PT symmetric nonlinear systems and some of their implications in optics

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Aim

- To show the occurrence of spontaneous symmetry breaking in nonlinearly damped $\mathcal{PT}$ symmetric systems.
- Understanding symmetry breaking from exact solution point of view.
- To illustrate the dynamics and the application of $\mathcal{PT}$ symmetrically coupled systems in the optical context (that is, in the construction of unidirectional optical devices).
- To show the ways to control blow-up responses by nonlinear $\mathcal{PT}$ couplings.

Plan

- Introduction
- Nonlinearly damped systems
- A remarkable scalar system- modified Emden equation
- Quantum version
- Bi -$\mathcal{PT}$ symmetric systems
- Systems that become $\mathcal{PT}$ symmetric through interaction
- Controlling blow-up responses by nonlinear coupling
- Localization of light with linear non-Hermitian defects
- Conclusion

Introduction

- Recent interest in $\mathcal{PT}$-symmetric systems
- $\mathcal{PT}$-symmetric systems as balanced loss/gain
- Bi-$\mathcal{PT}$ symmetry: Position dependent/ Nonlinear damping

  (a) Scalar systems:

  $\mathcal{PT}$: $x \to -x$, $t \to -t$.

  (b) Coupled systems:

  $\mathcal{PT}-1$: $x \to -x$, $y \to -y$, $t \to -t$.
  $\mathcal{PT}-2$: $x \to -y$, $y \to -x$, $t \to -t$.

- Spontaneous symmetry breaking

C. M. Bender, B. K. Berntson, D. Parker and E. Samuel, Am. J. Phys. **81** 173 (2013).

C. M. Bender, M. Gianfreda, S. K. Özdemir, B. Peng, and L. Yang, Phys. Rev. A **88** 062111 (2013).

J. Cuevas, P.G. Kevrekidis, A. Saxena and A. Khare, Phys. Rev. A **88** 032108 (2013).

Nonlinearly damped systems

(a) Scalar Case:

$$\ddot{x} + h(x, \dot{x}) + g(x) = 0.$$

$$\text{with } h(x, \dot{x}) = f(x)\dot{x}.$$

$$\Rightarrow \quad \ddot{x} + g(x) = -h(x, \dot{x}) = -f(x)\dot{x}.$$

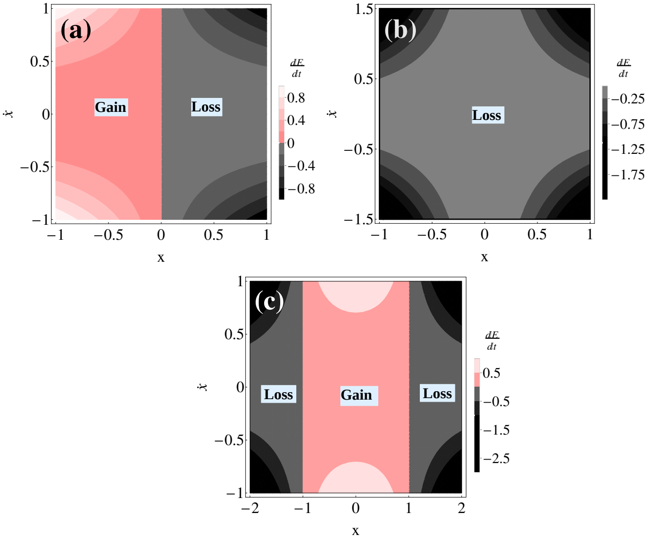
- If $f(x)$ - odd function: system is $\mathcal{PT}$-symmetric

$$\text{with} \qquad T(\dot{x}) = \frac{1}{2}\dot{x}^2; \qquad V(x) = \int g(x)dx.$$

$$\Rightarrow \quad E = \frac{1}{2}\dot{x}^2 + \int g(x)dx.$$

$$\Rightarrow \quad \frac{dE}{dt} = -\dot{x}h(x,\dot{x}) = -f(x)\dot{x}^2.$$

- If $\frac{dE}{dt} < 0$ - loss, $\frac{dE}{dt} > 0$ - gain


(a)
Gain
Loss
$\frac{dE}{dt}$
0.8
0.4
0
−0.4
−0.8
(b)
Loss
−0.25
−0.75
−1.25
−1.75
(c)
Loss
Gain
Loss
0.5
−0.5
−1.5
−2.5
x
$\dot{x}$

Examples:

1. $\mathcal{PT}$-symmetric conservative system - Modified Emden Equation (MEE):

$$\ddot{x} + \alpha x \dot{x} + \beta x^3 + \omega_0^2 x = 0$$

2. Non-$\mathcal{PT}$-symmetric damped system:

$$\ddot{x} + \alpha x^2 \dot{x} + \beta x^3 + \omega_0^2 x = 0$$

3. Limit cycle oscillator (van der Pol oscillator):

$$\ddot{x} + (x^2 - 1)\dot{x} + \omega_0^2 x = 0.$$

V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, J. Phys. A: Math. Theor. **40** 4717 (2007).

V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, J. Phys. A: Math. Gen. **37** 4527 (2004).

S. H. Strogatz, *Nonlinear Dynamics and Chaos*, (Perseus Book Publishing, USA, 1994).

- Consider coupled systems:

$$\begin{aligned} \ddot{x} + h_1(x,\dot{x}) + h_2(x,\dot{x}) + g(x) + \kappa y &= 0, \\ \ddot{y} + h_1(y,\dot{y}) - h_2(y,\dot{y}) + g(y) + \kappa x &= 0, \end{aligned}$$

$h_1(x,\dot{x})$ and $h_2(x,\dot{x})$ are odd and even functions in x, respectively. Defining

$$E = \frac{1}{2}\dot{x}^2 + \int g(x)dx + \frac{1}{2}\dot{y}^2 + \int g(y)dy + \kappa xy.$$

$$\Rightarrow \quad \frac{dE}{dt} = -\dot{x}[h_1(x,\dot{x}) + h_2(x,\dot{x})] - \dot{y}[h_1(y,\dot{y}) - h_2(y,\dot{y})].$$

Modified Emden Equation: $\mathcal{PT}$ symmetry breaking

- Consider the MEE equation

$$\ddot{x} + \alpha x \dot{x} + \beta x^3 + \lambda x = 0, \qquad \lambda = \omega_0^2,$$

$\Rightarrow \quad \mathcal{PT}-1$ symmetric.
Equivalently

$$\begin{aligned} \dot{x} &= x_1 \\ \dot{x}_1 &= -\alpha x x_1 - \beta x^3 - \lambda x. \end{aligned}$$

- Equilibrium/fixed points
  - (a) E_0: $(x^*, x_1^*) = (0, 0)$
  - (b) $E_{1,2}$: $(\pm\sqrt{\frac{-\lambda}{\beta}}, 0) \qquad (\lambda < 0,\ \beta > 0)$

V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, J. Phys. A: Math. Theor. **40** 4717 (2007).

- Jacobian matrix

$$J = \begin{bmatrix} 0 & 1 \\ -\alpha x_1^* - 3\beta {x^*}^2 - \omega_0^2 & -\alpha x^* \end{bmatrix}$$

- Eigenvalues of J for E_0:

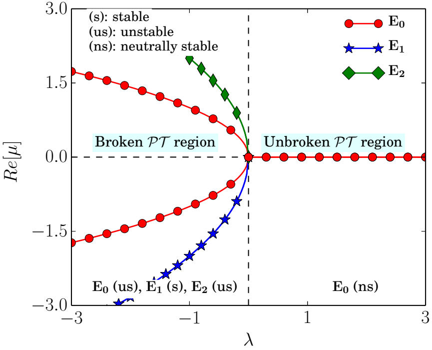
$$\mu_{1,2}^{(0)} = \pm i\sqrt{\lambda}$$

- Eigenvalues of E_1, E_2:

$$\begin{aligned} \mu_{1,2}^{(1)} &= \frac{1}{2\sqrt{\beta}}\left(-\alpha\sqrt{-\lambda} \pm \sqrt{-\lambda(\alpha^2 - 8\beta)}\right), \\ \mu_{1,2}^{(2)} &= \frac{1}{2\sqrt{\beta}}\left(\alpha\sqrt{-\lambda} \pm \sqrt{-\lambda(\alpha^2 - 8\beta)}\right) \end{aligned}$$

$Re[\mu]$ vs λ

$$\mathcal{PT}[E_0] = E_0, \quad \mathcal{PT}[E_1] = E_2, \quad \mathcal{PT}[E_2] = E_1$$



Integrable case: $\beta = \frac{\alpha^2}{9}$

We have

$$\ddot{x} + \alpha x \dot{x} + \frac{\alpha^2}{9} x^3 + \lambda x = 0$$

or

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\alpha x y - \frac{\alpha^2}{9} x^3 - \lambda x. \end{aligned}$$

V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, Phys. Rev. E **72** 066203 (2005).

- Nonstandard Lagrangian

$$L = \frac{27\lambda^2}{2\alpha^2}\left(\frac{1}{\alpha\dot{x} + \frac{\alpha^2}{3}x^2 + 3\lambda}\right) + \frac{3\lambda}{2\alpha}\dot{x} - \frac{9\lambda^2}{2\alpha^2}.$$

- Canonically conjugate momentum

$$p = -\frac{27\lambda^3}{2\alpha}\left(\frac{1}{(\alpha\dot{x} + \frac{\alpha^2}{3}x^2 + 3\lambda)^2}\right) + \frac{3\lambda}{2\alpha},$$

- Hamiltonian ('Energy')

$$\begin{aligned} H &= \frac{9\lambda^2}{2}\left(\frac{((\dot{x} + \frac{\alpha}{3}x^2)^2 + \lambda x^2)}{(\alpha\dot{x} + \frac{\alpha^2}{3}x^2 + 3\lambda)^2}\right) \\ &= \frac{9\lambda^2}{2\alpha^2}\left[2 - 2\left(1 - \frac{2\alpha p}{3\lambda}\right)^{\frac{1}{2}} + \frac{\alpha^2 x^2}{9\lambda} - \frac{2\alpha p}{3\lambda} - \frac{2k^3x^2p}{27\lambda^2}\right] \end{aligned}$$

General solution

$\lambda > 0$

$$
\begin{aligned}
x(t) &= \frac{A\sin(\omega_0 t + \delta)}{1 - A\frac{\alpha}{3\omega_0}\cos(\omega_0 t + \delta)}, \quad \omega_0 = \sqrt{\lambda} \\
\dot{x}(t) &= \frac{A\omega_0\cos(\omega_0 t + \delta)}{1 - A\frac{\alpha}{3\omega_0}\cos(\omega_0 t + \delta)} - \frac{\alpha}{3}x^2(t),
\end{aligned}
$$

so that $H = \mathcal{E} = \frac{1}{2}\omega_0^2 A^2$.

General solution

$\lambda < 0$

$$x(t) = \frac{3\sqrt{|\lambda|}(I_1 e^{\sqrt{|\lambda|}t} - e^{-\sqrt{|\lambda|}t})}{\alpha(I_1 I_2 + I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})}$$

$$\dot{x}(t) = \frac{3|\lambda|(I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})}{\alpha(I_1 I_2 + I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})} - \frac{\alpha}{3}x^2(t).$$

with $H = \mathcal{E} = \frac{18|\lambda|^2}{\alpha^2}\frac{1}{I_1 I_2^2}$,

(a) $\mathcal{PT}$ invariant solutions for $\lambda > 0$

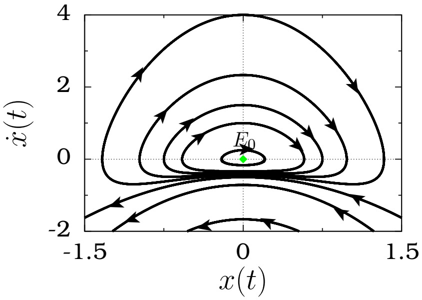
- Initial value problem obeying $\mathcal{PT}$ symmetry

$$x(0) = 0, \qquad \dot{x}(0) = B = \frac{3A\omega_0^2}{3\omega_0 - A\alpha}$$

$$\begin{aligned} x(t) &= \frac{A\sin(\omega_0 t)}{(1 - A\frac{\alpha}{3\omega_0}\cos\omega_0 t)}, \quad \omega_0 = \sqrt{\lambda}, \quad 0 \leq A < \frac{3\omega_0}{\alpha}, \\ \dot{x}(t) &= \frac{A\omega_0\cos\omega_0 t}{(1 - A\frac{\alpha}{3\omega_0}\cos\omega_0 t)} - \frac{\alpha}{3}x^2(t), \quad 0 \leq t < \infty, \end{aligned}$$

$$\begin{aligned} PT[x(t)] &= -x(-t) = x(t), \\ PT[\dot{x}(t)] &= \dot{x}(-t) = \dot{x}(t), \quad \text{for all } t \geq 0, \end{aligned}$$

Phase trajectories in the case of $\lambda > 0$



(b) $\mathcal{PT}$ breaking solutions for $\lambda < 0$

1. Symmetric solution

$$\begin{aligned} 1)\, x(t) &= x_0(t) = -x_0(-t) \equiv \mathcal{PT}[x_0(t)], \\ \dot{x}(t) &= \dot{x}_0(t) = \dot{x}_0(-t) \equiv \mathcal{PT}[\dot{x}_0(t)], \end{aligned}$$

2. Asymmetric solution

$$\begin{aligned} 2)\, x(t) &= x_1(t),\, \dot{x}(t) = \dot{x}_1(t) \quad \text{and} \\ 3)\, x(t) &= x_2(t),\, \dot{x}(t) = \dot{x}_2(t) \end{aligned}$$

$$\begin{aligned} \mathcal{PT}x_1(t) &= -x_1(-t) = x_2(t) \neq x_1(t), \\ \mathcal{PT}\dot{x}_1(t) &= \dot{x}_1(-t) = \dot{x}_2(t) \neq \dot{x}_1(t), \\ \mathcal{PT}x_2(t) &= -x_2(-t) = x_1(t) \neq x_2(t), \\ \mathcal{PT}\dot{x}_2(t) &= \dot{x}_2(-t) = \dot{x}_1(t) \neq \dot{x}_2(t), \quad \text{for all} \quad t \geq 0. \end{aligned}$$

- Symmetric solution satisfying the initial conditions $x_0(0) = 0$, $\dot{x}_0(0) = \text{constant}$

$$
\begin{aligned}
x_0(t) &= \frac{3\sqrt{|\lambda|}(e^{\sqrt{|\lambda|}t} - e^{-\sqrt{|\lambda|}t})}{\alpha(I_2 + e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})} \\
\dot{x}_0(t) &= \frac{3|\lambda|(e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})}{\alpha(I_2 + e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})} - \frac{\alpha}{3}x_0^2(t).
\end{aligned}
$$

- Asymmetric solution from initial condition $(x_1(0), \dot{x}_1(0)) = \left(\frac{3\sqrt{|\lambda|}}{\alpha}, \frac{6|\lambda|}{\alpha(I_1-1)}\right)$

$$
\begin{aligned}
x_1(t) &= \frac{3\sqrt{|\lambda|}(I_1 e^{\sqrt{|\lambda|}t} - e^{-\sqrt{|\lambda|}t})}{\alpha(-2 + I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})} \\
\dot{x}_1(t) &= \frac{3|\lambda|(I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})}{\alpha(-2 + I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})} - \frac{\alpha}{3}x_1^2(t).
\end{aligned}
$$

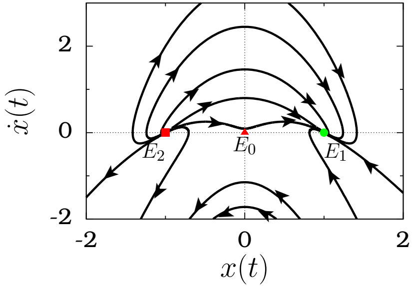
- Asymmetric solution from initial condition $(x_2(0), \dot{x}_2(0))$ $= (-\frac{3\sqrt{|\lambda|}}{\alpha}, \frac{6|\lambda|}{\alpha}\frac{I_1}{I_1-1})$

$$x_2(t) = \frac{3\sqrt{|\lambda|}(I_1 e^{\sqrt{|\lambda|}t} - e^{-\sqrt{|\lambda|}t})}{\alpha(-2I_1 + I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})}$$

and

$$\dot{x}_2(t) = \frac{3|\lambda|(I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})}{\alpha(-2I_1 + I_1 e^{\sqrt{|\lambda|}t} + e^{-\sqrt{|\lambda|}t})} - \frac{\alpha}{3}x_2^2(t).$$

- Phase trajectories



- Confirmation of fixed point analysis:
  $\mathcal{PT}$ symmetry $\Rightarrow$ $\mathcal{PT}$ symmetric neutrally stable centre type fixed point.

Quantum version of Integrable MEE equation

- Classical Hamiltonian:

$$H(x,p) = \frac{x^2}{2\;m(p)} + U(p), \qquad -\infty < p \leq \frac{3\omega^2}{2k},$$

where

$$m(p) = \frac{1}{\omega^2\;\left(1 - \frac{2k}{3\omega^2}p\right)} \text{ and } \quad U(p) = \frac{9\omega^4}{2k^2}\left(\sqrt{1 - \frac{2k}{3\omega^2}p} - 1\right)^2.$$

- Quantum Hamiltonian

$$H(\hat{x},\hat{p}) = \frac{1}{4}\left[m^{\alpha}(\hat{p})\hat{x}m^{\beta}(\hat{p})\hat{x}m^{\gamma}(\hat{p}) + m^{\gamma}(\hat{p})\hat{x}m^{\beta}(\hat{p})\hat{x}m^{\alpha}(\hat{p})\right] + U(\hat{p}),$$

- Time independent Schrödinger equation

$$H(\hat{x},\hat{p})\psi(x,p) = E\psi(x,p),$$

or

$$\begin{aligned}\frac{-\hbar^2}{2m}[\psi'' - \frac{m'}{m}\psi' &+ \left(\frac{1+\beta}{2}\right)\left(2\frac{m'^2}{m^2} - \frac{m''}{m}\right)\psi \\ &+ \frac{\alpha(\alpha+\beta+1)m'^2}{m^2}\psi] + U(p)\psi = E\psi,\end{aligned}$$

where prime stands for differentiation with respect to p.

V. Chithiika Ruby, M. Senthilvelan and M. Lakshmanan, J. Phys. A. Math. Theor. **45** 382002 (2012).

V. Chithiika Ruby, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, J. Math. Phys. **56** 012103 (2015).

Boundary conditions:

$$\psi = 0, \quad \text{for} \quad p \geq \frac{3\omega^2}{2k}$$

$$\psi(-\infty) = \psi\left(\frac{3\omega^2}{2k}\right) = 0$$

$\Rightarrow$

$$\psi'' - \frac{2k}{3\omega^2}\frac{1}{\left(1-\frac{2k}{3\omega^2}p\right)}\psi' + \frac{4k^2\alpha(\alpha+\beta+1)}{9\omega^4\left(1-\frac{2k}{3\omega^2}p\right)^2}\psi - \frac{2Ek^2 - 9\omega^4\left(\sqrt{1-\frac{2k}{3\omega^2}p}-1\right)^2}{\hbar^2\omega^2k^2\left(1-\frac{2k}{3\omega^2}p\right)}\psi = 0, \quad -\infty < p \leq \frac{3\omega^2}{2k}.$$

$$\frac{d^2\psi}{dy^2} + \frac{1}{y}\frac{d\psi}{dy} + \left(\tilde{E} + \frac{4\alpha(\alpha+\beta+1)}{y^2} - a(y-1)^2\right)\psi = 0, \quad 0 \le y < \infty,$$

$\Rightarrow$

$$\psi(y) = e^{-\frac{\sqrt{a}}{2}y^2+\sqrt{a}y}\,\phi(y)$$

$$\frac{d^2\phi}{dy^2} + \left(2\sqrt{a} - 2\sqrt{a}y + \frac{1}{y}\right)\frac{d\phi}{dy} + \left(\frac{4\alpha(\alpha+\beta+1)}{y^2} + \frac{\sqrt{a}}{y} + \tilde{E} - 2\sqrt{a}\right)\phi = 0.$$

$\Rightarrow$ Singular solution

Non-Hermitian Hamiltonian ($\mathcal{PT}$ symmetric)

$$H(\hat{x},\hat{p}) = \frac{1}{2}\left[m^{-1/4}(\hat{p})\hat{x}m^{-1/2}(\hat{p})\hat{x}m^{-1/4}(\hat{p})\right] + U(\hat{p}),$$

$$\begin{aligned}
\tilde{H} &= \frac{1}{\sqrt{m}}H\sqrt{m} = \frac{1}{2}\left[m^{-3/4}(\hat{p})\hat{x}m^{-1/2}(\hat{p})\hat{x}m^{1/4}(\hat{p})\right] + U(\hat{p}),\\
&= -\frac{\hbar^2}{2}\omega^2\left(1-\frac{2k}{3\omega^2}p\right)\left[\frac{d^2}{dp^2}+\frac{k^2}{12\omega^4}\frac{1}{\left(1-\frac{2k}{3\omega^2}p\right)^2}\right]\\
&\quad +\frac{9\omega^4}{2k^2}\left(\sqrt{1-\frac{2k}{3\omega^2}p}-1\right)^2.
\end{aligned}$$

$\Rightarrow$ Schrödinger equation

$$-\frac{\hbar^2\omega^2}{2}\left(1-\frac{2k}{3\omega^2}p\right)\Phi'' - \frac{\hbar^2k^2}{24\omega^2\left(1-\frac{2k}{3\omega^2}p\right)}\Phi + \frac{9\omega^4}{2k^2}\left(\sqrt{1-\frac{2k}{3\omega^2}p}-1\right)^2\Phi = E\Phi, \quad \left({}' = \frac{d}{dp}\right).$$

Bound state ($\mathcal{PT}$-invariant solution)

$$\Phi_n(p) = \begin{cases} \tilde{N}_n \left(1 - \frac{2k}{3\omega^2} p\right)^{1/4} \exp\left(-\frac{9\omega^3}{2\,\hbar\,k^2}\left(1 - \frac{2k}{3\omega^2} p - 2\sqrt{1 - \frac{2k}{3\omega^2} p}\right)\right) \\ \qquad \times H_n\left[\frac{3\omega^{3/2}}{\sqrt{\hbar}\,k}\left(\sqrt{1 - \frac{2k}{3\omega^2} p} - 1\right)\right], -\infty < p \leq \frac{3\omega^2}{2k}, \\ 0, \quad p \geq \frac{3\omega^2}{2k}, \end{cases}$$

and Energy eigenvalues

$$E_n = (n + \frac{1}{2})\,\hbar\,\omega, \qquad n = 0, 1, 2, ... \tag{3}$$

- Normalization.

The $p > \frac{3\omega_0^2}{2k}$ sector: Broken Symmetry

$$\Phi_n(p) = \begin{cases} \tilde{\mathcal{N}}_n \left(\frac{2k}{3\omega^2} p - 1\right)^{1/4} \exp\left(-\frac{9\omega^3}{2\,\hbar\,k^2}\left(\frac{2k}{3\omega^2} p - 1 + i\,2\sqrt{\frac{2k}{3\omega^2} p - 1}\right)\right) \\ \qquad\qquad \times H_n\left[\frac{3\omega^{3/2}}{\sqrt{\hbar}\,k}\left(\sqrt{\frac{2k}{3\omega^2} p - 1} + i\right)\right], \quad p \geq \frac{3\omega^2}{2k}, \\ 0, \quad -\infty < p \leq \frac{3\omega^2}{2k}, \end{cases}$$

and

$$E_n = -(n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, 3, ... \tag{4}$$

A Bi-$\mathcal{PT}$ symmetric system

- Consider

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 + \omega_0^2 x + \kappa y = 0,$$
$$\ddot{y} + \alpha y\dot{y} + \beta y^3 + \omega_0^2 y + \kappa x = 0.$$

$\Rightarrow$ Bi-$\mathcal{PT}$ symmetric.

- Equivalently

$$\begin{aligned}
\dot{x} &= x_1,\\
\dot{x}_1 &= -\alpha x x_1 - \beta x^3 - \omega_0^2 x - \kappa y,\\
\dot{y} &= y_1,\\
\dot{y}_1 &= -\alpha y y_1 - \beta y^3 - \omega_0^2 y - \kappa x.
\end{aligned}$$

S. Karthiga, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Phys. Rev. A **93**, 012102 (2016)

- Fixed points
  1. e_0: $(x^*, x_1^*, y^*, y_1^*) = (0, 0, 0, 0)$.
  2. $e_{1,2}$: $(x^*, x_1^*, y^*, y_1^*) = (\pm a_1^*, 0, \mp a_1^*, 0)$, where $a_1^* = \sqrt{\frac{\kappa - \omega_0^2}{\beta}}$.
  3. $e_{3,4}$: $(x^*, x_1^*, y^*, y_1^*) = (\pm a_2^*, 0, \pm a_2^*, 0)$, where $a_2^* = \sqrt{\frac{-\kappa - \omega_0^2}{\beta}}$.

	$\kappa < -\omega_0^2$	$-\omega_0^2 \leq \kappa \leq \omega_0^2$	$\kappa > \omega_0^2$
$\Omega = \omega_0^2 > 0$	e_0, e_3, e_4	e_0	e_0, e_1, e_2
$\Omega = \omega_0^2 = 0$	e_0, e_3, e_4	e_0	e_0, e_1, e_2
	$\kappa < \omega_0^2$	$\omega_0^2 \leq \kappa \leq -\omega_0^2$	$\kappa > -\omega_0^2$
$\Omega = \omega_0^2 < 0$	e_0, e_3, e_4	e_0, e_1, e_2 e_3, e_4	e_0, e_1, e_2

(a) Case $\Omega = \omega_0^2 > 0$

- We use the condition for $\mathcal{PT}$ symmetry

$$\mathcal{PT} e_i = e_i$$

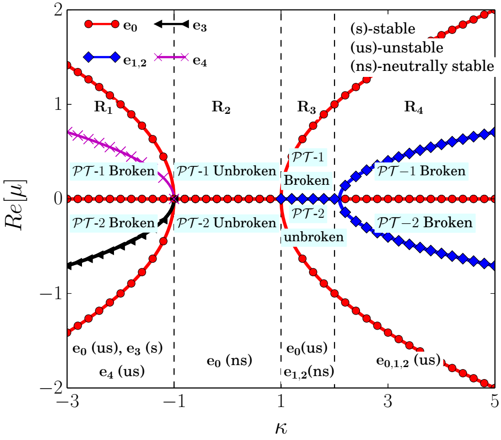
from which $\mathcal{PT}$ region is identified if the fixed point is neutrally stable.

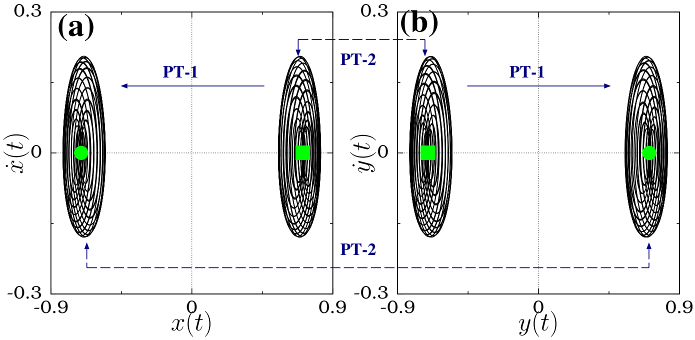
- Linear stability analysis

$$\text{Jacobian} \quad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c_{21} & -\alpha x^* & -\kappa & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa & 0 & c_{43} & -\alpha y^* \end{bmatrix}, \qquad (5)$$

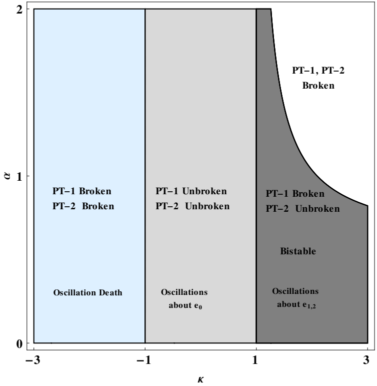
where $c_{21} = -\alpha x_1^* - 3\beta x^{*2} - \omega_0^2$, $c_{43} = -\alpha y_1^* - 3\beta y^{*2} - \omega_0^2$

- Calculate the eigenvalues



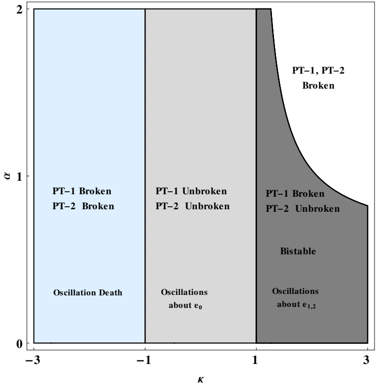

(a)
(b)
PT-1
PT-1
PT-2
PT-2
0.3
0
-0.3
-0.9
0
0.9
$\dot{x}(t)$
$x(t)$
$\dot{y}(t)$
$y(t)$

- Dynamics in the (κ, α) parameter space



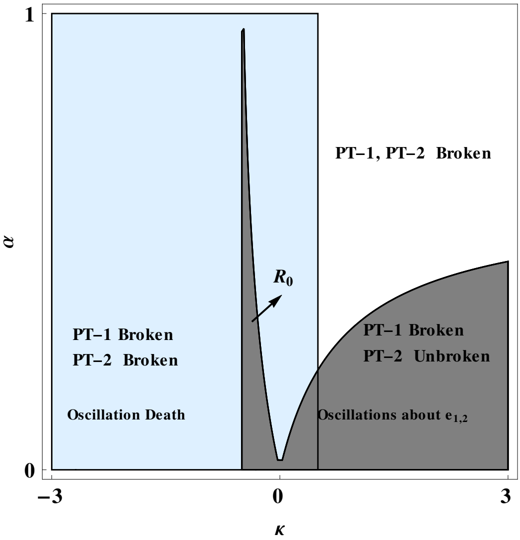
- In the range $-1 < \kappa < 1$, the $\mathcal{PT}$ symmetry remains unbroken by increasing the nonlinear loss/gain strength α.

- Dynamics in the (κ, α) parameter space



- But, in the range $\kappa > 1$, the $\mathcal{PT}$ symmetry is broken by increasing the nonlinear loss/gain strength α.

(b) Case $\Omega = \omega_0^2 < 0$



Nonlinear Plus Linear Damping

- Adding linear damping term

$$\begin{aligned} \ddot{x} + \gamma\dot{x} + \alpha x\dot{x} + \beta x^3 + \omega_0^2 x + \kappa y = 0, \\ \ddot{y} - \gamma\dot{y} + \alpha y\dot{y} + \beta y^3 + \omega_0^2 y + \kappa x = 0, \end{aligned} \tag{6}$$

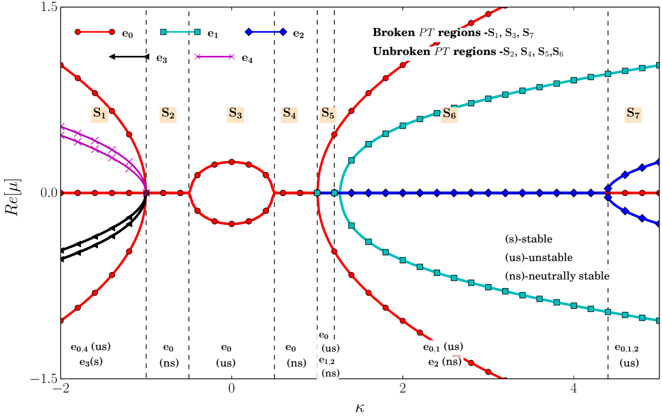
- $\mathcal{PT}-1$ symmetry: $x \to -x$, $y \to -y$, $t \to -t$ (absent)
- $\mathcal{PT}-2$ symmetry: $x \to -y$, $y \to -x$, $t \to -t$ (existing)

- Fixed points: same as that of the previous case

$$\text{Jacobian} \quad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c_{21} & -\gamma - \alpha x^* & -\kappa & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa & 0 & c_{43} & \gamma - \alpha y^* \end{bmatrix}, \tag{7}$$

where, $c_{21} = -\alpha x_1^* - 3\beta x^{*2} - \omega_0^2$, $c_{43} = -\alpha y_1^* - 3\beta y^{*2} - \omega_0^2$.

$Re[\mu]$ vs κ



$\mathcal{PT}$ restoration at higher loss/gain strength

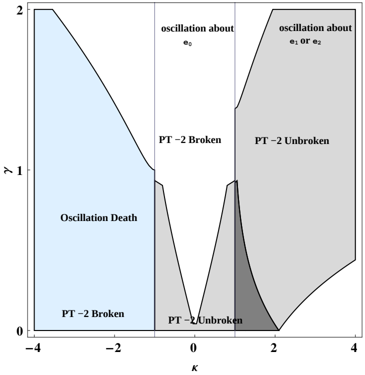


Figure: Dynamics in (κ, γ) space: For $\alpha = 1.0$, $\omega = 1$ and $\beta = 1$

- In the range $\kappa > 2$, the $\mathcal{PT}$ symmetry is broken for lower γ and becomes unbroken by increasing γ.

$\mathcal{PT}$ restoration at higher loss/gain strength

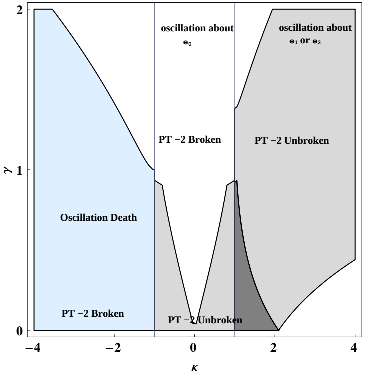
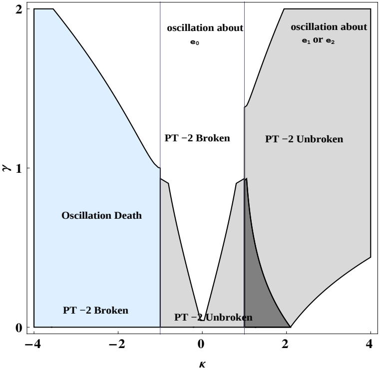
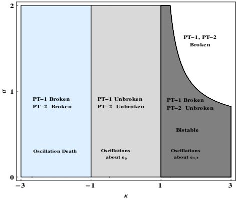


Figure: Dynamics in (κ, γ) space: For $\alpha = 1.0$, $\omega = 1$ and $\beta = 1$

- The above $\mathcal{PT}$ restoration at higher loss/gain strength arises due to the competition between the linear and nonlinear damping terms.

$\mathcal{PT}$ restoration at higher loss/gain strength





- The above behavior is absent when the nonlinear damping alone present in the system.

General case

$$\ddot{x} + \gamma\dot{x} + (-1)^n\alpha f(x)\dot{x} + \beta x^3 + \omega_0^2 x + \kappa y = 0,$$
$$\ddot{y} - \gamma\dot{y} + \alpha f(y)\dot{y} + \beta y^3 + \omega_0^2 y + \kappa x = 0, \qquad (8)$$

- $n = 0$ if $f(x)$ is an odd function in x
- $n = 1$ if $f(x)$ is an even function in x
- Bi-$\mathcal{PT}$ symmetric: when $\gamma = 0$ and $f(x) \to$ odd.
- Tailoring the form of $f(x)$, $\mathcal{PT}$ regions can be tailored.

(a) Case $f(x)$: odd

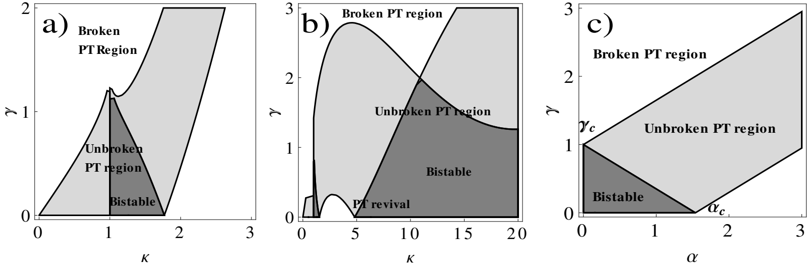


Figure: Phase diagram in (κ, γ) space: (a) $f(x) = x^3$, (b) $f(x) = sinx$ for $\alpha = 1.5$. (c): Phase diagram in (α, γ) space corresponding to $f(x) = sinx$ and $\kappa = 1.5$

(b) Case $f(x)$: even

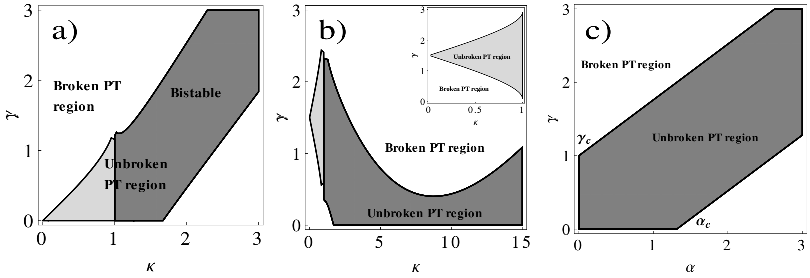


Figure: Phase diagram in (κ, γ) space: (a) $f(x) = x^2$ and (b) $f(x) = cosx$ with $\alpha = 1.5$. (c): Phase diagram in (α, γ) space: $f(x) = cosx$ with $\kappa = 1.5$

Systems that become $\mathcal{PT}$ symmetric through interaction

- Optical systems : Normally by introducing loss, gain terms - very delicate as they have to match exactly
- We are interested in identifying alternate $\mathcal{PT}$-symmetric systems $\Rightarrow$ through coupling & study the dynamics
- $\mathcal{PT}$ symmetric (loss/gain) $\Rightarrow$ special cases of dissipative systems & lie at the boundary between conservative & dissipative cases.
- What about $\mathcal{PT}$ symmetry coupling systems?
  Both dissipative and conservative systems!
- Nonreciprocal nature of $\mathcal{PT}$ symmetric loss/gain systems $\Rightarrow$ for unidirectional propagation of light
- Present case: Both in conservative & dissipative
  - Requirement for non-reciprocity : self-trapping nonlinearity.
  - Spontaneous symmetry breaking also occurs here
    - Pitchfork type bifurcation : Reciprocal light propagation
    - Tangent type bifurcation : Non-reciprocal & unidirectional light propagation

$\mathcal{PT}$ symmetric loss-gain system

$$\begin{aligned} i\frac{d\phi_1}{dz} &= i\gamma\phi_1 - k\phi_2 + \alpha G(\phi_1, \phi_2, \phi_1^*, \phi_2^*), \\ i\frac{d\phi_2}{dz} &= -i\gamma\phi_2 - k\phi_1 + \alpha G(\phi_2, \phi_1, \phi_2^*, \phi_1^*). \end{aligned}$$

- $k \Rightarrow$ Coupling due to evanescent fields
- $\mathcal{PT}$ symmetry: $\phi_1 \to -\phi_2$, $\phi_2 \to -\phi_1$, $i \to -i$, $z \to -z$

$$G(\phi_1, \phi_2, \phi_1^*, \phi_2^*) = -G(-\phi_1, -\phi_2, -\phi_1^*, -\phi_2^*)$$

Linear $\mathcal{PT}$ symmetric coupling

We consider

$$\begin{aligned} i\frac{d\phi_1}{dz} &= ia\phi_2 - k\phi_2 + \alpha G(\phi_1, \phi_2, \phi_1^*, \phi_2^*), \\ i\frac{d\phi_2}{dz} &= -ia\phi_1 - k\phi_1 + \alpha G(\phi_2, \phi_1, \phi_2^*, \phi_1^*), \end{aligned}$$

$a \Rightarrow$ magneto-optic coupling

Simple conservative case

$$\begin{aligned} i\frac{d\phi_1}{dz} &= -\beta|\phi_1|^2\phi_1 - k\phi_2 + ia\phi_2, \\ i\frac{d\phi_2}{dz} &= -\beta|\phi_2|^2\phi_2 - k\phi_1 - ia\phi_1. \end{aligned}$$

- Conservative: $\frac{dP}{dz} = 0,\ P = |\phi_1|^2 + |\phi_2|^2$
  (Total power is conserved $\Rightarrow$ Conservative.)

Light beam propagation: Comparison

- Light beam propagation with respect to propagation distance & compare with $\mathcal{PT}$ symmetric system with loss and gain
- **Linear case:**

$$i\frac{d\phi}{dz} = \mathcal{H}\phi,$$

$$\mathcal{H} = \begin{pmatrix} 0 & -k+ia \\ -k-ia & 0 \end{pmatrix} \quad \Rightarrow \zeta = \pm\sqrt{k^2+a^2}.$$

such that the modes evolve as $e^{\pm i\zeta z}$.

- Real eigenvalues $\Rightarrow$ always unbroken symmetry.
- In the linear $\mathcal{PT}$ symmetric system with loss-gain

$$\zeta = \pm\sqrt{k^2-\gamma^2}.$$

- Real eigenvalues for $k > \gamma \Rightarrow$ unbroken symmetry for $k > \gamma$.

Nonlinear case

With

$$\begin{aligned}\phi_1(z) &= R_1 e^{i(\omega z+\theta_1)},\\ \phi_2(z) &= R_2 e^{i(\omega z+\theta_2)}\end{aligned}$$

and $\delta = \theta_1 - \theta_2$

$$\dot{R}_1 = (k + \alpha_1 R_1^2) R_2 \sin\delta + a R_2 \cos\delta,$$
$$\dot{R}_2 = -(k + \alpha_1 R_2^2) R_1 \sin\delta - a R_1 \cos\delta,$$
$$\dot{\delta} = (\beta R_1 R_2 - k\cos\delta + a\sin\delta)\frac{R_1^2 - R_2^2}{R_1 R_2},\ \ \delta = \theta_1 - \theta_2.$$

(i) Symmetric Nonlinear Modes:

$$M_1 : \; R_1 \equiv R_1^* = \sqrt{P/2}, \quad \cos\delta^* = \frac{-k}{\sqrt{a^2+k^2}}, \quad P = R_1^2 + R_2^2$$

$$M_2 : \; R_1 \equiv R_1^* = \sqrt{P/2}, \quad \cos\delta^* = \frac{k}{\sqrt{a^2+k^2}},$$

$\sin\delta^* = \frac{-a\cos\delta^*}{k}$

(ii) Asymmetric Nonlinear Modes:

$$M_3 : \; R_1 \equiv R_1^* = \sqrt{\frac{P\beta + \sqrt{P^2\beta^2 - 4(a^2+k^2)}}{2\beta}},$$

$$M_4 : \; R_1 \equiv R_1^* = \sqrt{\frac{P\beta - \sqrt{P^2\beta^2 - 4(a^2+k^2)}}{2\beta}}.$$

$\sin\delta^* = \frac{-a}{\sqrt{a^2+k^2}}$ and $\cos\delta^* = \frac{k}{\sqrt{a^2+k^2}}$.

Linear stability

Eigenvalues

$$
\begin{aligned}
M_1:\ \lambda &= \pm i2\sqrt{\sqrt{a^2+k^2}\left(\frac{\beta P}{2}+\sqrt{a^2+k^2}\right)},\\
M_2:\ \lambda &= \pm 2\sqrt{\sqrt{a^2+k^2}\left(\frac{\beta P}{2}-\sqrt{a^2+k^2}\right)},\\
M_3:\ \lambda &= \pm i\sqrt{P^2\beta^2-4(a^2+k^2)},\\
M_4:\ \lambda &= \pm i\sqrt{P^2\beta^2-4(a^2+k^2)},.
\end{aligned}
$$

Stability results

M_1: always stable

M_2: stable when $\beta < \frac{2\sqrt{a^2+k^2}}{P}$

M_3 & M_4: stable when $\beta > \frac{2\sqrt{a^2+k^2}}{P}$

Stability & Pitchfork bifurcation at $\beta = \frac{2\sqrt{a^2+k^2}}{P}$

Stabilizes the asymmetric modes M_3 & M_4 & breaks the $\mathcal{PT}$ symmetry ($R_1^* \neq R_2^*$)

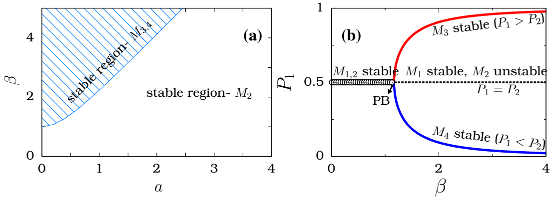


Figure: (a) Stable regions of symmetric as well as asymmetric modes for $P(0) = P = 1.0$, $k = 0.5$. Note that in the entire region, M_1 is stable. Fig. (b) Pitchfork bifurcation for $k = 0.5$, $P = 1.0$ and $a = 0.3$.

Reciprocal nature

Time reversibility $\Rightarrow$ Consider two input situations

$$\text{(a) } |\phi_1^{(1)}|^2(0) = 1,\ |\phi_2^{(1)}|^2(0) = 0$$
$$\text{(b) } |\phi_1^{(2)}|^2(0) = 0,\ |\phi_2^{(2)}|^2(0) = 1$$

Integrating the considered equations:

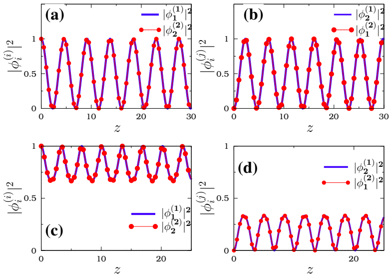


Figure: Reciprocal dynamics in the system for $k = 0.5$, $a = 0.5$. Figs. (a) and (b): $\beta = 1.0$. Figs. (c) and (d): $\beta = 3.0$.

General Case

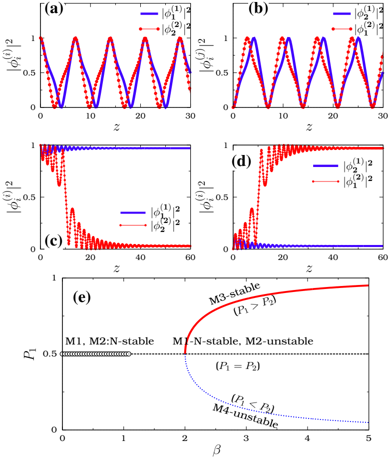
Do non-reciprocity is always absent in conservative systems?

Consider more general coupling:

$$
\begin{aligned}
i\frac{d\phi_1}{dz} &= -\beta|\phi_1|^2\phi_1 - k\phi_2 + ia\phi_2 + \alpha G(\phi_1,\phi_2,\phi_1^*,\phi_2^*),\\
i\frac{d\phi_2}{dz} &= -\beta|\phi_2|^2\phi_2 - k\phi_1 - ia\phi_1 + \alpha G(\phi_2,\phi_1,\phi_2^*,\phi_1^*),
\end{aligned}
$$

- Physically relevant nonlinearities satisfy
  $$G(\phi_1 e^{i\chi},\phi_2 e^{i\chi},\phi_1^* e^{-i\chi},\phi_2^* e^{-i\chi}) = e^{i\chi}G(\phi_1,\phi_2,\phi_1^*,\phi_2^*).$$
- The possible physically relevant nonlinearities preserving total power conservation are identified.
- Nonlinearities that can support reciprocal dynamics - pitchfork type symmetry breaking bifurcation
  $$G(\phi_1,\phi_2,\phi_1^*,\phi_2^*) = |\phi_2|^2\phi_1,\ \phi_2^2\phi_1^*,\ \phi_2^3\phi_1^{*2},\ |\phi_1|^4\phi_1,\ |\phi_2|^4\phi_1,\ |\phi_1|^2|\phi_2^2|\phi_1,$$
- Nonlinearities that can support non-reciprocal dynamics - tangent type symmetry breaking bifurcation
  $$G(\phi_1,\phi_2,\phi_1^*,\phi_2^*) = |\phi_1|^2|\phi_2^2|\phi_2,\ |\phi_2|^2\phi_1^2\phi_2^*.$$

Non-reciprocal dynamics in the conservative case for $G(\phi_1, \phi_2, \phi_1^*, \phi_2^*) = |\phi_1|^2|\phi_2|^2\phi_2$



Tangent like bifurcation occurs!

Tangent like bifurcation occurs!

- For low values of β, both M_1 & M_2 stable and then M_2 becomes unstable. In addition, at critical value of β, M_3 is stable, M_4 is unstable.
- Localization of light in the first waveguide regardless of whether the input beam is injected in the first waveguide or in the second waveguide.
- Unidirectional transport of light occurs due to tangent like symmetry breaking bifurcation.

Addition of non-conservative terms (like four wave mixing)

- Now include $\mathcal{PT}$ symmetric nonconservative forms in G:

$$G(\phi_1,\phi_2,\phi_1^*,\phi_2^*) = \phi_1^2\phi_2^*,\ |\phi_1|^2\phi_2,\ |\phi_2|^2\phi_2, |\phi_1|^4\phi_2,\ |\phi_2|^4\phi_2,\ \phi_1^3\phi_2^{*2},\ |\phi_1|^2\phi_1^2\phi_2,\ |\phi_1|^2\phi_2^2\phi_1^*,\ |\phi_2|^2\phi_2^2\phi_1^*.$$

- If we consider nonlinearities upto cubic order:

$$\alpha G(\phi_1,\phi_2,\phi_1^*,\phi_2^*) = \alpha_1\phi_1^2\phi_2^* + \alpha_2|\phi_1|^2\phi_2 + \alpha_3|\phi_2|^2\phi_2.$$

$$\frac{dP}{dz} = (-\alpha_1+\alpha_2-\alpha_3)S_2(P_1-P_2) \neq 0.$$

one of the Stoke's variables $S_2 = i(\phi_1\phi_2^* - \phi_1^*\phi_2)$ and $P_{1,2} = |\phi_{1,2}|^2$

What about the non-reciprocal nature of the non-conservative cases?

- All non-conservative cases are non-reciprocal and show unidirectional light transport in the symmetry broken region.
- Self trapping nonlinearity - essential for unidirectional light transport.

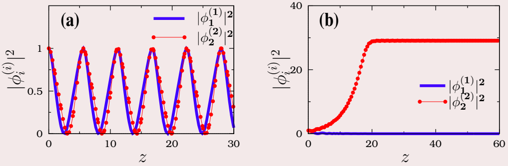


Figure: (Non-reciprocal dynamics in the unbroken and broken $\mathcal{PT}$ regions for the case $\alpha G = \alpha_1 \phi_1^2 \phi_2^*$

Nonlinear $\mathcal{PT}$ symmetric coupling

$$
\begin{aligned}
i\frac{d\phi_1}{dz} &= -\beta|\phi_1|^2\phi_1 - k\phi_2 + i\eta G(\phi_1, \phi_2, \phi_1^*, \phi_2^*), \\
i\frac{d\phi_2}{dz} &= -\beta|\phi_2|^2\phi_2 - k\phi_1 - i\eta G(\phi_2, \phi_1, \phi_2^*, \phi_1^*),
\end{aligned}
$$

- Except the cases $G = |\phi_2|^4\phi_1$ and $|\phi_1|^2\phi_1^2\phi_2^*$, all the other conservative and non-conservative cases are non-reciprocal and show unidirectional light transport in the presence of self-trapping nonlinearity (β).
- An interesting case which support unidirectional light transport even in the absence of self-trapping nonlinearity

$$
\begin{aligned}
i\dot{\phi}_1 &= -\beta|\phi_1|^2\phi_1 - k\phi_2 + i\eta|\phi_2|^2\phi_1, \\
i\dot{\phi}_2 &= -\beta|\phi_2|^2\phi_2 - k\phi_1 - i\eta|\phi_1|^2\phi_2.
\end{aligned}
$$

- The above nonlinear coupling is also useful in controlling blow-up responses in $\mathcal{PT}$ symmetric systems with loss-gain.

Controlling blow-up responses with nonlinear coupling

Model

Coupled waveguide system under consideration

$$
\begin{aligned}
i\frac{d\phi_1}{dz} &= \omega_1\phi_1 - i\gamma\phi_1 + i\alpha|\phi_1|^2\phi_1 - k\phi_2 - i\chi_1|\phi_2|^2\phi_1,\\
i\frac{d\phi_2}{dz} &= \omega_2\phi_2 + i\gamma\phi_2 - i\alpha|\phi_2|^2\phi_2 - k\phi_1 + i\chi_2|\phi_1|^2\phi_2,
\end{aligned}
$$

$\gamma \Rightarrow$ linear loss-gain strength
$\alpha \Rightarrow$ nonlinear loss-gain strength
$\chi_1, \chi_2 \Rightarrow$ coupling induced by stimulated Raman scattering, $\chi_1 = \frac{\omega_1}{\omega_2}\chi$ and $\chi_2 = \chi$
$\omega_1 > \omega_2$, $\omega_1 \Rightarrow$ Pump frequency, $\omega_2 \Rightarrow$ Stokes frequency.

Limiting case: $\mathcal{PT}$ symmetric case

$$
\begin{aligned}
i\frac{d\phi_1}{dz} &= \omega\phi_1 - i\gamma\phi_1 + i\alpha|\phi_1|^2\phi_1 - k\phi_2 - i\chi|\phi_2|^2\phi_1,\\
i\frac{d\phi_2}{dz} &= \omega\phi_2 + i\gamma\phi_2 - i\alpha|\phi_2|^2\phi_2 - k\phi_1 + i\chi|\phi_1|^2\phi_2.
\end{aligned}
$$

Equation of amplitude and phase

$\phi_1 = R_1 e^{-i\tilde{\omega}z + i\theta_1}$ and $\phi_2 = R_2 e^{-i\tilde{\omega}z + i\theta_2}$.

$$\begin{aligned}
\dot{R}_1 &= -\gamma R_1 + \alpha R_1^3 + kR_2 \sin\delta - \chi R_2^2 R_1, \\
\dot{R}_2 &= \gamma R_2 - \alpha R_2^3 - kR_1 \sin\delta + \chi R_1^2 R_2, \\
\dot{\delta} &= -\frac{k(R_1^2 - R_2^2)}{R_1 R_2} \cos\delta,
\end{aligned}$$

$\Rightarrow$ Symmetric mode

$$R_1^* = R_2^* = R^*, \quad \sin\delta^* = \frac{\gamma - (\alpha - \chi)R^{*2}}{k}$$

$\Rightarrow$ Asymmetric modes

$$\text{(i)} \quad R_1^* = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma}{\alpha} - \Delta}, \quad R_2^* = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma}{\alpha} + \Delta},$$

$$\text{(ii)} \quad R_1^* = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma}{\alpha} + \Delta}, \quad R_2^* = \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma}{\alpha} - \Delta}, \delta = \frac{(2n+1)\pi}{2}, \ n = 0, 1, 2, ...,$$

where $\Delta = \sqrt{\frac{\gamma^2}{\alpha^2} - \frac{4k^2}{(\alpha+\chi)^2}}$. If $k < \frac{\gamma(\alpha+\chi)}{2\alpha}$, asymmetric state is stable. Otherwise symmetric state is stable.

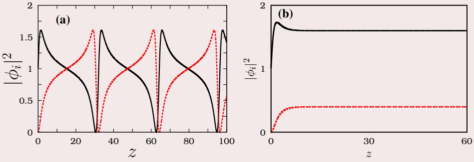


Figure: Dynamics of $|\phi_1|^2$ (red curve) and $|\phi_2|^2$ (black curve) in the (a) unbroken $\mathcal{PT}$ region ($\chi = 0.0$ and $k = 0.51$) and (b) broken $\mathcal{PT}$ region ($\chi = 0.0$ and $k = 0.4$). In both the figures $\gamma = 1.0$ and $\alpha = 0.5$

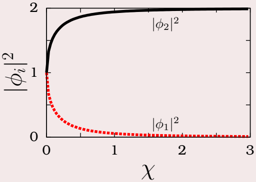


Figure: Power in the first and second waveguides for different values of χ and for $k = 0.5$, $\gamma = 1.0$ and $\alpha = 0.5$. From the figure, we can observe the isolation of light in the second waveguide with increase of χ.

Besides the above stable dynamics, the system also shows blow-up responses for certain higher input powers. Linear stability analysis is a local analysis that does not evidence the existence of such blow-up responses.

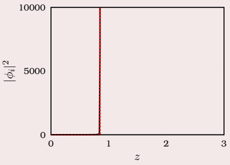


Figure: Blow-up response for $\phi_1(0) = \sqrt{2.5}$, $\phi_2(0) = 0.0$ and for $\chi = 0.0$, $k = 0.4$, $\gamma = 1.0$ and $\alpha = 0.5$

We use integrable nature of the system to show the ability of the nonlinear coupling in controlling blow-up responses.

Dynamical equations in Stokes variables

$p = |\phi_1|^2 + |\phi_2|^2$, $s_0 = |\phi_1|^2 - |\phi_2|^2$, $s_1 = \phi_1\phi_2^* + \phi_1^*\phi_2$, $s_2 = i(\phi_1\phi_2^* - \phi_1^*\phi_2)$ and $p^2 = s_0^2 + s_1^2 + s_2^2$

$$\begin{aligned} \dot{p} &= -2(\gamma - \alpha p)s_0, \\ \dot{s_0} &= -2(\gamma - \alpha p)p - [(\alpha + \chi)s_1]\, s_1 - [(\alpha + \chi)s_2 + 2k]\, s_2, \\ \dot{s_1} &= [(\alpha + \chi)s_1]\, s_0, \\ \dot{s_2} &= [(\alpha + \chi)s_2 + 2k]\, s_0. \end{aligned} \tag{9}$$

The above set of equations has two integrals of motion (I_1, I_2). With two integrals of motion and using $p^2 = s_0^2 + s_1^2 + s_2^2$, the problem can be reduced to the particle in a potential problem

$$\frac{\dot{p}^2}{2} + V(p, E) = 0,$$

$$V(p, E) = \widetilde{V}(p) = V(p) - E = 2(\gamma - \alpha p)^2 \left[s_1^2(p) + s_2^2(p) - p^2\right]. \tag{10}$$

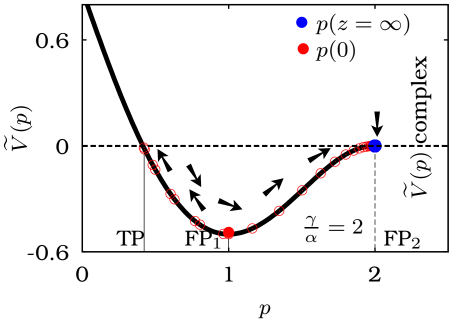


Figure: Dynamics (stable dynamics) in the broken $\mathcal{PT}$ symmetric region for $\chi = 0.0$: Figure shows the potential curve of $\widetilde{V}(p)$ for $\gamma = 1.0$, $\alpha = 0.5$, $k = 0.4$, and $\chi = 0.0$. It is obtained for the initial condition $\phi_1 = 1.0 + 0.0i$ and $\phi_2 = 0.0$. The empty circles along the $\widetilde{V}(p)$ curve represent the dynamics of p with respect to z.

Control of blow-up responses by χ

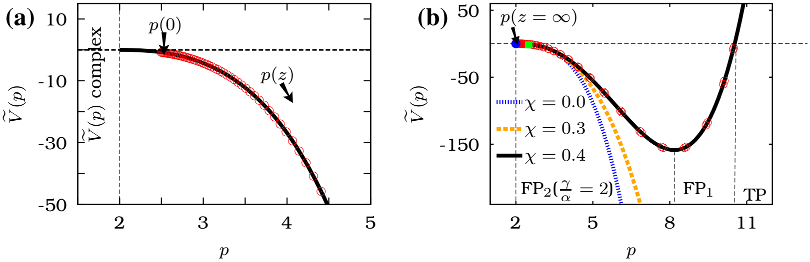


Figure: Blow-up responses in the broken $\mathcal{PT}$ symmetric region: Fig. (a) is plotted for the same values of parameters as that of previous figure but for initial conditions $\phi_1 = \sqrt{2.5}$ and $\phi_2 = 0.0$ and it also shows the existence of blow-up solution for certain input power configurations. Fig. (b) shows the curves of $\widetilde{V}(p)$ for different values of χ, namely, $0.0, 0.3$ and 0.4, for the same initial conditions and parametric values as considered in Fig. (a).

Conclusion

- $\mathcal{PT}$ symmetry breaking in the interesting class of nonlinear/position dependently damped systems have been studied.
- The existence of Bi-$\mathcal{PT}$ symmetric systems is shown. Tailoring of $\mathcal{PT}$ regions for proper choice of position dependent damping is also discussed.
- The possibilities to achieve unidirectional light transport with loss-gain free systems, that is, in the $\mathcal{PT}$ symmetrically coupled systems are shown.
- The ability to control blow-up responses in a $\mathcal{PT}$ symmetric system with a nonlinear coupling is demonstrated.

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