

Time-dependent non-Hermitian systems a status update

Andreas Fring

Non-Hermitian Hamiltonians in Physics (PHHQ18)
International Centre for Theoretical Sciences, Bangalore, India
June 04 - 13, 2018

Time-dependent non-Hermitian systems a status update

Andreas Fring

Non-Hermitian Hamiltonians in Physics (PHHQ18)
International Centre for Theoretical Sciences, Bangalore, India
June 04 - 13, 2018

based on:

- A. Fring, T. Frith, Physical Review A 95, 010102(R) (2017)
- A. Fring, T. Frith, Physics Letters A 381 (2017) 2318-2323
- A. Fring, T. Frith, European Physics Journal Plus (2018) 133:57(9)
- A. Fring, T. Frith, J. of Phys A: Math. and Theor. (2018)

Outline

Outline

- Theoretical framework (key equations)

Outline

Outline

- Theoretical framework (key equations)
- The role of $H(t)$
 - governs unitary time-evolution
 - not observable and not the energy operator

Outline

Outline

- Theoretical framework (key equations)
- The role of $H(t)$
 - governs unitary time-evolution
 - not observable and not the energy operator
- Solutions procedures for Dyson operator

Outline

Outline

- Theoretical framework (key equations)
- The role of $H(t)$
 - governs unitary time-evolution
 - not observable and not the energy operator
- Solutions procedures for Dyson operator
- Special features and concrete models

Outline

Outline

- Theoretical framework (key equations)
- The role of $H(t)$
 - governs unitary time-evolution
 - not observable and not the energy operator
- Solutions procedures for Dyson operator
- Special features and concrete models
- Conclusions

Theoretical framework (key equations):

Time-dependent Schrödinger eqn for $h(t) = h^\dagger(t)$, $H(t) \neq H^\dagger(t)$

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \quad \text{and} \quad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

Theoretical framework (key equations):

Time-dependent Schrödinger eqn for $h(t) = h^\dagger(t)$, $H(t) \neq H^\dagger(t)$

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \quad \text{and} \quad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

Time-dependent Dyson operator

$$\phi(t) = \eta(t)\Psi(t)$$

⇒ Time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

Theoretical framework (key equations):

Time-dependent Schrödinger eqn for $h(t) = h^\dagger(t)$, $H(t) \neq H^\dagger(t)$

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \quad \text{and} \quad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

Time-dependent Dyson operator

$$\phi(t) = \eta(t)\Psi(t)$$

⇒ Time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

⇒ Time-dependent quasi-Hermiticity relation

$$H^\dagger\rho(t) - \rho(t)H = i\hbar\partial_t\rho(t)$$

[from conjugating Dyson relation and $\rho(t) := \eta^\dagger(t)\eta(t)$]

$H(t)$ governs unitary time-evolution:

Hermitian:

$$\phi(t) = u(t, t')\phi(t'), \quad u(t, t') = T \exp \left[-i \int_{t'}^t ds h(s) \right]$$

with

$$h(t)u(t, t') = i\hbar\partial_t u(t, t'), \quad u(t, t')u(t', t'') = u(t, t''), \quad u(t, t) = \mathbb{I}$$

$$\left\langle u(t, t')\phi(t') \left| u(t, t')\tilde{\phi}(t') \right\rangle = \left\langle \phi(t) \left| \tilde{\phi}(t) \right\rangle\right.$$

$H(t)$ governs unitary time-evolution:

Hermitian:

$$\phi(t) = u(t, t')\phi(t'), \quad u(t, t') = T \exp \left[-i \int_{t'}^t ds h(s) \right]$$

with

$$h(t)u(t, t') = i\hbar\partial_t u(t, t'), \quad u(t, t')u(t', t'') = u(t, t''), \quad u(t, t) = \mathbb{I}$$
$$\left\langle u(t, t')\phi(t') \left| u(t, t')\tilde{\phi}(t') \right. \right\rangle = \left\langle \phi(t) \left| \tilde{\phi}(t) \right. \right\rangle$$

Non-Hermitian:

$$\Psi(t) = U(t, t')\Psi(t'), \quad U(t, t') = T \exp \left[-i \int_{t'}^t ds H(s) \right]$$

$$H(t)U(t, t') = i\hbar\partial_t U(t, t'), \quad U(t, t')U(t', t'') = U(t, t''), \quad U(t, t) = \mathbb{I}$$
$$\left\langle U(t, t')\Psi(t') \left| U(t, t')\tilde{\Psi}(t') \right. \right\rangle_{\rho} = \left\langle \Psi(t) \left| \tilde{\Psi}(t) \right. \right\rangle_{\rho}$$

Relation between $u(t, t')$ and $U(t, t')$:

$$U(t, t') = \eta^{-1}(t)u(t, t')\eta(t')$$

or the generalized Duhamel's formula

$$\begin{aligned} U(t, t') &= u(t, t') - \int_{t'}^t \frac{d}{ds} [U(t, s)u(s, t')] ds \\ &= u(t, t') - i\hbar \int_{t'}^t U(t, s) [H(s) - h(s)] u(s, t') ds \end{aligned}$$

Relation between $u(t, t')$ and $U(t, t')$:

$$U(t, t') = \eta^{-1}(t)u(t, t')\eta(t')$$

or the generalized Duhamel's formula

$$\begin{aligned} U(t, t') &= u(t, t') - \int_{t'}^t \frac{d}{ds} [U(t, s)u(s, t')] ds \\ &= u(t, t') - i\hbar \int_{t'}^t U(t, s) [H(s) - h(s)] u(s, t') ds \end{aligned}$$

Relation between Green's functions:

$$G_h(t, t') := -iu(t, t')\theta(t - t') \quad G_H(t, t') := -iU(t, t')\theta(t - t')$$

$$G_U(t, t') = G_u(t, t') + i \int_{-\infty}^{\infty} G_U(t, s) [H(s) - h(s)] G_u(s, t') ds$$

$H(t)$ is nonobservable and not the energy operator

Observables $o(t)$ in the Hermitian system are self-adjoint.

Observables $\mathcal{O}(t)$ in the non-Hermitian $\mathcal{O}(t)$ are quasi Hermitian

$$o(t) = \eta(t)\mathcal{O}(t)\eta^{-1}(t).$$

$H(t)$ is nonobservable and not the energy operator

Observables $o(t)$ in the Hermitian system are self-adjoint.

Observables $\mathcal{O}(t)$ in the non-Hermitian $\mathcal{O}(t)$ are quasi Hermitian

$$o(t) = \eta(t)\mathcal{O}(t)\eta^{-1}(t).$$

Then we have

$$\langle \phi(t) | o(t) \phi(t) \rangle = \langle \Psi(t) | \rho(t) \mathcal{O}(t) \Psi(t) \rangle .$$

$H(t)$ is nonobservable and not the energy operator

Observables $o(t)$ in the Hermitian system are self-adjoint.

Observables $\mathcal{O}(t)$ in the non-Hermitian $\mathcal{O}(t)$ are quasi Hermitian

$$o(t) = \eta(t)\mathcal{O}(t)\eta^{-1}(t).$$

Then we have

$$\langle \phi(t) | o(t) \phi(t) \rangle = \langle \Psi(t) | \rho(t) \mathcal{O}(t) \Psi(t) \rangle.$$

Since $H(t)$ is not quasi/pseudo Hermitian it is not an observable.

The observable energy operator is

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t).$$

Three scenarios:

1. $\partial_t \eta = 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

Technically reduces to time-independent case.

[C. Figueira de Morisson Faria, A. Fring; J. of Phys. A 39 (2006) 9269]

Three scenarios:

1. $\partial_t \eta = 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

Technically reduces to time-independent case.

[C. Figueira de Morisson Faria, A. Fring; J. of Phys. A 39 (2006) 9269]

2. $\partial_t \eta \neq 0$, $\partial_t H = 0$, $\partial_t h \neq 0$

Alternative representation:

- Heisenberg picture: time-dependent observables
- Schrödinger picture: time-dependent states
- Metric picture: time-dependent metric operators

Three scenarios:

1. $\partial_t \eta = 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

Technically reduces to time-independent case.

[C. Figueira de Morisson Faria, A. Fring; J. of Phys. A 39 (2006) 9269]

2. $\partial_t \eta \neq 0$, $\partial_t H = 0$, $\partial_t h \neq 0$

Alternative representation:

- Heisenberg picture: time-dependent observables
- Schrödinger picture: time-dependent states
- Metric picture: time-dependent metric operators

3. $\partial_t \eta \neq 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

- Solve full quasi-Hermiticity relation for $\rho(t)$
 $\Rightarrow \eta(t)$ from $\rho(t) := \eta^\dagger(t)\eta(t)$
- Solve full time-dependent Dyson equation $\eta(t)$
 $\Rightarrow \rho(t)$ from $\rho(t) := \eta^\dagger(t)\eta(t)$

Making sense of the broken \mathcal{PT} -regime:

Two-level system

$$H = -\frac{1}{2} [\omega \mathbb{I} + \lambda \sigma_z + i\kappa \sigma_x]$$

with eigensystem

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \quad \varphi_{\pm} = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

Making sense of the broken \mathcal{PT} -regime:

Two-level system

$$H = -\frac{1}{2} [\omega \mathbb{I} + \lambda \sigma_z + i\kappa \sigma_x]$$

with eigensystem

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \quad \varphi_{\pm} = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

Making sense of the broken \mathcal{PT} -regime:

Two-level system

$$H = -\frac{1}{2} [\omega \mathbb{I} + \lambda \sigma_z + i\kappa \sigma_x]$$

with eigensystem

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \quad \varphi_{\pm} = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

with \mathcal{PT} -symmetry $\mathcal{PT} = \tau \sigma_z$; $\tau : i \rightarrow -i$

$$[\mathcal{PT}, H] = 0, \quad \text{and} \quad \mathcal{PT}\varphi_{\pm} = e^{i\phi}\varphi_{\pm} \quad \text{for} \quad |\lambda| > |\kappa|$$

Making sense of the broken \mathcal{PT} -regime:

Two-level system

$$H = -\frac{1}{2} [\omega \mathbb{I} + \lambda \sigma_z + i\kappa \sigma_x]$$

with eigensystem

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \quad \varphi_{\pm} = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

with \mathcal{PT} -symmetry $\mathcal{PT} = \tau \sigma_z$; $\tau : i \rightarrow -i$

$$[\mathcal{PT}, H] = 0, \quad \text{and} \quad \mathcal{PT}\varphi_{\pm} = e^{i\phi}\varphi_{\pm} \quad \text{for} \quad |\lambda| > |\kappa|$$

with broken \mathcal{PT} -symmetry $\mathcal{PT} = \tau \sigma_z$; $\tau : i \rightarrow -i$

$$[\mathcal{PT}, H] = 0, \quad \mathcal{PT}\varphi_{\pm} \neq e^{i\phi}\varphi_{\pm} \quad |\lambda| < |\kappa|$$



Making sense of the broken \mathcal{PT} -regime:

Two-level system

$$H = -\frac{1}{2} [\omega \mathbb{I} + \lambda \sigma_z + i\kappa \sigma_x]$$

with eigensystem

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \quad \varphi_{\pm} = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

with \mathcal{PT} -symmetry $\mathcal{PT} = \tau \sigma_z$; $\tau : i \rightarrow -i$

$$[\mathcal{PT}, H] = 0, \quad \text{and} \quad \mathcal{PT}\varphi_{\pm} = e^{i\phi}\varphi_{\pm} \quad \text{for} \quad |\lambda| > |\kappa|$$

with broken \mathcal{PT} -symmetry $\mathcal{PT} = \tau \sigma_z$; $\tau : i \rightarrow -i$

$$[\mathcal{PT}, H] = 0, \quad \mathcal{PT}\varphi_{\pm} \neq e^{i\phi}\varphi_{\pm} \quad |\lambda| < |\kappa|$$



Claim: This system has real energies for $|\lambda(t)| < |\kappa(t)|!$

Two-dimensional system with infinite dimensional Hilbert space

$$H_K = aK_1 + bK_2 + i\lambda K_3, \quad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

$$K_1 = \frac{1}{2}(p_x^2 + x^2), \quad K_2 = \frac{1}{2}(p_y^2 + y^2), \quad K_3 = \frac{1}{2}(xy + p_x p_y)$$

$$K_4 = \frac{1}{2}(xp_y - yp_x)$$

$$\begin{aligned} [K_1, K_2] &= 0, & [K_1, K_3] &= iK_4, & [K_1, K_4] &= -iK_3, \\ [K_2, K_3] &= -iK_4, & [K_2, K_4] &= iK_3, & [K_3, K_4] &= i(K_1 - K_2)/2 \end{aligned}$$

Two-dimensional system with infinite dimensional Hilbert space

$$H_K = aK_1 + bK_2 + i\lambda K_3, \quad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

$$K_1 = \frac{1}{2}(p_x^2 + x^2), \quad K_2 = \frac{1}{2}(p_y^2 + y^2), \quad K_3 = \frac{1}{2}(xy + p_x p_y)$$

$$K_4 = \frac{1}{2}(xp_y - yp_x)$$

$$\begin{aligned} [K_1, K_2] &= 0, & [K_1, K_3] &= iK_4, & [K_1, K_4] &= -iK_3, \\ [K_2, K_3] &= -iK_4, & [K_2, K_4] &= iK_3, & [K_3, K_4] &= i(K_1 - K_2)/2 \end{aligned}$$

- H_K is \mathcal{PT} -symmetric: $[\mathcal{PT}_\pm, H_{xy}] = 0$

$$\mathcal{PT}_\pm : x \rightarrow \pm x, \quad y \rightarrow \mp y, \quad p_x \rightarrow \mp p_x, \quad p_y \rightarrow \pm p_y, \quad i \rightarrow -i$$

Two-dimensional system with infinite dimensional Hilbert space

$$H_K = aK_1 + bK_2 + i\lambda K_3, \quad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

$$K_1 = \frac{1}{2}(p_x^2 + x^2), \quad K_2 = \frac{1}{2}(p_y^2 + y^2), \quad K_3 = \frac{1}{2}(xy + p_x p_y)$$

$$K_4 = \frac{1}{2}(xp_y - yp_x)$$

$$\begin{aligned} [K_1, K_2] &= 0, & [K_1, K_3] &= iK_4, & [K_1, K_4] &= -iK_3, \\ [K_2, K_3] &= -iK_4, & [K_2, K_4] &= iK_3, & [K_3, K_4] &= i(K_1 - K_2)/2 \end{aligned}$$

- H_K is \mathcal{PT} -symmetric: $[\mathcal{PT}_\pm, H_{xy}] = 0$

$$\mathcal{PT}_\pm : x \rightarrow \pm x, y \rightarrow \mp y, p_x \rightarrow \mp p_x, p_y \rightarrow \pm p_y, i \rightarrow -i$$

- H_K is quasi-Hermitian: $h_K = \eta H_K \eta^{-1}$

$$h_K = \frac{1}{2}(a+b)(K_1 + K_2) + \frac{1}{2}\sqrt{(a-b)^2 - \lambda^2}(K_1 - K_2)$$

$$\text{with } \eta = e^{2\theta K_4}, \quad \theta = \operatorname{arctanh}[\lambda/(b-a)]$$

Spontaneously broken \mathcal{PT} -symmetry for $a = b$:

Eigenenergies:

$$E_{n,m} = E_{m,n}^* = a(1 + n + m) + i\frac{\lambda}{2}(n - m)$$

Eigenfunctions:

$$\varphi_{n,m}(x, y) = \frac{e^{-\frac{x^2}{2} - \frac{y^2}{2}}}{2^{n+m} \sqrt{n! m! \pi}} \left[\sum_{k=0}^n \binom{n}{k} H_k(x) H_{n-k}(y) \right] \\ \times \left[\sum_{l=0}^m (-1)^l \binom{m}{l} H_l(y) H_{m-l}(x) \right]$$

Spontaneously broken \mathcal{PT} -symmetry for $a = b$:

Eigenenergies:

$$E_{n,m} = E_{m,n}^* = a(1 + n + m) + i\frac{\lambda}{2}(n - m)$$

Eigenfunctions:

$$\varphi_{n,m}(x, y) = \frac{e^{-\frac{x^2}{2} - \frac{y^2}{2}}}{2^{n+m} \sqrt{n! m! \pi}} \left[\sum_{k=0}^n \binom{n}{k} H_k(x) H_{n-k}(y) \right] \\ \times \left[\sum_{l=0}^m (-1)^l \binom{m}{l} H_l(y) H_{m-l}(x) \right]$$

Claim: This system has real energies for $a(t), \lambda(t)$!

Time-dependent system:

$$H(t) = \frac{a(t)}{2} (p_x^2 + p_y^2 + x^2 + y^2) + i \frac{\lambda(t)}{2} (xy + p_x p_y), \quad a(t), \lambda(t) \in \mathbb{R}$$

Ansatz:

$$\eta(t) = \prod_{i=1}^4 e^{\gamma_i(t) K_i}, \quad \gamma_i \in \mathbb{R}$$

Time-dependent system:

$$H(t) = \frac{a(t)}{2} (p_x^2 + p_y^2 + x^2 + y^2) + i \frac{\lambda(t)}{2} (xy + p_x p_y), \quad a(t), \lambda(t) \in \mathbb{R}$$

Ansatz:

$$\eta(t) = \prod_{i=1}^4 e^{\gamma_i(t) K_i}, \quad \gamma_i \in \mathbb{R}$$

Time-dependent Dyson equations is satisfied when

Constraint:

$$\dot{\gamma}_1 = \gamma_2 = q_1, \quad \dot{\gamma}_3 = -\lambda \cosh \gamma_4, \quad \dot{\gamma}_4 = \lambda \tanh \gamma_3 \sinh \gamma_4,$$

$$h(t) = a(t) (K_1 + K_2) + \frac{\lambda(t)}{2} \frac{\sinh \gamma_4}{\cosh \gamma_3} (K_1 - K_2)$$

Time-dependent system:

$$H(t) = \frac{a(t)}{2} (p_x^2 + p_y^2 + x^2 + y^2) + i \frac{\lambda(t)}{2} (xy + p_x p_y), \quad a(t), \lambda(t) \in \mathbb{R}$$

Ansatz:

$$\eta(t) = \prod_{i=1}^4 e^{\gamma_i(t) K_i}, \quad \gamma_i \in \mathbb{R}$$

Time-dependent Dyson equations is satisfied when

Constraint:

$$\dot{\gamma}_1 = \dot{\gamma}_2 = q_1, \quad \dot{\gamma}_3 = -\lambda \cosh \gamma_4, \quad \dot{\gamma}_4 = \lambda \tanh \gamma_3 \sinh \gamma_4,$$

$$h(t) = a(t) (K_1 + K_2) + \frac{\lambda(t)}{2} \frac{\sinh \gamma_4}{\cosh \gamma_3} (K_1 - K_2)$$

Solution: $\gamma_4 = \operatorname{arcsinh}(\kappa \operatorname{sech} \gamma_3)$, $\chi(t) := \cosh \gamma_3$, $\kappa = \text{const}$
with dissipative Ermakov-Pinney equation

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda} \dot{\chi} - \lambda^2 \chi = \frac{\kappa^2 \lambda^2}{\chi^3}$$

Lewis Riesenfeld invariants:

$$\frac{dl_{\mathcal{H}}(t)}{dt} = \partial_t l_{\mathcal{H}}(t) - i\hbar [l_{\mathcal{H}}(t), \mathcal{H}(t)] = 0, \quad \text{for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

Lewis Riesenfeld invariants:

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar [I_{\mathcal{H}}(t), \mathcal{H}(t)] = 0, \quad \text{for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

The invariants I_H is quasi-Hermitian:

$$I_h(t) = \eta(t) I_H(t) \eta^{-1}(t)$$

Lewis Riesenfeld invariants:

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar [I_{\mathcal{H}}(t), \mathcal{H}(t)] = 0, \quad \text{for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

The invariants I_H is quasi-Hermitian:

$$I_h(t) = \eta(t) I_H(t) \eta^{-1}(t)$$

Solution to time-dependent Schrödinger equation:

$$\begin{aligned} I_{\mathcal{H}}(t) |\phi_{\mathcal{H}}(t)\rangle &= \Lambda |\phi_{\mathcal{H}}(t)\rangle, & |\Psi_{\mathcal{H}}(t)\rangle &= e^{i\hbar\alpha(t)} |\phi_{\mathcal{H}}(t)\rangle \\ \dot{\alpha} &= \langle \phi_{\mathcal{H}}(t) | i\hbar\partial_t - \mathcal{H}(t) | \phi_{\mathcal{H}}(t) \rangle, & \dot{\Lambda} &= 0 \end{aligned}$$

Lewis Riesenfeld invariants:

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar [I_{\mathcal{H}}(t), \mathcal{H}(t)] = 0, \quad \text{for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

The invariants I_H is quasi-Hermitian:

$$I_h(t) = \eta(t) I_H(t) \eta^{-1}(t)$$

Solution to time-dependent Schrödinger equation:

$$\begin{aligned} I_{\mathcal{H}}(t) |\phi_{\mathcal{H}}(t)\rangle &= \Lambda |\phi_{\mathcal{H}}(t)\rangle, & |\Psi_{\mathcal{H}}(t)\rangle &= e^{i\hbar\alpha(t)} |\phi_{\mathcal{H}}(t)\rangle \\ \dot{\alpha} &= \langle \phi_{\mathcal{H}}(t) | i\hbar\partial_t - \mathcal{H}(t) | \phi_{\mathcal{H}}(t) \rangle, & \dot{\Lambda} &= 0 \end{aligned}$$

Procedure:

1. Construct $I_h(t)$
2. Construct $I_H(t)$
3. Find $\eta(t)$ from similarity transformation

With Ansätze:

$$l_H(t) = \sum_{i=1}^4 \alpha_i(t) K_i, \quad l_h(t) = \sum_{i=1}^4 \beta_i(t) K_i, \quad h(t) = \sum_{i=1}^4 b_i(t) K_i,$$

where $\alpha_i = \alpha_i^r + i\alpha_i^i \in \mathbb{C}$, $b_i, \beta_i, \alpha_i^r, \alpha_i^i \in \mathbb{R}$.

With Ansätze:

$$l_H(t) = \sum_{i=1}^4 \alpha_i(t) K_i, \quad l_h(t) = \sum_{i=1}^4 \beta_i(t) K_i, \quad h(t) = \sum_{i=1}^4 b_i(t) K_i,$$

where $\alpha_i = \alpha_i^r + i\alpha_i^i \in \mathbb{C}$, $b_i, \beta_i, \alpha_i^r, \alpha_i^i \in \mathbb{R}$. we find

$$\gamma_3 = \arctan \left[\frac{\tanh \left[q_2 - \int_0^t \lambda(s) ds \right]}{\sqrt{1 - q_3^2 \operatorname{sech} \left[q_2 - \int_0^t \lambda(s) ds \right]^2}} \right]$$
$$\gamma_4 = -\operatorname{arccot} \left[\frac{1}{q_3} \cosh \left[q_2 - \int_0^t \lambda(s) ds \right] \right]$$

$$q_2, q_3 = \text{const}$$

With γ_3 we obtain a solution to the Ermakov-Pinney equation

$$\chi(t) = \cosh \gamma_3 = \sqrt{\frac{\cosh^2 \left[q_2 - \int_0^t \lambda(s) ds \right] - q_3^2}{1 - q_3^2}}$$

where $\kappa = q_3 / \sqrt{1 - q_3^2}$, $|q_3| < 1$.

With γ_3 we obtain a solution to the Ermakov-Pinney equation

$$\chi(t) = \cosh \gamma_3 = \sqrt{\frac{\cosh^2 \left[q_2 - \int_0^t \lambda(s) ds \right] - q_3^2}{1 - q_3^2}}$$

where $\kappa = q_3 / \sqrt{1 - q_3^2}$, $|q_3| < 1$.

We did not solve any 2nd order differential equation directly!

With γ_3 we obtain a solution to the Ermakov-Pinney equation

$$\chi(t) = \cosh \gamma_3 = \sqrt{\frac{\cosh^2 \left[q_2 - \int_0^t \lambda(s) ds \right] - q_3^2}{1 - q_3^2}}$$

where $\kappa = q_3 / \sqrt{1 - q_3^2}$, $|q_3| < 1$.

We did not solve any 2nd order differential equation directly!

Explicit form of the Hermitian Hamiltonian:

$$h(t) = f_+(t)K_1 + f_-(t)K_2$$

with

$$f_{\pm}(t) = a(t) \pm \frac{q_3 \sqrt{1 - q_3^2} \lambda(t)}{1 + \cosh \left[2q_2 - 2 \int_0^t \lambda(s) ds \right] - 2q_3^2}$$

Solution for time-dependent Schrödinger equation

Solution for $\tilde{h}(t) = a(t)K_1$, $a(t) \in \mathbb{R}$

in [I. A. Pedrosa, Phys. Rev. A 55(4), 3219 (1997)] :

Solution for time-dependent Schrödinger equation

Solution for $\tilde{h}(t) = a(t)K_1$, $a(t) \in \mathbb{R}$

in [I. A. Pedrosa, Phys. Rev. A 55(4), 3219 (1997)] :

$$\tilde{\varphi}_n(x, t) = \frac{e^{i\alpha_n(t)}}{\sqrt{\varkappa(t)}} \exp \left[\left(\frac{i}{a(t)} \frac{\dot{\varkappa}(t)}{\varkappa(t)} - \frac{1}{\varkappa^2(t)} \right) \frac{x^2}{2} \right] H_n \left[\frac{x}{\varkappa(t)} \right]$$

with phase

$$\alpha_n(t) = - \left(n + \frac{1}{2} \right) \int_0^t \frac{a(s)}{\varkappa^2(s)} ds$$

where

$$\ddot{\varkappa} - \frac{\dot{a}}{a} \dot{\varkappa} + a^2 \varkappa = \frac{a^2}{\varkappa^3}$$

Solution for time-dependent Schrödinger equation

Solution for $\tilde{h}(t) = a(t)K_1$, $a(t) \in \mathbb{R}$

in [I. A. Pedrosa, Phys. Rev. A 55(4), 3219 (1997)] :

$$\tilde{\varphi}_n(x, t) = \frac{e^{i\alpha_n(t)}}{\sqrt{\varkappa(t)}} \exp \left[\left(\frac{i}{a(t)} \frac{\dot{\varkappa}(t)}{\varkappa(t)} - \frac{1}{\varkappa^2(t)} \right) \frac{x^2}{2} \right] H_n \left[\frac{x}{\varkappa(t)} \right]$$

with phase

$$\alpha_n(t) = - \left(n + \frac{1}{2} \right) \int_0^t \frac{a(s)}{\varkappa^2(s)} ds$$

where

$$\ddot{\varkappa} - \frac{\dot{a}}{a} \dot{\varkappa} + a^2 \varkappa = \frac{a^2}{\varkappa^3}$$

The dissipative Ermakov-Pinney equation has re-emerged!

Solution for time-dependent Schrödinger equation

Solution for $\tilde{h}(t) = a(t)K_1$, $a(t) \in \mathbb{R}$

in [I. A. Pedrosa, Phys. Rev. A 55(4), 3219 (1997)] :

$$\tilde{\varphi}_n(x, t) = \frac{e^{i\alpha_n(t)}}{\sqrt{\varkappa(t)}} \exp \left[\left(\frac{i}{a(t)} \frac{\dot{\varkappa}(t)}{\varkappa(t)} - \frac{1}{\varkappa^2(t)} \right) \frac{x^2}{2} \right] H_n \left[\frac{x}{\varkappa(t)} \right]$$

with phase

$$\alpha_n(t) = - \left(n + \frac{1}{2} \right) \int_0^t \frac{a(s)}{\varkappa^2(s)} ds$$

where

$$\ddot{\varkappa} - \frac{\dot{a}}{a} \dot{\varkappa} + a^2 \varkappa = \frac{a^2}{\varkappa^3}$$

The dissipative Ermakov-Pinney equation has re-emerged!

We compute

$$\langle \tilde{\varphi}_n(x, t) | K_1 | \tilde{\varphi}_m(x, t) \rangle = 2^{n-2} n! (2n+1) \sqrt{\pi} \frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} \delta_{n,m}$$

$$\langle \tilde{\varphi}_n(x, t) | \tilde{\varphi}_n(x, t) \rangle = 2^n n! \sqrt{\pi} := N$$

right hand side does not depend on t :

$$\frac{d}{dt} \left[\frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} \right] = \frac{2\dot{\varkappa}}{a^2} \left(\ddot{\varkappa} - \frac{\dot{a}}{a} \dot{\varkappa} + a^2 \varkappa - \frac{a^2}{\varkappa^3} \right) = 0$$

right hand side does not depend on t :

$$\frac{d}{dt} \left[\frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} \right] = \frac{2\dot{\varkappa}}{a^2} \left(\ddot{\varkappa} - \frac{\dot{a}}{a} \dot{\varkappa} + a^2 \varkappa - \frac{a^2}{\varkappa^3} \right) = 0$$

take previous solution

$$\varkappa(t) = \sqrt{\tilde{\kappa} \cos \left[2 \int_0^t a(s) ds \right] + \sqrt{1 + \tilde{\kappa}^2}}$$

$$\frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} = 2\sqrt{1 + \tilde{\kappa}^2}$$

right hand side does not depend on t :

$$\frac{d}{dt} \left[\frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} \right] = \frac{2\dot{\varkappa}}{a^2} \left(\ddot{\varkappa} - \frac{\dot{a}}{a} \dot{\varkappa} + a^2 \varkappa - \frac{a^2}{\varkappa^3} \right) = 0$$

take previous solution

$$\varkappa(t) = \sqrt{\tilde{\kappa} \cos \left[2 \int_0^t a(s) ds \right] + \sqrt{1 + \tilde{\kappa}^2}}$$

$$\frac{a^2(1 + \varkappa^4) + \varkappa^2 \dot{\varkappa}^2}{a^2 \varkappa^2} = 2\sqrt{1 + \tilde{\kappa}^2}$$

For $\hat{\varphi}_n(x, t) = \tilde{\varphi}_m(x, t) / \sqrt{N}$ we compute

$$\langle \hat{\varphi}_n(x, t) | K_1 | \hat{\varphi}_m(x, t) \rangle = \left(n + \frac{1}{2} \right) \sqrt{1 + \tilde{\kappa}^2} \delta_{n,m}$$

Solution for

$$h(t) = f_+(t)K_1 + f_-(t)K_2$$

$$f_{\pm}(t) = a(t) \pm \frac{q_3 \sqrt{1 - q_3^2} \lambda(t)}{1 + \cosh \left[2q_2 - 2 \int_0^t \lambda(s) ds \right] - 2q_3^2}$$

Solution for

$$h(t) = f_+(t)K_1 + f_-(t)K_2$$

$$f_{\pm}(t) = a(t) \pm \frac{q_3 \sqrt{1 - q_3^2} \lambda(t)}{1 + \cosh \left[2q_2 - 2 \int_0^t \lambda(s) ds \right] - 2q_3^2}$$

$$\Psi_h^{n,m}(x, y, t) = \hat{\varphi}_n^+(x, t) \hat{\varphi}_m^-(y, t)$$

with $a \rightarrow f^{\pm}$, $\varkappa \rightarrow \varkappa_{\pm}$, $\tilde{\kappa} \rightarrow \tilde{\kappa}_{\pm}$, $\alpha_n \rightarrow \alpha_n^{\pm}$

Solution for

$$h(t) = f_+(t)K_1 + f_-(t)K_2$$

$$f_{\pm}(t) = a(t) \pm \frac{q_3 \sqrt{1 - q_3^2} \lambda(t)}{1 + \cosh \left[2q_2 - 2 \int_0^t \lambda(s) ds \right] - 2q_3^2}$$

$$\Psi_h^{n,m}(x, y, t) = \hat{\varphi}_n^+(x, t) \hat{\varphi}_m^-(y, t)$$

with $a \rightarrow f^{\pm}$, $\varkappa \rightarrow \varkappa_{\pm}$, $\tilde{\kappa} \rightarrow \tilde{\kappa}_{\pm}$, $\alpha_n \rightarrow \alpha_n^{\pm}$

gives **real instantaneous energy expectation values**

$$\begin{aligned} E^{n,m}(t) &= \langle \Psi_h^{n,m}(t) | h(t) | \Psi_h^{n,m}(t) \rangle = \langle \Psi_H^{n,m}(t) | \rho(t) \tilde{H}(t) | \Psi_H^{n,m}(t) \rangle \\ &= f_+(t) \left(n + \frac{1}{2} \right) \sqrt{1 + \tilde{\kappa}_+^2} + f_-(t) \left(m + \frac{1}{2} \right) \sqrt{1 + \tilde{\kappa}_-^2} \end{aligned}$$

for any given fields $a(t)$, $\lambda(t) \in \mathbb{R}$, constants $\tilde{\kappa}_{\pm} \in \mathbb{R}$, $|q_3| < 1$

Symmetry ensuring reality of $E(t)$

Back to time-dependent two-level system

$$H(t) = -\frac{1}{2} [\omega \mathbb{I} + \alpha \kappa(t) \sigma_z + i \kappa(t) \sigma_x] \quad h(t) = -\frac{1}{2} [\omega \mathbb{I} + \chi(t) \sigma_z]$$

either known $\kappa(t)$ unknown $\chi(t)$ or unknown $\kappa(t)$ known $\chi(t)$.

Symmetry ensuring reality of $E(t)$

Back to time-dependent two-level system

$$H(t) = -\frac{1}{2} [\omega \mathbb{I} + \alpha \kappa(t) \sigma_z + i \kappa(t) \sigma_x] \quad h(t) = -\frac{1}{2} [\omega \mathbb{I} + \chi(t) \sigma_z]$$

either known $\kappa(t)$ unknown $\chi(t)$ or unknown $\kappa(t)$ known $\chi(t)$.

Energy operator

$$\tilde{H}(t) = -\frac{1}{2} \left\{ \omega \mathbb{I} + \frac{\chi}{\delta} \left[i(\alpha\xi - 1) \sigma_x + i \left(\hat{\xi} \sqrt{1 - \alpha^2} \right) \sigma_y + (\xi - \delta) \sigma_z \right] \right\}$$

with instantaneous expectation values

$$\tilde{E}_{\pm}(t) = \langle \psi_{\pm}(t) | \tilde{H}(t) \eta^2 \psi_{\pm}(t) \rangle = -\frac{1}{2} [\omega \pm \chi(t)]$$

Time-dependent $\widetilde{\mathcal{PT}}$ -symmetry

Solve

$$\left[\widetilde{\mathcal{PT}}, \tilde{H}\right] = 0, \quad \widetilde{\mathcal{PT}}\tilde{\varphi}_{\pm} = e^{i\tilde{\omega}_{\pm}}\tilde{\varphi}_{\pm}, \quad \widetilde{\mathcal{PT}}^2 = \mathbb{I}.$$

Time-dependent $\widetilde{\mathcal{PT}}$ -symmetry

Solve

$$\left[\widetilde{\mathcal{PT}}, \tilde{H}\right] = 0, \quad \widetilde{\mathcal{PT}}\tilde{\varphi}_{\pm} = e^{i\tilde{\omega}_{\pm}}\tilde{\varphi}_{\pm}, \quad \widetilde{\mathcal{PT}}^2 = \mathbb{I}.$$

We find

$$\widetilde{\mathcal{PT}} = \frac{1}{\sqrt{(\xi - \delta)^2 + (\alpha^2 - 1)\hat{\xi}^2}} \left[i \left(\sqrt{1 - \alpha^2\hat{\xi}} \right) \sigma_y + (\xi - \delta)\sigma_z \right] \tau$$

$$\tilde{\varphi}_{\pm} \sim \begin{pmatrix} (1 \mp 1)\delta - \xi \\ \sqrt{1 - \alpha^2\hat{\xi}} + i(1 - \alpha\xi) \end{pmatrix}$$

$$\tilde{\omega}_+ = \arctan \left[\frac{2\sqrt{1 - \alpha^2}(1 - \alpha\xi)\hat{\xi}}{1 + \xi(\xi - 2\alpha + \xi\alpha^2) + (\alpha^2 - 1)\hat{\xi}^2} \right],$$

$$\tilde{\omega}_- = \arctan \left[\frac{\sqrt{1 - \alpha^2}(1 - \alpha\xi)\hat{\xi}}{2\delta^2 - 3\delta\xi + \xi^2 + (\alpha^2 - 1)\hat{\xi}^2} \right] + \pi$$

Systems solved so far:

- non-Hermitian Swanson model
- one-site lattice Yang-Lee model
- non-Hermitian spin $1/2$, 1 and $3/2$ models
- two dimensional system with infinite Hilbert space
- general Lie algebraic Hamiltonians (to appear)

Conclusions

- Non-Hermitian quantum mechanical systems with time-dependent metric operators are consistent.

Conclusions

- Non-Hermitian quantum mechanical systems with time-dependent metric operators are consistent.
- The time-dependent Dyson equation can be solved exactly.

Conclusions

- Non-Hermitian quantum mechanical systems with time-dependent metric operators are consistent.
- The time-dependent Dyson equation can be solved exactly.
- The dual nature of the Hamiltonian is no longer valid.
 - $H(t)$ governs the time evolution.
 - $H(t)$ is not an observable.

Conclusions

- Non-Hermitian quantum mechanical systems with time-dependent metric operators are consistent.
- The time-dependent Dyson equation can be solved exactly.
- The dual nature of the Hamiltonian is no longer valid.
 - $H(t)$ governs the time evolution.
 - $H(t)$ is not an observable.
- Instantaneous energies can be real even in the broken \mathcal{PT} -regime of the non-Hermitian Hamiltonian.

Conclusions

- Non-Hermitian quantum mechanical systems with time-dependent metric operators are consistent.
- The time-dependent Dyson equation can be solved exactly.
- The dual nature of the Hamiltonian is no longer valid.
 - $H(t)$ governs the time evolution.
 - $H(t)$ is not an observable.
- Instantaneous energies can be real even in the broken \mathcal{PT} -regime of the non-Hermitian Hamiltonian.
- Metric representation allows to solve $h(t)$ -system

Conclusions

- Non-Hermitian quantum mechanical systems with time-dependent metric operators are consistent.
- The time-dependent Dyson equation can be solved exactly.
- The dual nature of the Hamiltonian is no longer valid.
 - $H(t)$ governs the time evolution.
 - $H(t)$ is not an observable.
- Instantaneous energies can be real even in the broken \mathcal{PT} -regime of the non-Hermitian Hamiltonian.
- Metric representation allows to solve $h(t)$ -system
- See Tom's poster for more details

Thank you for your attention
आपका बहुत बहुत धन्यवाद