Extended kq-representation and bi-coherent states

Fabio Bagarello

D.E.I.M., Università di Palermo

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Organization of the talk

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F.B., JMAA, 2017, and F.B.+others in Proc. Royal Soc. A, 2017

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Let $\hat{q}_0$ and $\hat{p}_0$ be the self-adjoint operators:

$$\hat{q}_0\,\varphi(x) = x\,\varphi(x), \qquad \hat{p}_0\,\varphi(x) = -i\varphi'(x),$$

for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$. Notice that $\mathcal{S}(\mathbb{R})$ is not the maximal domain of these operators, which are

$$D_{max}(\hat{q}_0) = \{f(x) \in \mathcal{L}^2(\mathbb{R}) :\ xf(x) \in \mathcal{L}^2(\mathbb{R})\}, \quad D_{max}(\hat{p}_0) = \{f(x) \in \mathcal{L}^2(\mathbb{R}) :\ f'(x) \in \mathcal{L}^2(\mathbb{R})\}.$$

Of course, $\mathcal{S}(\mathbb{R}) \subset D_{max}(\hat{q}_0) \cap D_{max}(\hat{p}_0)$, and is dense in $\mathcal{L}^2(\mathbb{R})$.

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Working with $\mathcal{S}(\mathbb{R})$ is *better* than $D_{max}(\hat{q}_0)$ and $D_{max}(\hat{p}_0)$, since, in particular, $\mathcal{S}(\mathbb{R}) \subseteq D^\infty(\hat{q}_0) \cap D^\infty(\hat{p}_0)$.

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Neither $\hat{q}_0$ nor $\hat{p}_0$ admit square integrable eigenvectors:

$$\hat{q}_0\,\xi_{x_0}(x) = x_0\,\xi_{x_0}(x), \qquad \hat{p}_0\,\theta_{p_0}(x) = p_0\,\theta_{p_0}(x),$$

where x_0 and p_0 are real numbers, and

$$\xi_{x_0}(x) = \delta(x - x_0), \qquad \theta_{p_0}(x) = \frac{1}{\sqrt{2\pi}}\,e^{ip_0x}.$$

Of course, $\xi_{x_0}(x), \theta_{p_0}(x) \in \mathcal{S}'(\mathbb{R})$, the set of tempered distributions.

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$$\langle \xi_{x_0}, \xi_{y_0} \rangle = \delta(x_0 - y_0), \qquad \int_{\mathbb{R}} dx_0 \, |\xi_{x_0}\rangle\langle \xi_{x_0}| = \mathbb{1}.$$

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Answer:– Since

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \overline{\varphi(x)} \, \psi(x) \, dx = (\overline{\varphi} * \tilde{\psi})(0)$$

for each $\varphi(x), \psi(x) \in \mathcal{S}(\mathbb{R})$, where $\tilde{\psi}(x) = \psi(-x)$, we define

$$\langle F, G \rangle = (\overline{F} * \tilde{G})(0),$$

for those $F, G \in \mathcal{S}'(\mathbb{R})$ for which the RHS makes sense (i.e., for compactly supported distributions).

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We deduce, $\forall \varphi(x) \in \mathcal{S}(\mathbb{R})$, the following equalities

$$\left(\overline{F} * \tilde{G}, \varphi \right) = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{F(x)} \, \tilde{G}(y) \, \varphi(x+y) \, dx \, dy = \langle F, G * \varphi \rangle \, ,$$

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Now, if we take $F = \xi_{x_0}$ and $G = \xi_{y_0}$,

$$\left(\overline{\xi}_{x_0} * \tilde{\xi}_{y_0}, \varphi\right) = \langle \xi_{x_0}, \xi_{y_0} * \varphi \rangle = \int_{\mathbb{R}} \xi_{x_0}(x)\varphi(x - y_0)dx = \varphi(x_0 - y_0) = (\xi_{t_0}, \varphi)\,,$$

where $t_0 = x_0 - y_0$, for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$.
Hence $\left(\overline{\xi}_{x_0} * \tilde{\xi}_{y_0}\right)(x) = \xi_{t_0}(x)$, and therefore

$$\langle \xi_{x_0}, \xi_{y_0} \rangle = \left(\overline{\xi}_{x_0} * \tilde{\xi}_{y_0}\right)(0) = \xi_{t_0}(0) = \delta(x_0 - y_0),$$

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As for the second property of ξ_{x_0}, for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$, $\varphi(x_0) = \langle \xi_{x_0}, \varphi \rangle$. Then we have

$$\varphi(x) = \int_{\mathbb{R}} \delta(x - x_0)\varphi(x_0)\, dx_0 = \int_{\mathbb{R}} \xi_{x_0}(x) \langle \xi_{x_0}, \varphi \rangle \, dx_0,$$

as we had to prove. Hence the resolution of the identity makes sense (at least) on $\mathcal{S}(\mathbb{R})$.

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The operators $\hat{q}_0$ and $\hat{p}_0$ satisfy

$$[\hat{q}_0, \hat{p}_0]\varphi(x) = i\varphi(x),$$

$\forall\varphi(x) \in \mathcal{S}(\mathbb{R})$. We define the unitary operators

$$\tau_1 = e^{i\alpha\hat{q}_0}, \qquad \tau_2 = e^{-i\alpha\hat{p}_0}.$$

Then, if $\alpha^2 = 2\pi L$, for some $L = 1, 2, 3, \ldots$, $[\tau_1, \tau_2] = 0$ (in the sense of bounded operators).

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Also, if $f(x) \in \mathcal{L}^2(\mathbb{R})$, then

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Now, the kq-representation makes use of the fact that, since τ_1 and τ_2 commute, they can be diagonalized simultaneously. However, the common eigenstates,

$$\rho_{kq}(x) = \frac{1}{\sqrt{\alpha}} \sum_{n\in\mathbb{Z}} e^{ikn\alpha} \delta(x - q - n\alpha), \qquad k, q \in [0, \alpha[,$$

are tempered distributions of $\mathcal{S}'(\mathbb{R})$: they are *generalized eigenstates* of τ_1 and τ_2, with

$$\tau_1 \rho_{kq}(x) = e^{i\alpha q} \rho_{kq}(x), \qquad \tau_2 \rho_{kq}(x) = e^{-i\alpha k} \rho_{kq}(x),$$

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Now, let

$$\square = \{(k,q) \in \mathbb{R}^2 : k, q \in [0, \alpha[\},$$

it is possible to check that

$$\int\int_{\square} \overline{\rho_{kq}(x)}\rho_{kq}(x')dk\,dq = \delta(x - x'),$$

and that

$$\int_{\mathbb{R}} \overline{\rho_{kq}(x)}\rho_{k'q'}(x)dx = \delta(k - k')\delta(q - q').$$

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Remark:– of course, these and the previous eigenvalue equations should be understood *in the sense of distributions*.

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$\rho_{kq}(x)$ can also be used to define a new representation of the wave functions by means of the integral transform $Z : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\Box)$, defined as follows:

$$h(k,q) := \langle \rho_{kq}, H \rangle =: (ZH)(k,q),$$

for all functions $H(x) \in \mathcal{S}(\mathbb{R})$, and then extended by continuity to all of $\mathcal{L}^2(\mathbb{R})$. The result is a function $h(k,q) \in \mathcal{L}^2(\Box)$.

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$$W(z) = e^{zc^\dagger - \overline{z}\, c},$$

a *standard* coherent state is the vector

$$\Phi(z) = W(z)e_0 = e^{-|z|^2/2} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}}\, e_k.$$

The vector $\Phi(z)$ is well defined (i.e., the series converge), and normalized $\forall\, z \in \mathbb{C}$. In fact $W(z)$ is unitary (or $\langle e_k, e_l \rangle = \delta_{k,l}$).

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$$c\, \Phi(z) = z \Phi(z), \qquad \text{and} \qquad \frac{1}{\pi} \int_{\mathbb{C}} d^2 z |\Phi(z)\rangle\langle \Phi(z)| = \mathbb{1}.$$

In particular, the second equality shows that $\mathcal{C} = \{\Phi(z),\, z \in \mathbb{C}\}$ is complete in $\mathcal{L}^2(\mathbb{R})$.

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It is also well known that $\Phi(z)$ saturates the Heisenberg uncertainty relation: $\Delta x \Delta p = \frac{1}{2}$.

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Can we extract out of the set $\mathcal{C} = \{\Phi(z),\, z \in \mathbb{C}\}$ (infinitely) many vectors, $\Phi(\hat{z}_j)$, $j = 1, 2, 3 \ldots$, getting now a complete set?

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Using kq-representation the main steps are the following:

1. we *discretize* $\mathbb{C}$ by considering a lattice defined by $z_{\underline{n}} = \frac{a}{\sqrt{2}}(n_2 + in_1)$, $n_j \in \mathbb{Z}$. Here $a^2 = 2\pi L$, $L = 1, 2, 3, \ldots$;

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2. we consider the unitary operator $W(z_{\underline{n}})=e^{z_{\underline{n}}c^\dagger-\overline{z_{\underline{n}}}\,c}$. We have

$$W(z_{\underline{n}})=(-1)^{Ln_1n_2}\tau_1^{n_1}\tau_2^{n_2},\qquad \tau_1=e^{i\hat{p}_0a},\,\tau_2=e^{i\hat{q}_0a};$$

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$$W(z_{\underline{n}}) = (-1)^{L n_1 n_2} \tau_1^{n_1} \tau_2^{n_2}, \qquad \tau_1 = e^{i\hat{p}_0 a},\, \tau_2 = e^{i\hat{q}_0 a};$$

3. Next we take the vectors in $\mathcal{C}_{num}$ as follows:

$$\Phi(z_{\underline{n}}) = W(z_{\underline{n}}) e_0$$

Completeness of standard coherent states

Sketch of the proof:

Let $f \in \mathcal{L}^2(\mathbb{R})$ be orthogonal to all the $\Phi(z_{\underline{n}})$:

$$\langle f, \Phi(z_{\underline{n}}) \rangle = \int_{\mathbb{R}} \overline{f(x)}\, \Phi(z_{\underline{n}}, x)\, dx = 0, \qquad \forall n_1, n_2 \in \mathbb{Z}.$$

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By means of the kq-representation we can rewrite this equality as

$$\int \int_{\square} \langle f, \rho_{kq} \rangle \langle \rho_{kq}, e_0 \rangle \; e^{ikn_1 a + iqn_2 a}\, dk\, dq = 0$$

for all $n_1, n_2 \in \mathbb{Z}$.

Completeness of standard coherent states

Sketch of the proof:

Let $f \in \mathcal{L}^2(\mathbb{R})$ be orthogonal to all the $\Phi(z_{\underline{n}})$:

$$\langle f, \Phi(z_{\underline{n}}) \rangle = \int_{\mathbb{R}} \overline{f(x)}\, \Phi(z_{\underline{n}}, x)\, dx = 0, \qquad \forall n_1, n_2 \in \mathbb{Z}.$$

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The set $\{e^{ikn_1 a + iqn_2 a},\, n_j \in \mathbb{Z}\}$ is complete in $\mathcal{L}^2(\square)$, **if** $L = 1$. Hence

$$\langle f, \rho_{kq} \rangle \langle \rho_{kq}, e_0 \rangle = 0$$

a.e. in $\square$. But $\langle \rho_{kq}, e_0 \rangle \neq 0$ in $\square$. Hence $\langle f, \rho_{kq} \rangle = 0$ a.e., and then $f = 0$.

$\square$

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Because of its possible relevance for *PT*- (or *Pseudo, Cripto, Quasi-Hermitian*) Quantum Mechanics, where Hamiltonians need not being self-adjoint, but have real eigenvalues:

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In some situations, $H \neq H^\dagger$ is similar to some $H_0 = H_0^\dagger$. Formally we have:

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3. introduce a generalized kq-representation for $\hat{q}$ and $\hat{p}$;
4. use the generalized kq-representation to prove completeness of (a discrete subset of) bi-coherent states.

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(ii) T, T^{-1}, $T^\dagger$ and $(T^{-1})^\dagger = (T^\dagger)^{-1}$ all map $\mathcal{S}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$.

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Remark 1:– This in particular means that, if $\{\varphi_n(x)\}$ is a sequence in $\mathcal{S}(\mathbb{R})$ $\tau_{\mathcal{S}}$-converging to $\varphi(x)$, then both $\{T^{-1}\varphi_n(x)\}$ and $\{T^{\dagger}\varphi_n(x)\}$ converge in the same topology.

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Remark 2:– $\mathcal{S}(\mathbb{R})$-stable operators will be used to deform $\hat{q}_0$ and $\hat{p}_0$, as shown after.

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Example 1: Let $u, v \in \mathcal{S}(\mathbb{R})$ be such that $\langle u, v \rangle = 1$. We define

$$P_{u,v} f := \langle u, f \rangle \, v.$$

Consider $\alpha, \beta \in \mathbb{C}$ satisfying $\alpha + \beta + \alpha\beta = 0$. Then the operator

$$T = \mathbb{1} + \alpha P_{u,v}$$

is invertible, with $T^{-1} = \mathbb{1} + \beta P_{u,v}$. Unless $u = v$ and $\alpha \in \mathbb{R}$, T is neither Hermitian, nor unitary. We have $T^\dagger = \mathbb{1} + \overline{\alpha} P_{v,u} \neq T^{-1}$. Then $(T^{-1})^\dagger = \mathbb{1} + \overline{\beta} P_{v,u} = (T^\dagger)^{-1}$.

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Moreover, if $\{\varphi_n \in \mathcal{S}(\mathbb{R})\}$ $\tau_{\mathcal{S}}$-converges to $\varphi \in \mathcal{S}(\mathbb{R})$ then, for each $F \in \mathcal{S}'(\mathbb{R})$,

$$\left\langle F, T^\dagger \varphi_n \right\rangle = \langle F, \varphi_n + \overline{\alpha}\langle v, \varphi_n\rangle u\rangle = \langle F, \varphi_n\rangle + \overline{\alpha}\langle v, \varphi_n\rangle\langle F, u\rangle \longrightarrow$$

$$\langle F, \varphi\rangle + \overline{\alpha}\langle v, \varphi\rangle\langle F, u\rangle = \langle F, \varphi + \overline{\alpha}\langle v, \varphi\rangle u\rangle = \left\langle F, T^\dagger \varphi \right\rangle.$$

Similarly, $\left\langle F, T^{-1}\varphi_n\right\rangle \to \left\langle F, T^{-1}\varphi\right\rangle$, and therefore both $T^\dagger$ and T^{-1} map $\mathcal{S}(\mathbb{R})$ into itself with continuity: $\Rightarrow T$ is fully $\mathcal{S}(\mathbb{R})$-stable.

Loosing self-adjointness

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Example 2: For convenience we introduce what we call T^{-1}:

$$T^{-1} = \mathbb{1} - i(\hat{p}_0)^2,$$

whose domain contains $\mathcal{S}(\mathbb{R})$. Its inverse can be obtained computing first the Green function for T^{-1}, $(T^{-1}G)(x) = \delta(x)$. We get

$$T(\varphi(x)) = \frac{i}{\sqrt{2}(1+i)} \int_{\mathbb{R}} \varphi(x-s) e^{-|s|\frac{\sqrt{2}}{2}(1+i)} ds,$$

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Remark:– Then we have bounded and unbounded examples of fully $\mathcal{S}(\mathbb{R})$-stable operators....for what?

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let T be a $\mathcal{S}(\mathbb{R})$-stable operator, and let us consider the operators

$$\hat{q}\,\varphi = T\hat{q}_0 T^{-1}\varphi, \qquad \hat{p}\,\varphi = T\hat{p}_0 T^{-1}\varphi,$$

for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$. Of course, $\hat{q}$ and $\hat{p}$ map $\mathcal{S}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$, so that they are, in particular, densely defined.

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It is possible to check that their adjoints satisfy the following:

$$\hat{q}^\dagger\varphi = (T^{-1})^\dagger\hat{q}_0T^\dagger\varphi, \qquad \hat{p}^\dagger\varphi = (T^{-1})^\dagger\hat{p}_0T^\dagger\varphi,$$

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It is clear that, in general, $\hat{q} \neq \hat{q}^\dagger$ and $\hat{p} \neq \hat{p}^\dagger$. It is also clear that

$$[\hat{q}, \hat{p}]\varphi(x) = i\varphi(x),$$

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for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$.

That's why we call them *non self-adjoint momentum and position operators*.

Loosing self-adjointness

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Given a tempered distribution $\eta_{x_0}(x) \in \mathcal{S}'(\mathbb{R})$, and its set $\mathcal{F}_\eta = \{\eta_{x_0}(x),\, x_0 \in \mathbb{R}\}$, then

Definition:– $\mathcal{F}_\eta$ is called *well-behaved* if:

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for all $x_0 \in \mathbb{R}$;

2. a second family of generalized vectors $\mathcal{F}^\eta = \{\eta^{x_0}(x) \in \mathcal{S}'(\mathbb{R}),\, x_0 \in \mathbb{R}\}$ exists such that

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Loosing self-adjointness

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Remarks:– (1) These properties extend those for $\xi_{x_0}(x)$;

(2) A similar definition can be introduced for $\hat{p}$, and its generalized eigenstates.

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let $\{\varphi_n(x) \in \mathcal{S}(\mathbb{R})\} \to \varphi(x) \in \mathcal{S}(\mathbb{R})$ in $\tau_{\mathcal{S}}$. Then, for instance,

$$\langle TF, \varphi_n \rangle = \left\langle F, T^\dagger \varphi_n \right\rangle \to \left\langle F, T^\dagger \varphi \right\rangle = \langle TF, \varphi \rangle,$$

since, if $\varphi_n(x)$ τ_S-converges, then $(T^\dagger\varphi_n)(x)$ converges as well, in the same topology. Hence TF is continuous.

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are tempered distributions. Moreover, $\eta_{x_0}(x) \in D(\hat{q})$ and $\eta^{x_0}(x) \in D(\hat{q}^\dagger)$, and we have:

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Of course, if $b = a^\dagger$ we recover ordinary bosons.

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Now, if (a, b) satisfy Definition 1, then $\varphi_0 \in D^\infty(b)$ and $\Psi_0 \in D^\infty(a^\dagger)$. Hence...

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n \varphi_0, \qquad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n \Psi_0, \tag{2}$$

$n \geq 0$, can be defined and they all belong to $\mathcal{D}$. We introduce $\mathcal{F}_\Psi = \{\Psi_n,\, n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\, n \geq 0\}$. Once again, since $\mathcal{D}$ is stable under the action of $a^\sharp$ and $b^\sharp$, we deduce that both φ_n and Ψ_n belong to the domains of $a^\sharp$, $b^\sharp$ and $N^\sharp$ (here $N = ba$).

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The following lowering and raising relations hold:

$$\left\{\begin{array}{ll} b\,\varphi_n = \sqrt{n+1}\varphi_{n+1}, & n \geq 0,\\ a\,\varphi_0 = 0, \quad a\varphi_n = \sqrt{n}\,\varphi_{n-1}, & n \geq 1,\\ a^\dagger \Psi_n = \sqrt{n+1}\Psi_{n+1}, & n \geq 0,\\ b^\dagger \Psi_0 = 0, \quad b^\dagger \Psi_n = \sqrt{n}\,\Psi_{n-1}, & n \geq 1, \end{array}\right. \tag{3}$$

as well as the following eigenvalue equations:

$$N\varphi_n = n\varphi_n, \quad N^\dagger \Psi_n = n\Psi_n, \quad n \geq 0.$$

A consequence: if $\langle \varphi_0, \Psi_0 \rangle = 1$, then

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But sometimes... these are formal equalities and definitions. In fact:

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Assumption $\mathcal{D}$-pbs 3.– $\mathcal{F}_\varphi$ is a Riesz basis for $\mathcal{H}$.

Then a pair $(S,\mathcal{F}_e=\{e_n,\, n\geq 0\})$ exists, with $S,S^{-1}\in B(\mathcal{H})$, such that $\varphi_n=Se_n$. $\mathcal{F}_\Psi$ is also a Riesz basis for $\mathcal{H}$, and $\Psi_n=(S^{-1})^\dagger e_n$.

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$$\varphi(z) = U(z)\varphi_0, \qquad \Psi(z) = V(z)\,\Psi_0,$$

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Definition:

A pair of vectors $(\eta(z), \xi(z))$, $z \in \mathbb{C}$, are called Riesz bi-coherent states (RBCSs) if there exist a standard coherent state $\Phi(z)$, $z \in \mathbb{C}$, and a bounded operator T with bounded inverse T^{-1} such that

$$\eta(z) = T\Phi(z), \qquad \xi(z) = (T^{-1})^{\dagger}\Phi(z).$$

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(3) If a subset $\mathcal{D} \subset \mathcal{H}$ exists, dense in $\mathcal{H}$ and invariant under the action of $T^\sharp$, $(T^{-1})^\sharp$ and $c^\sharp$, and if the standard coherent state $\Phi(z)$ belongs to $\mathcal{D}$, then two operators a and b exist, leaving $\mathcal{D}$ stable, satisfying $[a, b] = 1\!\!1$, such that

$$a\, \eta(z) = z \eta(z), \qquad b^\dagger \xi(z) = z \xi(z)$$

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Bi-coherent states, with nice properties, can also be introduced also if $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are not Riesz bases, at least under very mild assumptions on the growth of $\|\varphi_n\|$ and $\|\Psi_n\|$.

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Let $\mathcal{F}_\varphi=\{\varphi_n,\, n\geq 0\}$ and $\mathcal{F}_\Psi=\{\Psi_n,\, n\geq 0\}$ be biorthogonal $\mathcal{D}$-quasi bases for some dense subspace $\mathcal{D}$ of $\mathcal{H}$. Let $\{\alpha_n\}$ be a sequence satisfying the inequalities

$$0=\alpha_0<\alpha_1<\alpha_2<\ldots,$$

and $\overline{\alpha}=\lim_{n,\infty}\alpha_n$, with $\overline{\alpha}\leq\infty$. We further consider two operators, a and $b^\dagger$, which act as lowering operators respectively on $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ in the following way:

$$a\,\varphi_n=\alpha_n\varphi_{n-1},\qquad b^\dagger\,\Psi_n=\alpha_n\Psi_{n-1},$$

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$$\lim_{n\to\infty} \frac{M_n(\varphi)}{M_{n+1}(\varphi)} = M(\varphi), \qquad \lim_{n\to\infty} \frac{M_n(\Psi)}{M_{n+1}(\Psi)} = M(\Psi),$$

where $M(\varphi)$ and $M(\Psi)$ could be infinity, such that, for all $n \geq 0$,

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Then the following series:

$$N(|z|)=\left(\sum_{k=0}^\infty\frac{|z|^{2k}}{(\alpha_k!)^2}\right)^{-1/2},$$

$$\varphi(z)=N(|z|)\sum_{k=0}^\infty\frac{z^k}{\alpha_k!}\varphi_k, \qquad \Psi(z)=N(|z|)\sum_{k=0}^\infty\frac{z^k}{\alpha_k!}\Psi_k,$$

are all convergent inside the circle $C_\rho(0)$ centered in the origin of the complex plane and of radius $\rho=\overline{\alpha}\min\left(1,\frac{M(\varphi)}{r_\varphi},\frac{M(\Psi)}{r_\Psi}\right)$.

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for all $k \geq 0$. Then, for all $f, g \in \mathcal{D}$, calling $d\nu(z, \overline{z}) = d\lambda(r) d\theta$, we have

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Remark:– We can apply the above result also to *deformed quons*, i.e. to operators a and b satisfying, in particular, the following q-mutation rule:

$$[a, b]_q f = abf - qbaf = f,$$

for $f \in \mathcal{D}$ and $q \in [-1, 1]$.

Extended kq-representation

Starting point:

$$\hat{q} = \frac{1}{\sqrt{2}}(a+b) = T\hat{q}_0 T^{-1}, \qquad \hat{p} = \frac{1}{\sqrt{2}\,i}(a-b) = T\hat{p}_0 T^{-1},$$

where T is fully $\mathcal{S}(\mathbb{R})$-stable.

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Remarks:– (1) The dependence of $\varphi_{\underline{n}}(x)$ on z is all through $\underline{n}$.

(2) A similar construction can be repeated for the *companion coherent state*, $\Psi_{\underline{n}}(x)$.

(3) Since $\varphi_0(x) = Te_0(x)$, $e_0(x) \in \mathcal{S}(\mathbb{R})$ and $ce_0 = 0$ ($[c, c^\dagger] = 1\!\!1$), $\varphi_{\underline{n}}(x) \in \mathcal{S}(\mathbb{R})$ for all $\underline{n}$. Hence, using formula (5), we conclude that...

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for all $n_1, n_2 \in \mathbb{Z}$. Hence, if $a^2 = 2\pi$,

$$\left\langle f, \varphi_{kq} \right\rangle \left\langle \Psi_{kq}, \varphi_0 \right\rangle = 0 \quad \Rightarrow \quad \left\langle f, \varphi_{kq} \right\rangle = 0,$$

a.e. in $\square$. Hence

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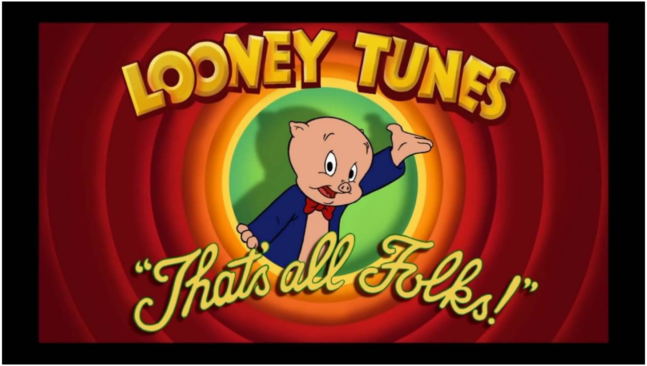
LOONEY TUNES
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