Nonlocality and degeneracy for the Hirota equation

Julia Cen, Francisco Correa, Andreas Fring

Non-Hermitian Physics - PHHQP XVIII, ICTS Bengaluru
4-13 June 2018

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J. Cen, F. Correa and A. Fring, arXiv:1710.11560, Integrable nonlocal Hirota equations

J. Cen and A. Fring, arXiv:1804.02013, Asymptotic and scattering behaviour for degenerate multi-solitons in the Hirota equation

Content

- Motivation
- New integrable nonlocal Hirota equations from $\mathcal{PT}$-symmetric reductions
- $\mathcal{P}$arity transformed conjugate pair
  - Hirota method
  - A new kind of solution
- Degenerate multi-solitons for local Hirota equation
  - Darboux-Crum transformation
  - Solution properties
- Conclusions

- Why Hirota equation

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$$iq_t + \sigma\left(q_{xx} - 2|q|^2 q\right) + i\rho\left(q_{xxx} - 6|q|^2 q_x\right) = 0$$

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$\mathcal{PT}$-symmetric extension preserving integrability

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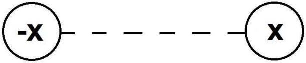

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Mathematically interesting and overlooked

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Hirota equation as zero curvature condition

$$U_t - V_x + [U, V] = 0$$

Hirota equation as zero curvature condition

$$U_t - V_x + [U, V] = 0$$

with

$$U = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

$$\begin{aligned}
A &= -i\sigma q r - 2i\sigma\lambda^2 + \rho(r q_x - q r_x - 4i\lambda^3 - 2i\lambda q r) \\
B &= i\sigma q_x + 2\sigma\lambda q + \rho(2q^2 r - q_{xx} + 2i\lambda q_x + 4\lambda^2 q) \\
C &= -i\sigma r_x + 2\sigma\lambda r + \rho(2q r^2 - r_{xx} - 2i\lambda r_x + 4\lambda r)
\end{aligned}$$

gives

$$\boxed{\begin{aligned}
i q_t + \sigma\left(q_{xx} - 2q^2 r\right) + i\rho\left(q_{xxx} - 6q r q_x\right) &= 0 \\
i r_t + \sigma\left(-r_{xx} + 2q r^2\right) + i\rho\left(r_{xxx} - 6q r r_x\right) &= 0
\end{aligned}}$$

New integrable nonlocal Hirota equations

$$\boxed{\begin{array}{c}\text{local}\\ r = q^*(x,t)\end{array}}$$

New integrable nonlocal Hirota equations

local
$r = q^*(x,t)$

$\mathcal{P}$arity transformed conjugate pair	$\mathcal{T}$ime-reversed pair	$\mathcal{PT}$-symmetric pair
$r = q^*(-x,t)$ $\sigma \in \mathbb{R}, \rho \in i\mathbb{R}$	$r = q^*(x,-t)$ $\sigma \in i\mathbb{R}, \rho \in i\mathbb{R}$	$r = q^*(-x,-t)$ $\sigma \in i\mathbb{R}, \rho \in \mathbb{R}$
$\mathcal{P}$arity transformed conjugate real pair	$\mathcal{T}$ime-reversed real pair	$\mathcal{PT}$-symmetric conjugate pair
$r = q(-x,t)$ $\sigma \in \mathbb{R}, \rho \in i\mathbb{R}$	$r = q(x,-t)$ $\sigma \in i\mathbb{R}, \rho \in i\mathbb{R}$	$r = q(-x,-t)$ $\sigma \in \mathbb{C}, \rho \in \mathbb{C}$

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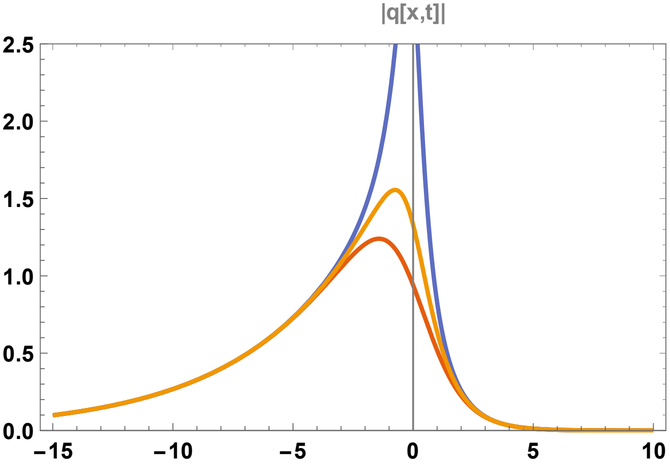
[Ablowitz, Musslimani, Phys. Rev. Lett. 110 (2013) 064105]

Non-standard solution for $r = -q^*(-x, t)$

$$q(x,t) = \frac{\beta e^{\alpha x - i\alpha^2(\alpha\theta-\sigma)t+\gamma_\alpha}}{1+e^{\beta x - i\beta(3\alpha^2\theta+\beta(\beta\theta+\sigma)-\alpha(3\beta\theta+2\sigma))t+\gamma_\beta}}$$



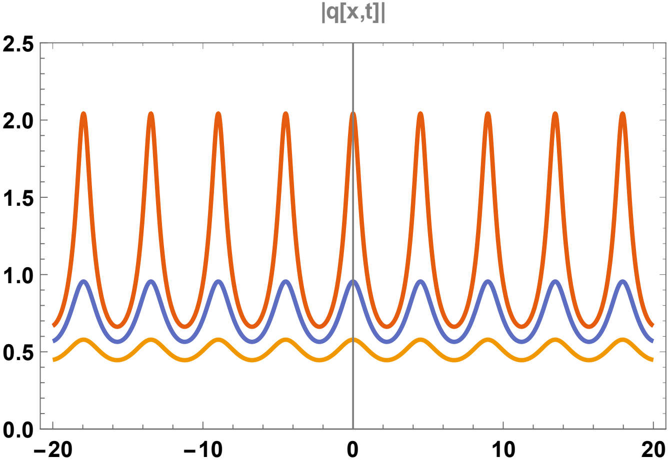
$t = 0$

$t = 1$

$t = 2$

Standard solution for $r = -q^*(-x, t)$

$$q(x,t) = \frac{c e^{\alpha x - i\alpha^2(\alpha\theta - \sigma)t + \gamma}}{1 + \frac{|c|^2}{(\alpha-\alpha^*)^2} e^{\alpha x - \alpha^* x + (-i\alpha^3\delta + i\alpha^{*3}\theta + i\alpha^2\sigma - i\alpha^{*2}\sigma)t + \gamma + \gamma^*}}$$



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Hirota method for $r = -q^*(-x, t)$

Let $\tilde{q}(x, t) = q(-x, t)$

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Take $q(x,t) = \frac{g(x,t)}{f(x,t)}$ and auxiliary function $h(x,t)$ to obtain bilinear equations

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$$
\begin{aligned}
iD_t g \cdot f + \sigma D_x^2 g \cdot f - \theta D_x^3 g \cdot f &= 0 \\
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which can be solved for n-soliton by:

1. $g = \sum_{k=0}^{\infty} \epsilon^{2k+1} g_{2k+1}$, $f = \sum_{k=0}^{\infty} \epsilon^{2k} f_{2k}$ and $h = \sum_{k=0}^{\infty} \epsilon^{k} h_k$
2. $g_1 = \sum_{l=1}^{n} c e^{\alpha_l x - i\alpha_l^2(\alpha_l\theta - \sigma)t + \gamma_l}$, a finite sum
3. ϵ arbitrary or ϵ=1

Degenerate n-soliton with Darboux-Crum transformations

From Z-C for an auxiliary function Ψ

$$\Psi_x = U\Psi,\ \Psi_t = V\Psi \qquad \text{with} \qquad U = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}$$

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Identify Hamiltonian

$$H\Psi = -\lambda\Psi \qquad \text{with} \qquad H = \begin{pmatrix} -i\partial_x & iq \\ -ir & i\partial_x \end{pmatrix}$$

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Darboux-Crum iteration

$$H_n = -i\sigma_3\partial_x + \begin{pmatrix} 0 & iq_n \\ -ir_n & 0 \end{pmatrix} \qquad \text{where} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{q_n = 2\frac{\det W_n^q}{\det W_n} \qquad r_n = -2\frac{\det W_n^r}{\det W_n}}$$

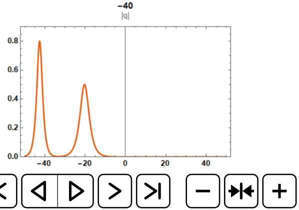
Degenerate 2-soliton with Darboux-Crum transformations

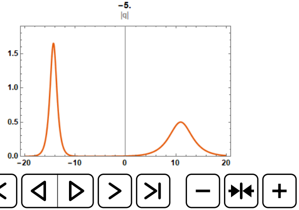
$$W_2^q = \begin{pmatrix} \phi & \varphi'' & \varphi' & \varphi \\ \phi^* & \varphi^{*\prime\prime} & \varphi^{*\prime} & \varphi^* \\ \partial_\lambda \phi & (\partial_\lambda \varphi)'' & (\partial_\lambda \varphi)' & (\partial_\lambda \varphi) \\ \partial_{\lambda^*} \phi^* & (\partial_{\lambda^*} \varphi^*)'' & (\partial_{\lambda^*} \varphi^*)' & \partial_{\lambda^*} \varphi^*) \end{pmatrix}$$

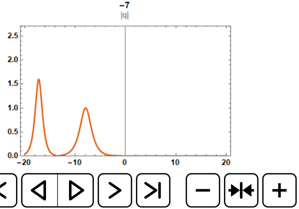
$$W_2 = \begin{pmatrix} \phi' & \phi & \varphi' & \varphi \\ \phi^{*\prime} & \phi^* & \varphi^{*\prime} & \varphi^* \\ (\partial_\lambda \phi)' & \partial_\lambda \phi & (\partial_\lambda \varphi)' & \partial_\lambda \varphi \\ (\partial_{\lambda^*} \phi^*)' & \partial_{\lambda^*} \phi^* & (\partial_{\lambda^*} \varphi^*)' & \partial_{\lambda^*} \varphi^* \end{pmatrix}$$

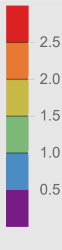
$$\Psi_1 = \begin{pmatrix} \varphi \\ \phi \end{pmatrix}, \Psi_2 = \begin{pmatrix} \phi^* \\ -\varphi^* \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} \varphi &= e^{-i\left[\lambda x + 2\lambda^2(\sigma + 2\rho\lambda)t\right] + \gamma_\varphi} \\ \phi &= e^{i\left[\lambda x + 2\lambda^2(\sigma + 2\rho\lambda)t\right] + \gamma_\phi} \end{aligned}$$

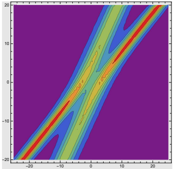
Scattering behaviour of non-degenerate 2-soliton



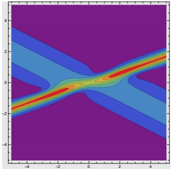




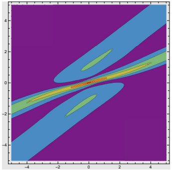




Bounce and Exchange



Merge and Split



Absorb and Emit

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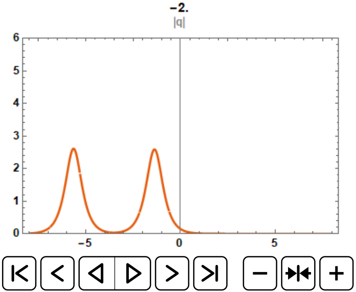
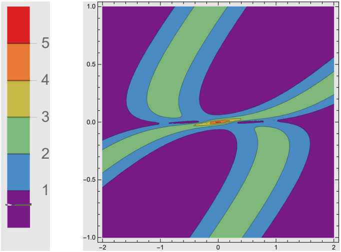
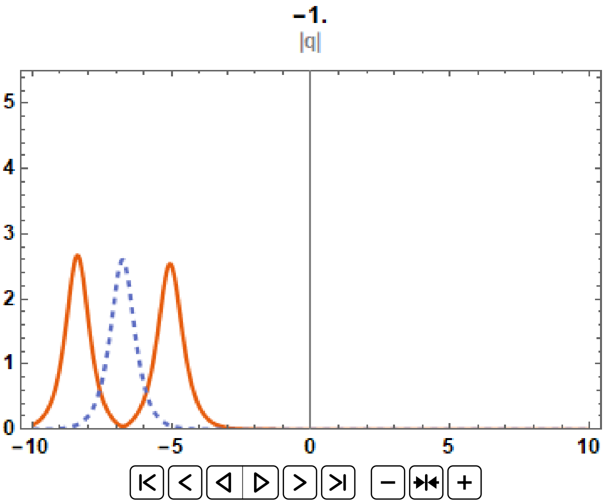




Asymptotic behaviour of degenerate 2-soliton



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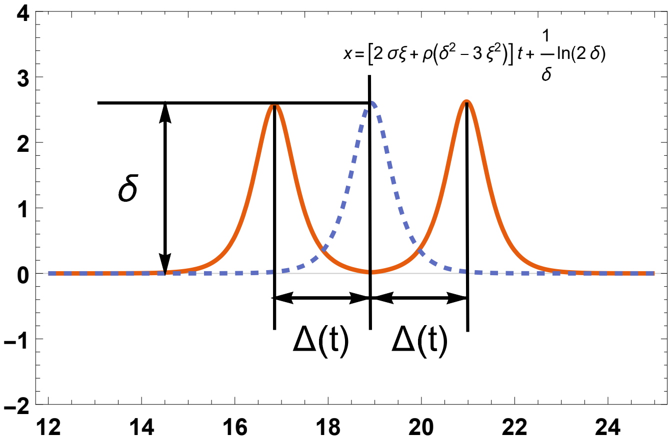

$$\Delta(t) = \frac{1}{\delta} \ln\left[2\delta\,|t|\,\sqrt{\rho^2\delta^2 + (\sigma - 3\rho\xi)^2}\right]$$

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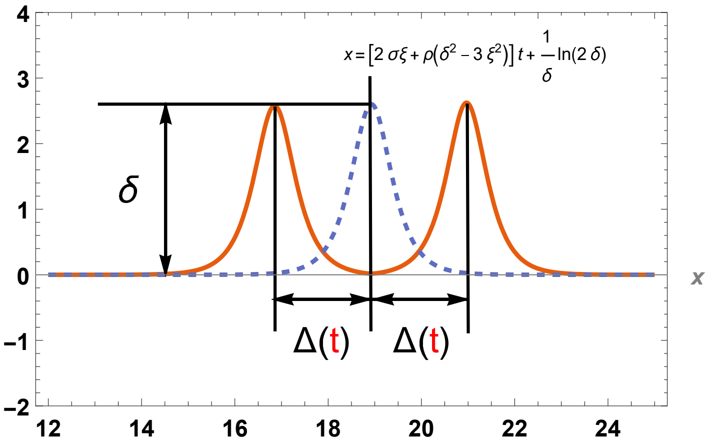

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Energy of degenerate 2-soliton

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Charges for 1-soliton solution $q_1(x,t)$

$$Q_m(q_1) = -|\delta| \sum_{k=0}^{m} \frac{m!\delta^k (i\xi)^{m-k}}{(k+1)!(m-k)!}\left[1+(-1)^k\right] \qquad \begin{array}{c} Q_{2m} \in \mathbb{R} \\ Q_{2m-1} \in i\mathbb{R} \end{array}$$

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Hamiltonian/energy for 1-soliton solution

$$H(q_1) = \sigma Q_2(q_1) + i\rho Q_3(q_1)$$

$$\mathcal{PT} : x \to -x,\ t \to -t,\ i \to -i$$

$$\mathcal{PT} H = H$$

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For degenerate 2-soliton

$$E(q_2) = 2E(q_1)$$

and degenerate n-soliton

$$E(q_n) = nE(q_1)$$

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Thank you!