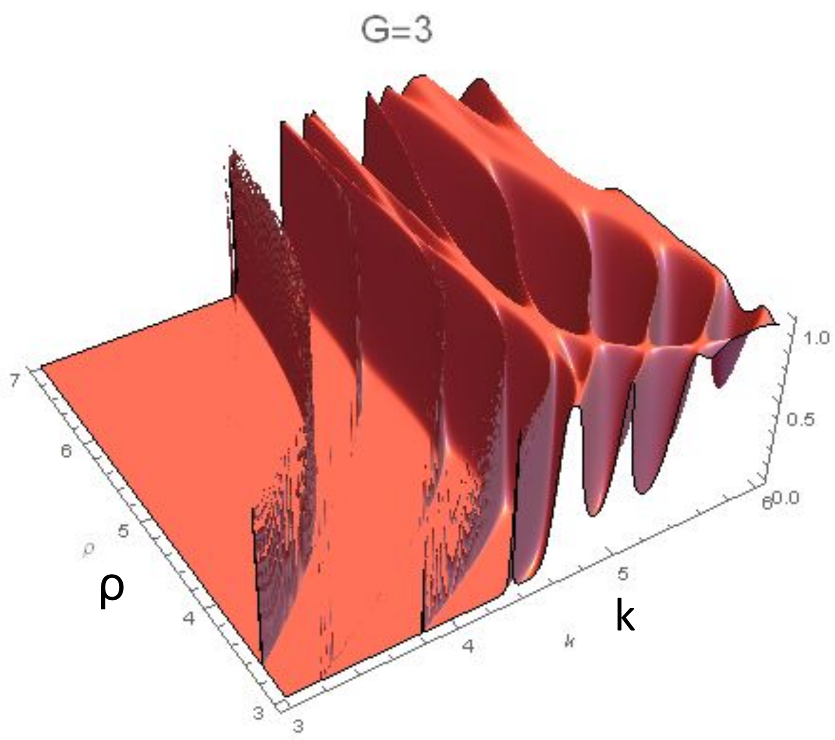


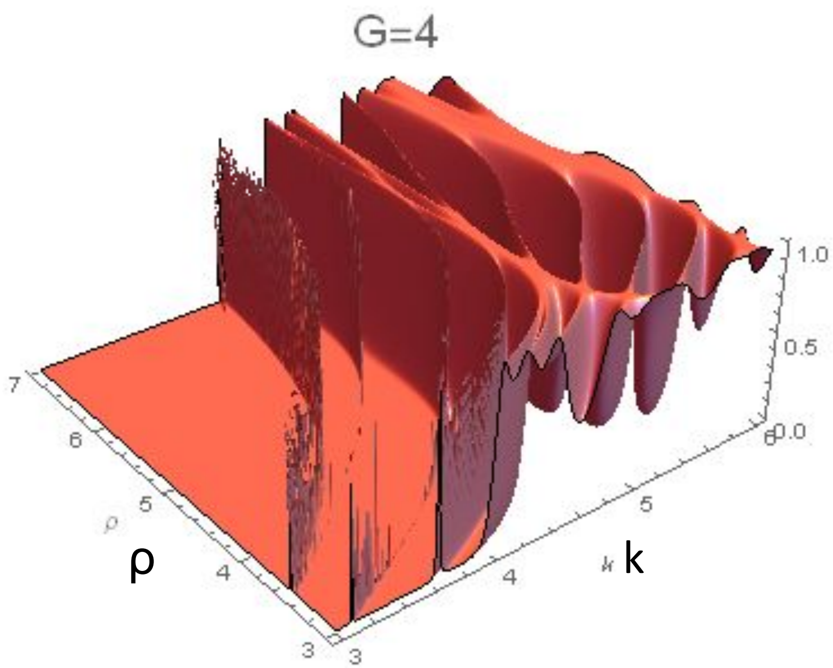
Super Periodic Potential: Applications

Tunneling amplitude for general Smith-Volterra-Cantor (SVC-ρ) potential of arbitrary stage G



$1/\rho^G$ fraction from the middle is taken out at stage G in general SVC fractal.





'Tunneling from general Smith-Volterra-Cantor potential' **by M.Hasan & B.P. Mandal (in preparation)**

Super Periodic Potential: General features

$$T(N_1, N_2, ..., N_n; k) = \frac{1}{1 + \left(|m_{12}| U_{N_1-1}(\xi_1) U_{N_2-1}(\xi_2) .... [U_{N_g-1}\{\xi_g(k = k_g^*)\} = 0] .... U_{N_j-1}(\xi_j) ..... U_{N_n-1}(\xi_n) \right)^2}$$

If $k=k_g^*$ is the transmission peak of SPP of order g then,

.

- ✓ *This implies that the transmission peaks of SPP of order 'g', is also the transmission peak of SPP of order 'j' for any j> g.*
- ✓ True for SPP of any unit cell with order j such that j >g, {j,g} ϵ I^+.

Super Periodic Potential: General features

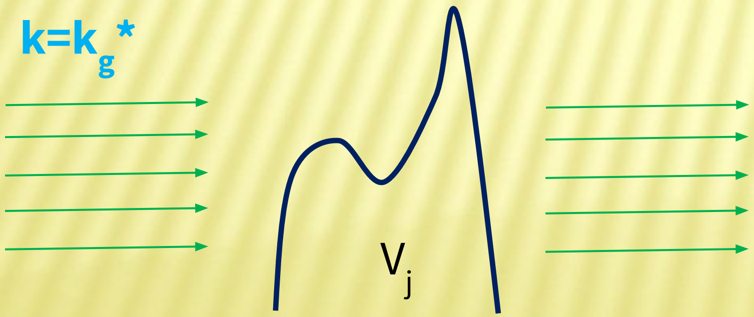
$$T(N_1, N_2, ..., N_n; k) = \frac{1}{1 + \left(|m_{12}| U_{N_1 - 1}(\xi_1) U_{N_2 - 1}(\xi_2) .... [U_{N_g - 1}\{\xi_g(k = k_g^*)\} = 0] .... U_{N_j - 1}(\xi_j) ..... U_{N_n - 1}(\xi_n) \right)^2}$$

If $k=k_g^*$ is the transmission peak of SPP of order g then,

.

- ✓ *This implies that the transmission peaks of SPP of order ‘g’, is also the transmission peak of SPP of order ‘j’ for any j> g.*
- ✓ True for SPP of any unit cell with order j such that j >g, {j,g} ϵ I^+.

At $k=k_g^*$



Super Periodic Potential: General features

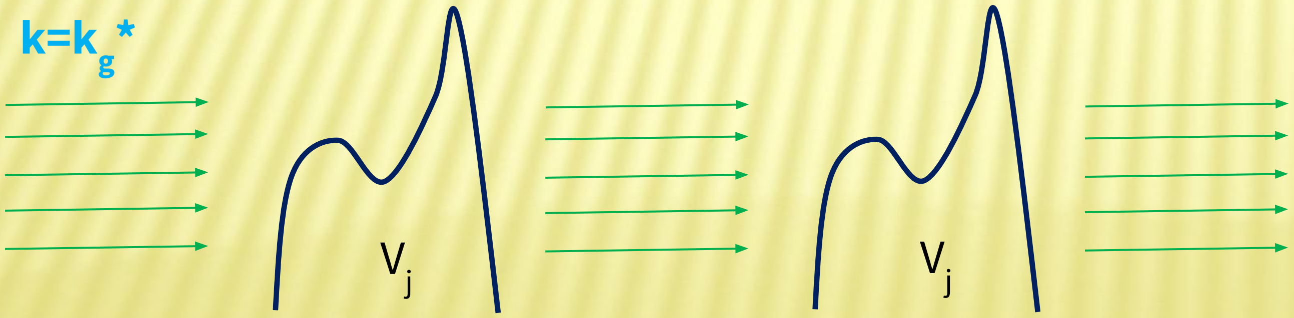
$$T(N_1, N_2, \ldots, N_n; k) = \frac{1}{1 + \left(|m_{12}| U_{N_1-1}(\xi_1) U_{N_2-1}(\xi_2) \ldots\ldots [U_{N_g-1}\{\xi_g(k = k_g^*)\} = 0] \ldots U_{N_j-1}(\xi_j) \ldots U_{N_n-1}(\xi_n) \right)^2}$$

If $k=k_g^*$ is the transmission peak of SPP of order g then,

.

- ✓ *This implies that the transmission peaks of SPP of order 'g', is also the transmission peak of SPP of order 'j' for any j> g.*
- ✓ True for SPP of any unit cell with order j such that j >g, {j,g} ϵ I^+.

At $k=k_g^*$



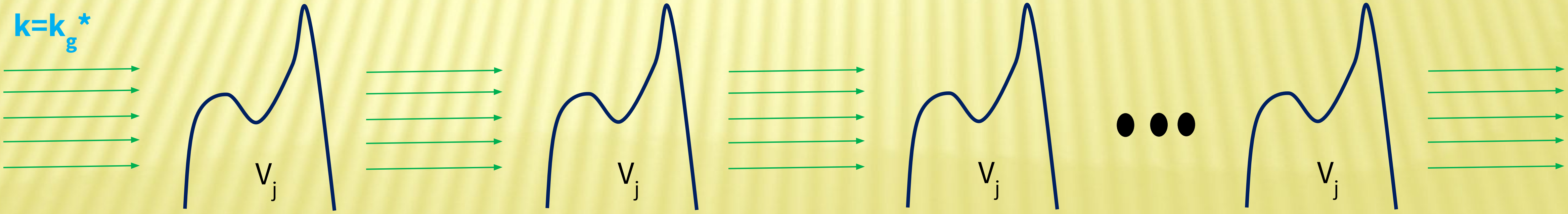
Super Periodic Potential: General features

$$T(N_1, N_2, \ldots, N_n; k) = \frac{1}{1 + \left(|m_{12}| U_{N_1-1}(\xi_1) U_{N_2-1}(\xi_2) \ldots\ldots [U_{N_g-1}\{\xi_g(k = k_g^*)\} = 0] \ldots U_{N_j-1}(\xi_j) \ldots U_{N_n-1}(\xi_n) \right)^2}$$

If $k=k_g^*$ is the transmission peak of SPP of order g then,

- ✓ *This implies that the transmission peaks of SPP of order 'g', is also the transmission peak of SPP of order 'j' for any j> g.*
- ✓ True for SPP of any unit cell with order j such that $j > g$, $\{j,g\} \in I^+$.

At $k=k_g^*$



Super Periodic Potential: General features

Splitting of transmission peaks

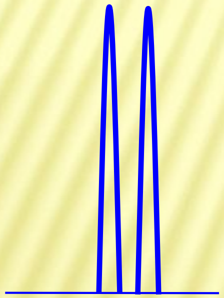
$$T(N_1,N_2;k)=\frac{1}{1+\left(|m_{12}|U_{N_1-1}(\xi_1)U_{N_2-1}(\xi_2)\right)^2}$$

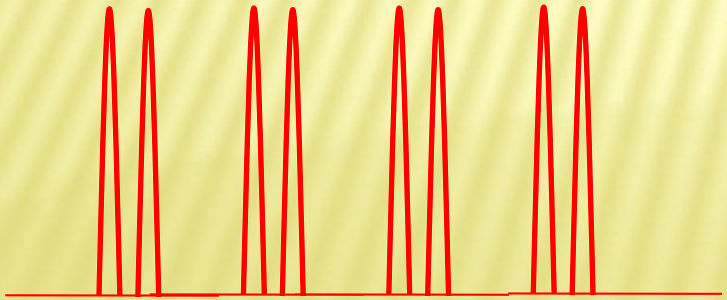
$$m_{12}=i\beta \quad , \beta=\frac{V_0}{k}$$

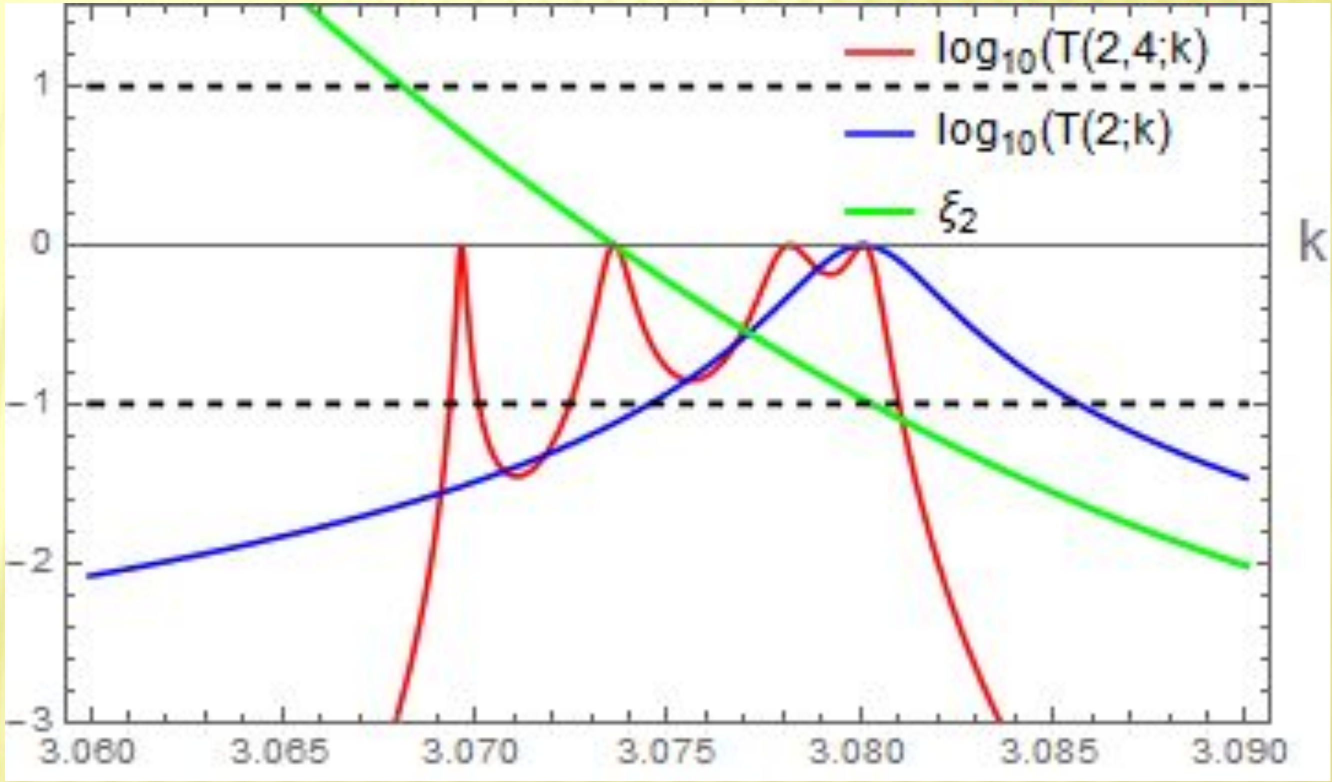
$$\xi_1=\cos ks_1+\beta\sin ks_1$$

$$\xi_2=\sqrt{1+\beta^2}U_{N_1-1}(\xi_1)\ \cos f(k,V_0)-U_{N_1-2}(\xi_1)\cos k(N_1 s_1-s_2)$$

potential parameters V=50 (δ potential), s_1=1, s_2=6, N_1=2, N_2=4







Super Periodic Potential: General features

Splitting of transmission peaks

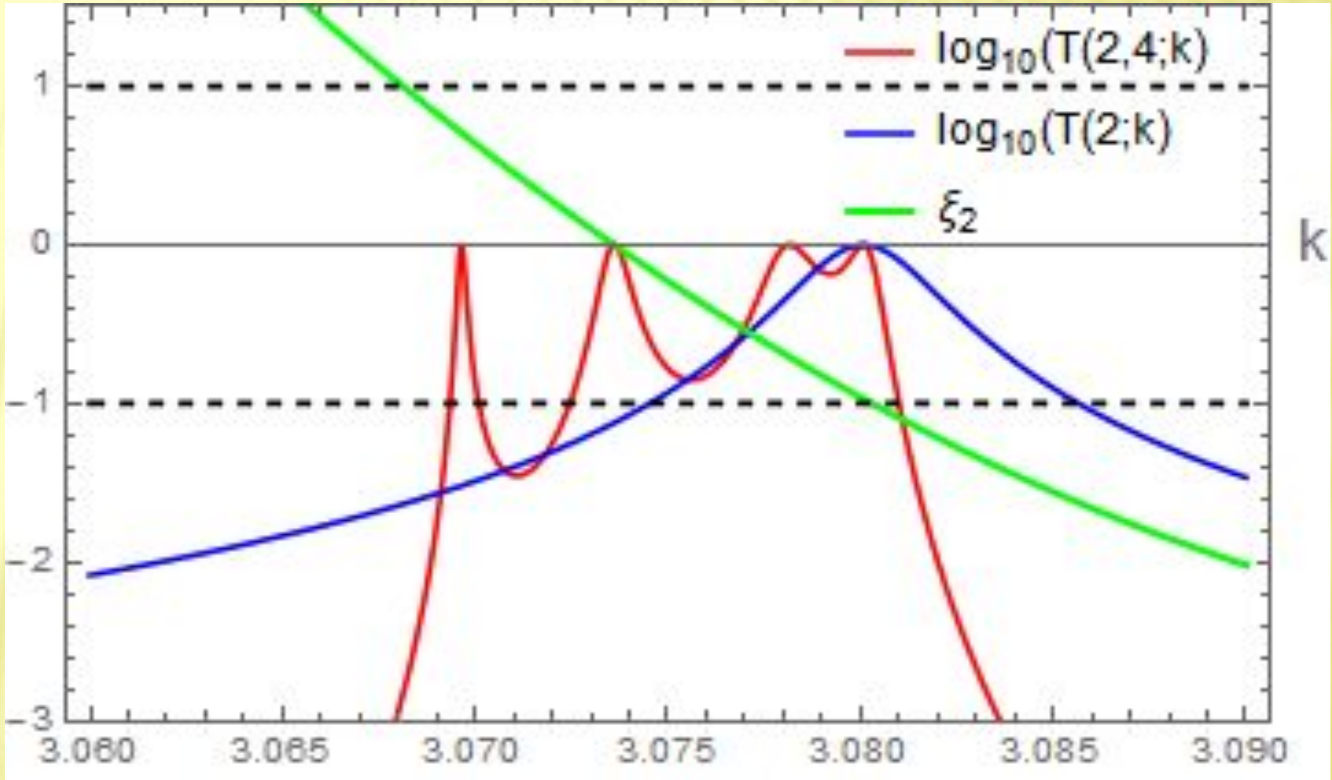
$$T(N_1,N_2;k)=\frac{1}{1+\left(|m_{12}|U_{N_1-1}(\xi_1)U_{N_2-1}(\xi_2)\right)^2}$$

$$m_{12}=i\beta \ \ , \ \beta=\frac{V_0}{k}$$

$$\xi_1=\cos ks_1+\beta\sin ks_1$$

$$\xi_2=\sqrt{1+\beta^2}U_{N_1-1}(\xi_1)\ \cos f(k,V_0)-U_{N_1-2}(\xi_1)\cos k(N_1s_1-s_2)$$

potential parameters V=50 (δ potential), s_1=1, s_2=6, N_1=2, N_2=4



✓At transmission peak, k=k* of T(N_1) , U_{N1-1} (ξ_1*) =0

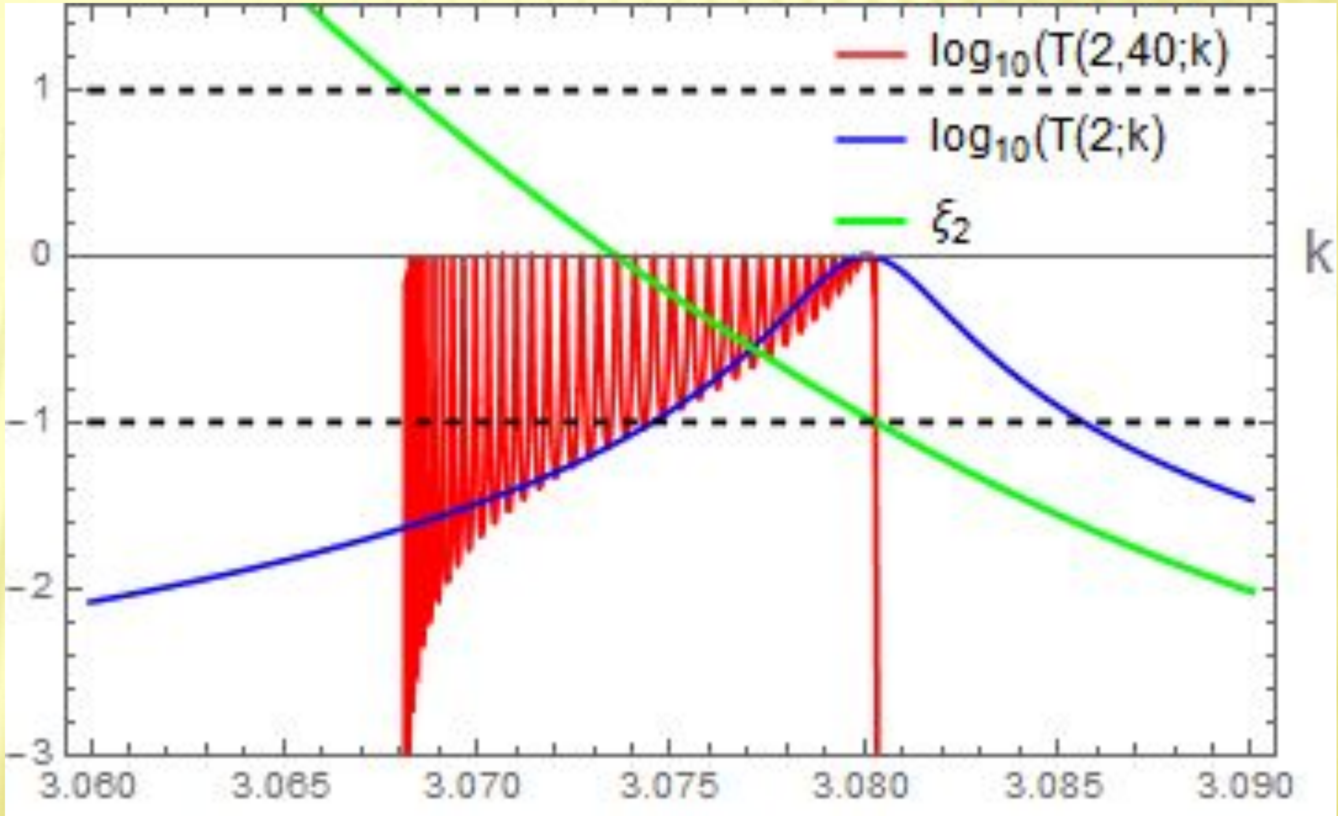
- ✓ Near k=k* , ξ_2 changes sign due to cosine term with k.
- ✓ U_{N2-1} has N_2-1 simple roots when ξ_2 ϵ [-1,1]. This results in N_2-1 transmission peaks near k=k*.
- ✓ The overall result is the (N_2-1)+1=N_2 transmission peaks due to super periodicity .

Super Periodic Potential: General features

Splitting of transmission peaks: Resonance band

- The transmission peaks arising out of periodicity of the system splits into transmission peaks due to super periodicity.

potential parameters V=50 (δ potential), $s_1=1$, $s_2=6$, $N_1=2$, $N_2=4$

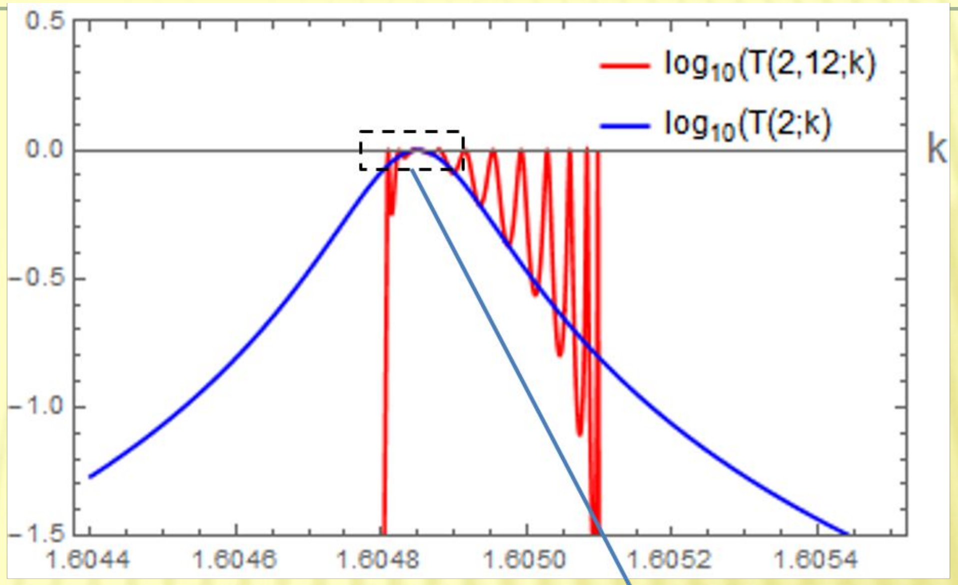


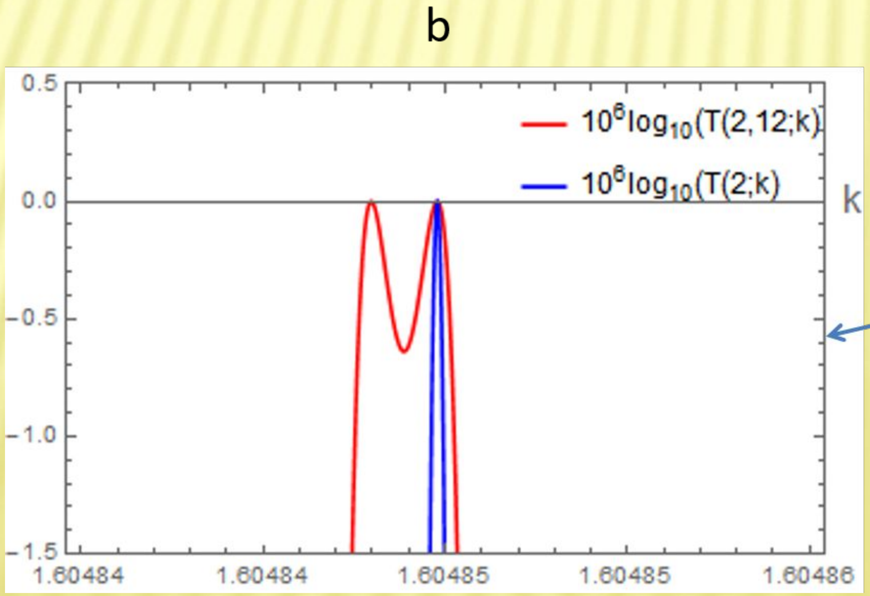
Super Periodic Potential: General features

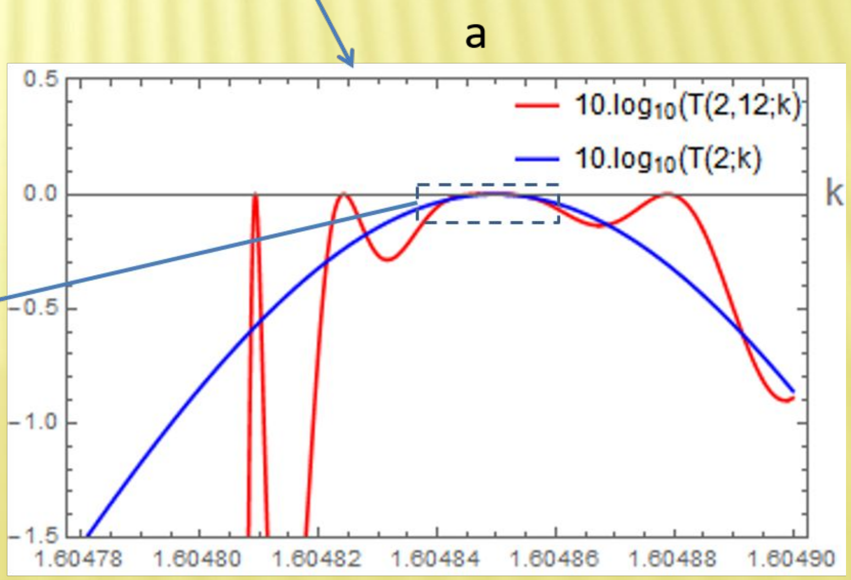
Splitting of transmission peaks: Resonance band

Potential parameters:

Unit cell: Rectangular barrier
$V_0=10$, $b=1$, $c_1=1.5$, $c_2=2.0$
$N_1=2$, $N_2=12$







CONCLUSIONS AND FUTURE PLANS

- Starting with a unit cell, super periodicity is the most general form of periodicity.
- Our observations suggest that in general, symmetric self-similarity (in 1D) is the special case of super periodicity (in 1D). *This is yet to be proven mathematically.*

CONCLUSIONS AND FUTURE PLANS

- ➢ Starting with a unit cell, super periodicity is the most general form of periodicity.
- ➢ Our observations suggest that in general, symmetric self-similarity (in 1D) is the special case of super periodicity (in 1D). *This is yet to be proven mathematically.*

- ✓ **By using SPP formalism we have derived the expression for tunneling amplitude for (general) Cantor and SVC potential.**
- ✓ **Super periodicity modifies the band structure through allowed energy in the forbidden zone.**
- ✓ **Transmission peaks that arises due to periodicity splits due to super periodicity.**

CONCLUSIONS AND FUTURE PLANS

- ➢ Starting with a unit cell, super periodicity is the most general form of periodicity.
- ➢ Our observations suggest that in general, symmetric self-similarity (in 1D) is the special case of super periodicity (in 1D). *This is yet to be proven mathematically.*

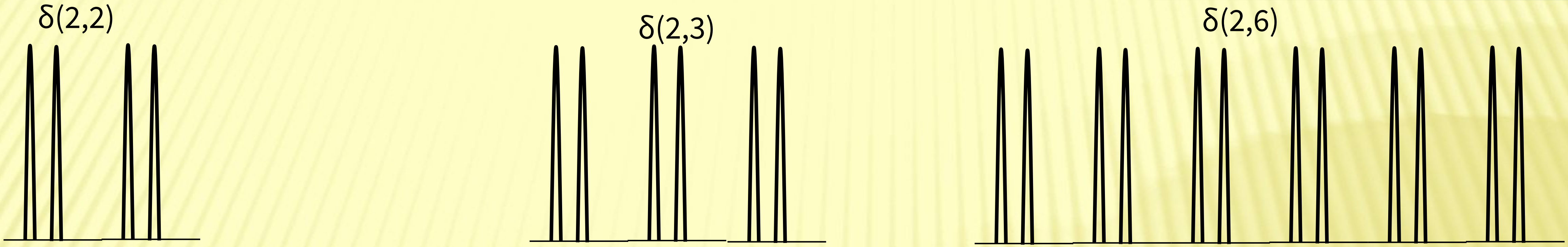
- ✓ **By using SPP formalism we have derived the expression for tunneling amplitude for (general) Cantor and SVC potential.**
- ✓ **Super periodicity modifies the band structure through allowed energy in the forbidden zone.**
- ✓ **Transmission peaks that arises due to periodicity splits due to super periodicity.**

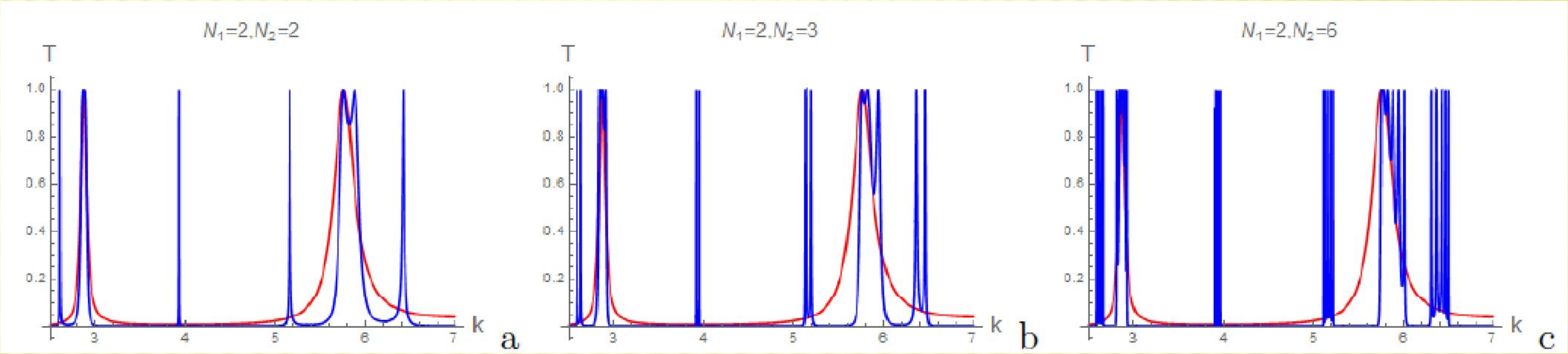
Future plans:

- **Extension of SPP to non-Hermitian potentials.**
- **Relation between the distribution of CPA points and SPP structure.**
- **Extension of SPP to fractional quantum mechanics**

Thank you

Super Periodic δ Potential:
Modulation of band structure and splitting of transmission peaks





- Red curve is transmission probability for double delta potential.
- Blue curve is the transmission probability for the case when double delta potential as a whole repeats.

Super periodic δ potential of order 2 : Resonance band

Transmission probability: $T(N_1,N_2;k)=\frac{1}{1+\left(|m_{12}|U_{N_1-1}(\xi_1)U_{N_2-1}(\xi_2)\right)^2}$

- ✓ If k=k* is a transmission peaks of Dirac comb then at k=k*,

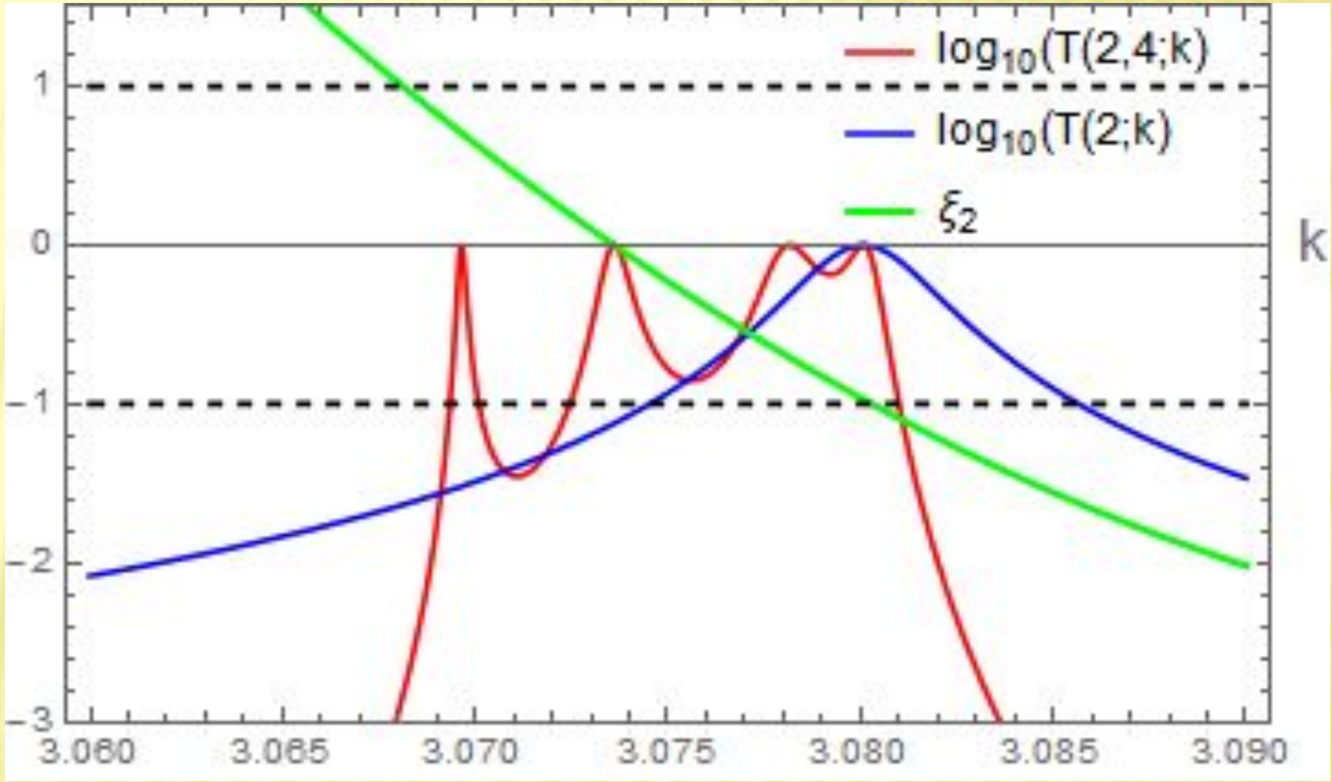
  $\xi_2^*=-U_{N_1-2}(\xi_1^*)\cos[k^*(N_1s_1-s_2)]$

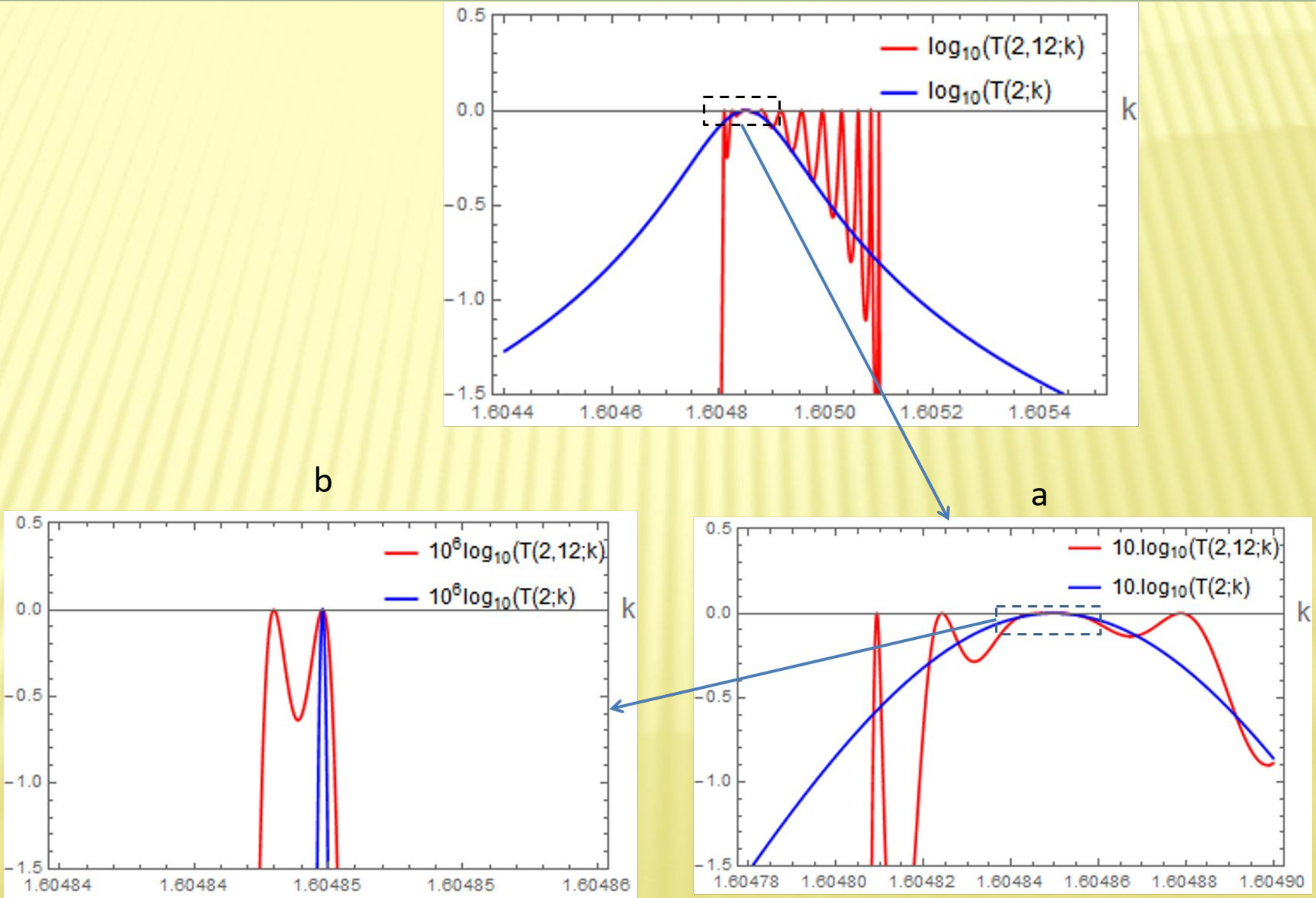
- ✓ Near k=k* , ξ2 changes sign due to cosine term with k.

- ✓ U_{N2-1} has N_2-1 simple roots when ξ2 ϵ [-1,1]. This results in N_2-1 transmission peaks near k=k*.

- ✓ The overall results is (N_2-1)+1=N_2 transmission peaks due to super periodicity .

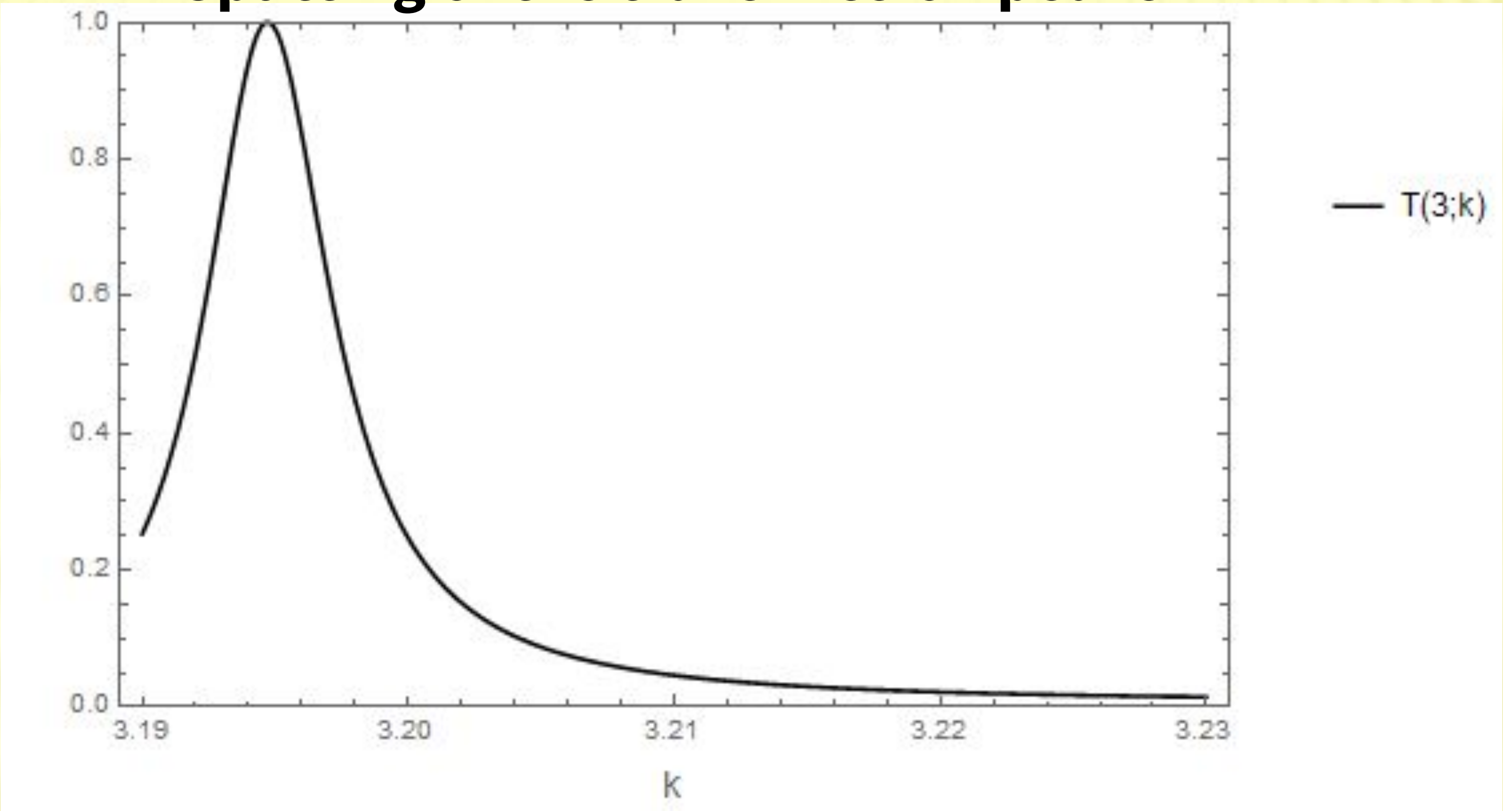
Potential parameters V=50, s_1=1, s_2=6



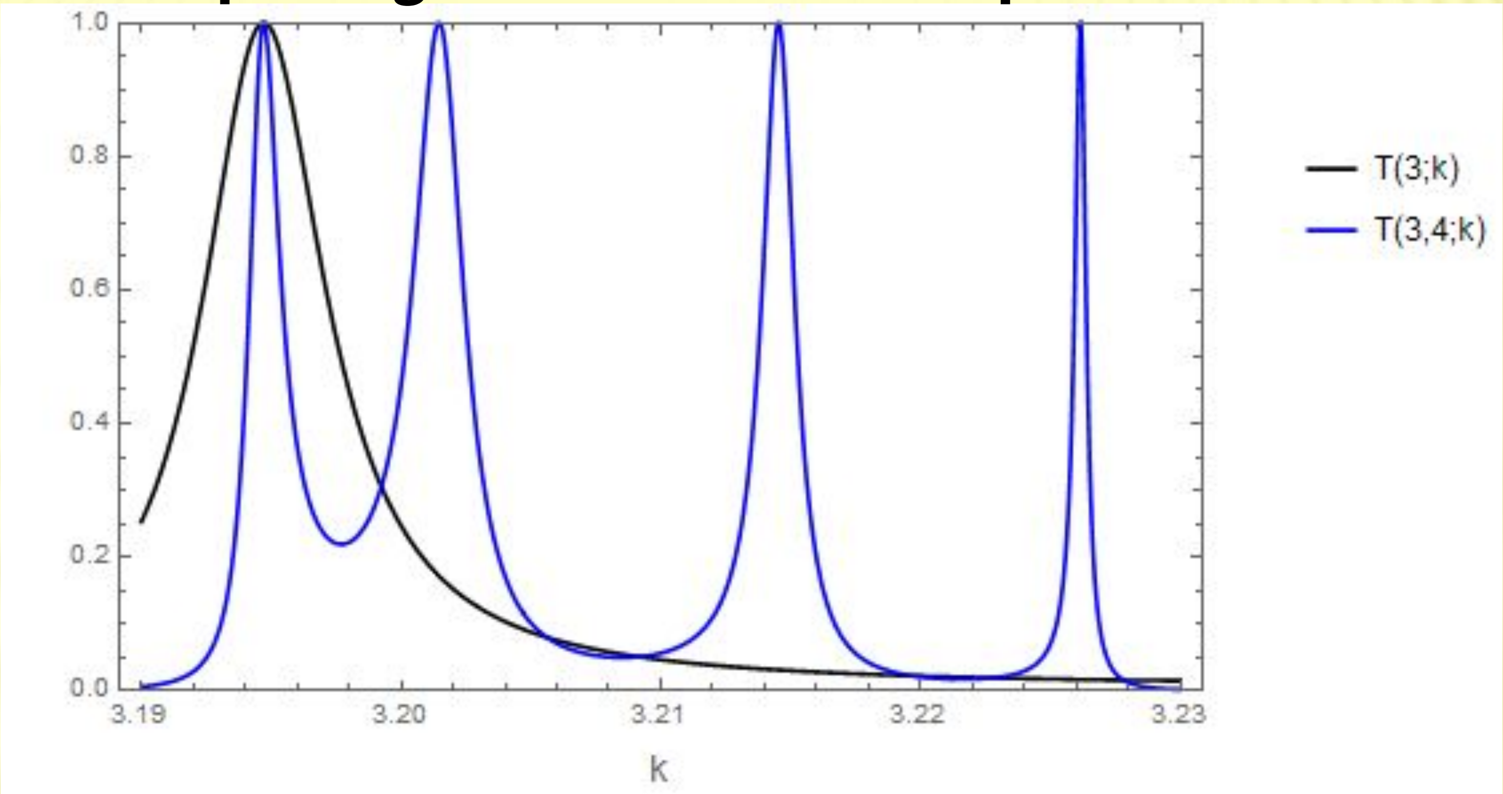


$\log_{10}(T(2,12;k)$
$\log_{10}(T(2;k)$
k
0.5
0.0
-0.5
-1.0
-1.5
1.6044
1.6046
1.6048
1.6050
1.6052
1.6054
b
$10^6\log_{10}(T(2,12;k)$
$10^6\log_{10}(T(2;k)$
1.60484
1.60484
1.60485
1.60485
1.60486
a
$10.\log_{10}(T(2,12;k)$
$10.\log_{10}(T(2;k)$
1.60478
1.60480
1.60482
1.60484
1.60486
1.60488
1.60490

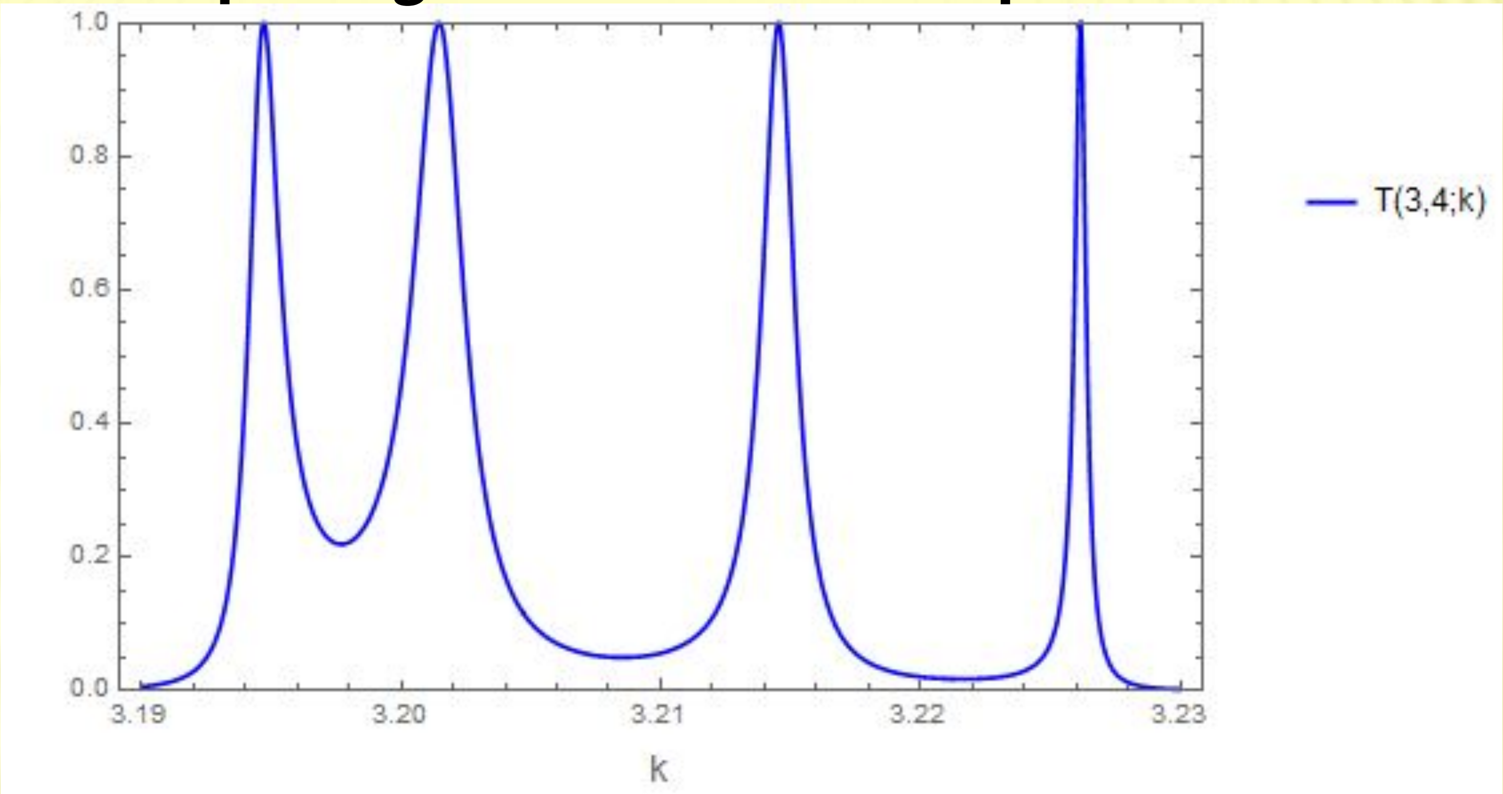
Super periodic δ potential of order 3:
Splitting of the transmission peaks



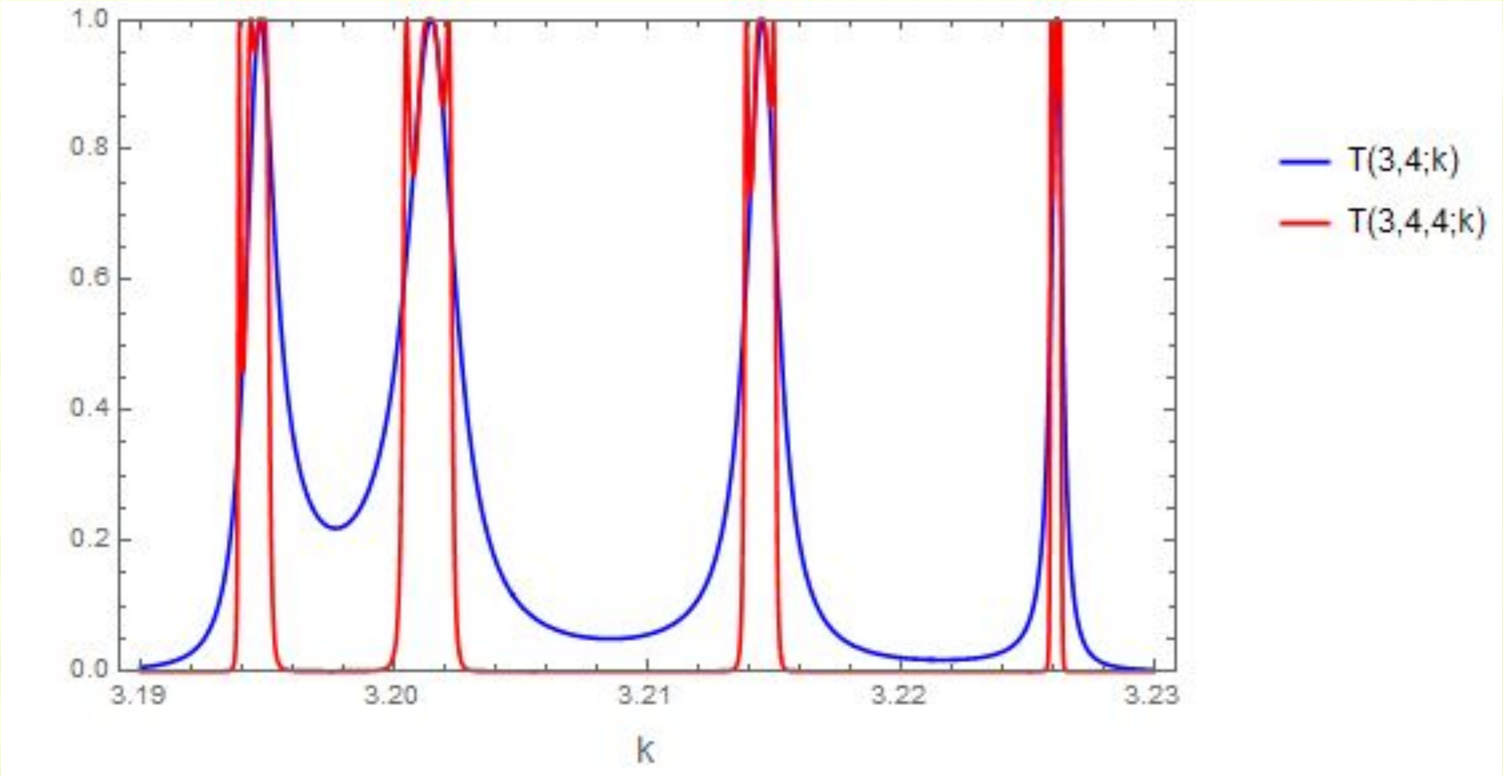
Super periodic δ potential of order 3:
Splitting of the transmission peaks



Super periodic δ potential of order 3:
Splitting of the transmission peaks

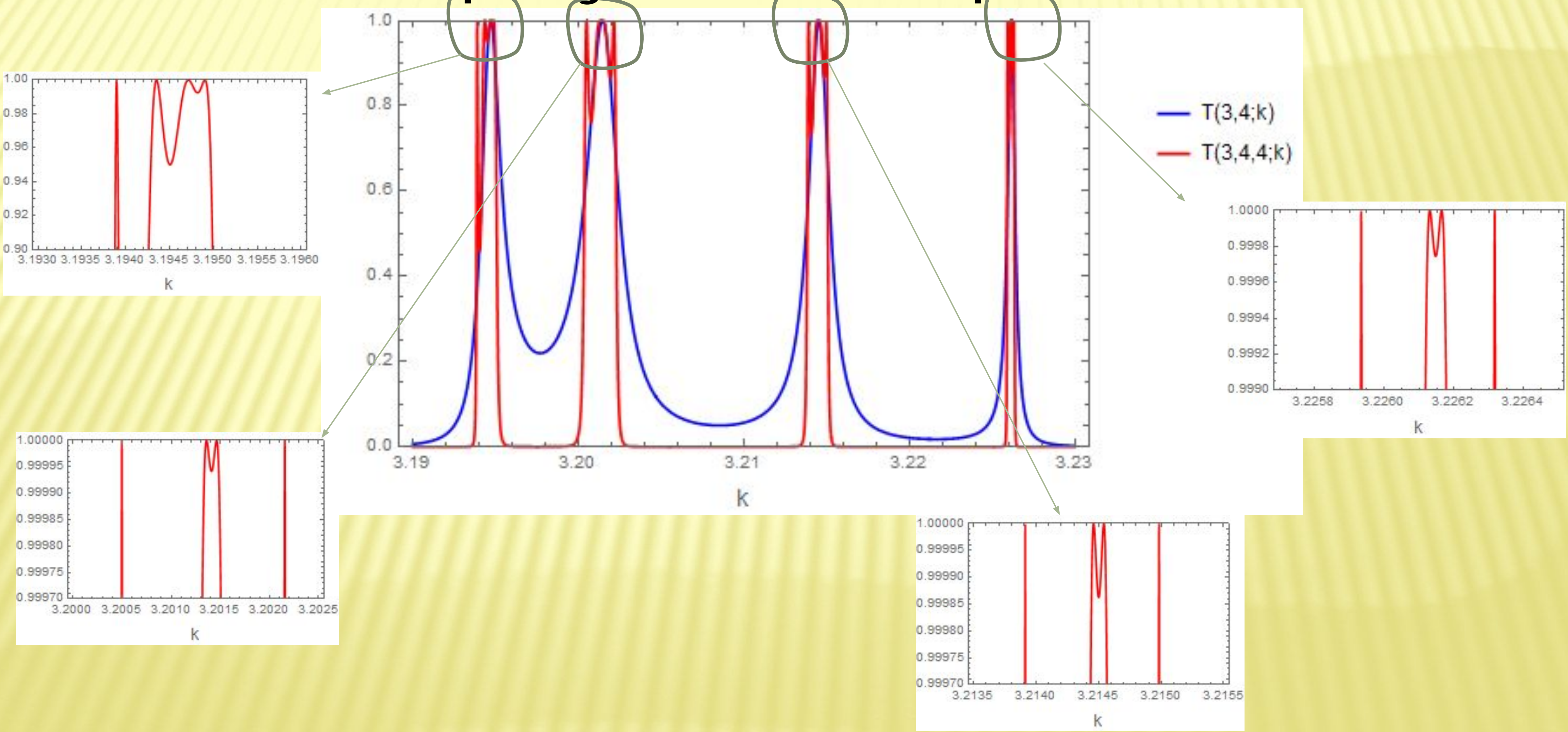


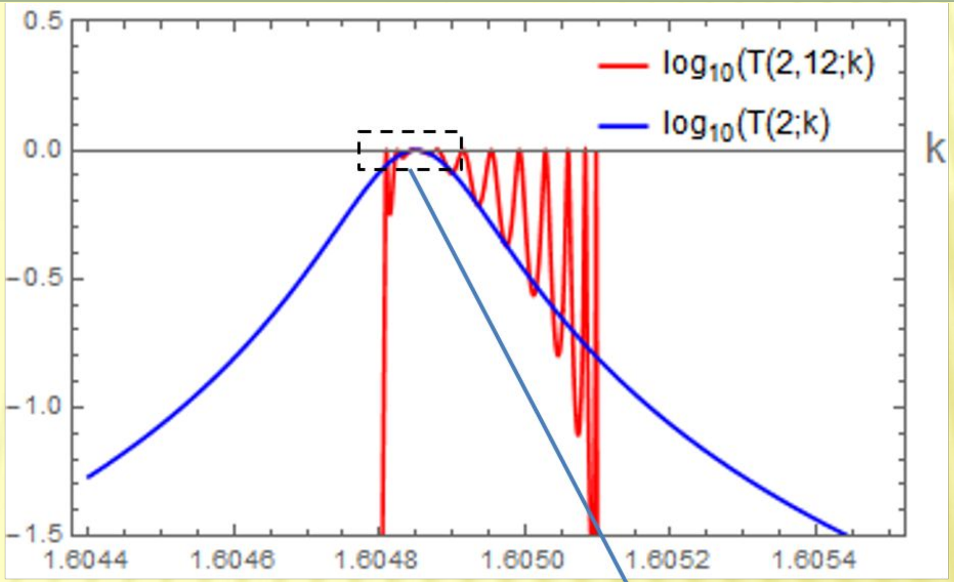
Super periodic δ potential of order 3:
Splitting of the transmission peaks



Super periodic δ potential of order 3:

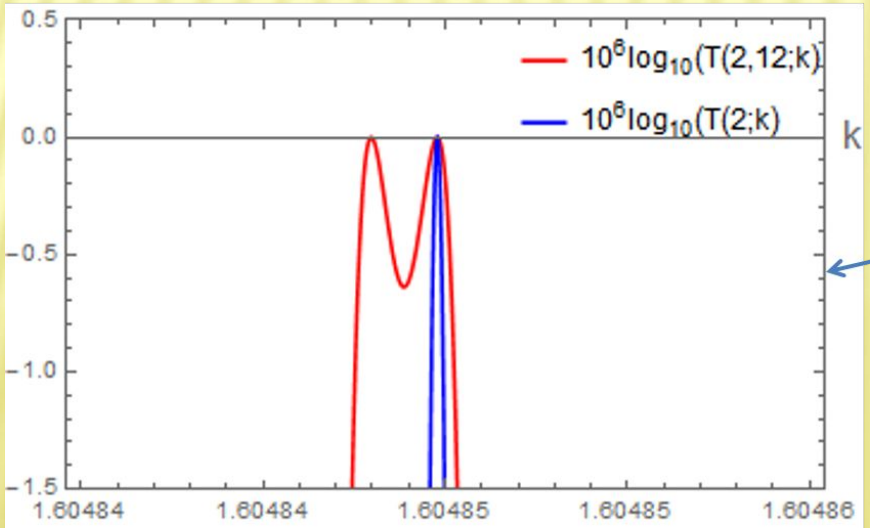
Splitting of the transmission peaks





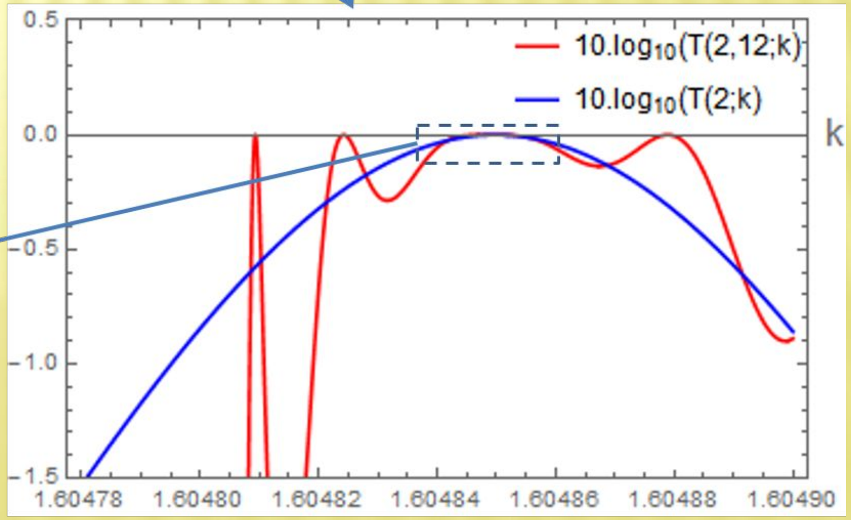
0.5
0.0
−0.5
−1.0
−1.5
1.6044
1.6046
1.6048
1.6050
1.6052
1.6054
log10(T(2,12;k)
log10(T(2;k)
k

b



0.5
0.0
−0.5
−1.0
−1.5
1.60484
1.60484
1.60485
1.60485
1.60486
10^6log10(T(2,12;k)
10^6log10(T(2;k)
k

a



0.5
0.0
−0.5
−1.0
−1.5
1.60478
1.60480
1.60482
1.60484
1.60486
1.60488
1.60490
10.log10(T(2,12;k)
10.log10(T(2;k)
k

Thanking You