Networks Games 2:
the price of anarchy, stability, and learning

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Plan for Lectures

Yesterday: Congestion games are potential games

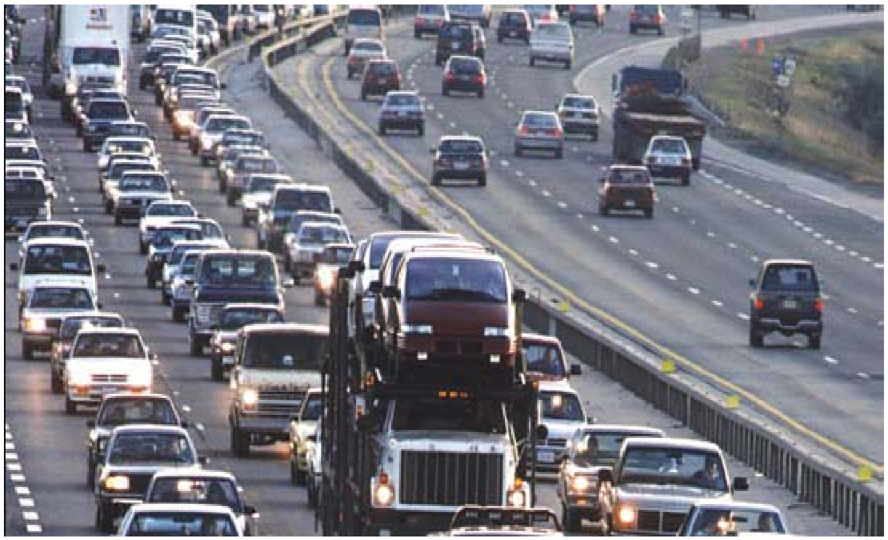
- ∃ Pure equilibria (min of potential)
- Min of potential has OK quality
- ⇒ Price of stability (or anarchy when unique)
- (λ,μ)-smooth and stronger Price of anarchy bounds

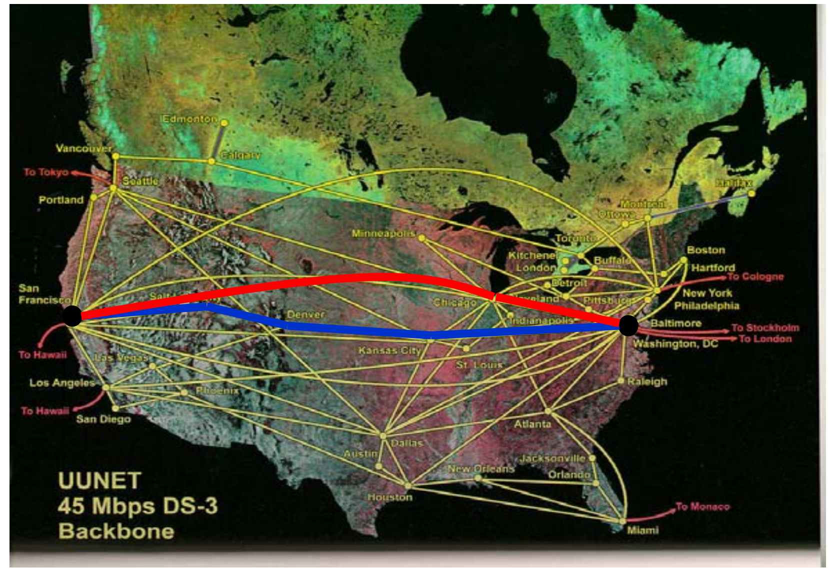
Today and Wednesday

- Learning in games (why and how?)
- solutions reached via learning worst case
- (λ,μ)-smooth bounds
- When better outcome can be expected

Thursday: contagion in financial networks

Example: Routing Game



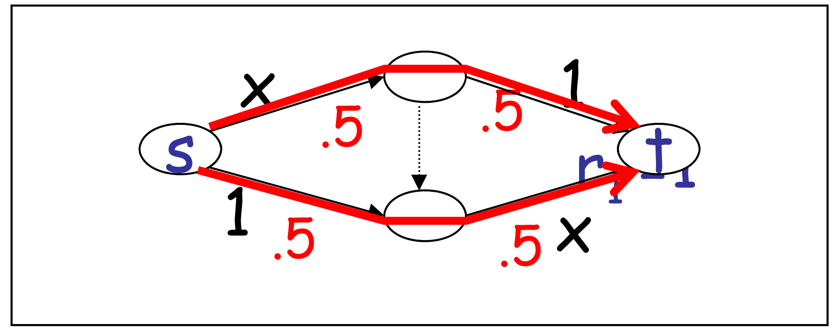


- Traffic subject to congestion delays
- cars and packets follow shortest path

Large number of participants!!

Model of Routing Game

- A directed graph $G = (V,E)$
- source-sink pairs s_i,t_i for $i=1,..,k$
- rate $r_i \geq 0$ of traffic between s_i and t_i for each $i=1,..,k$

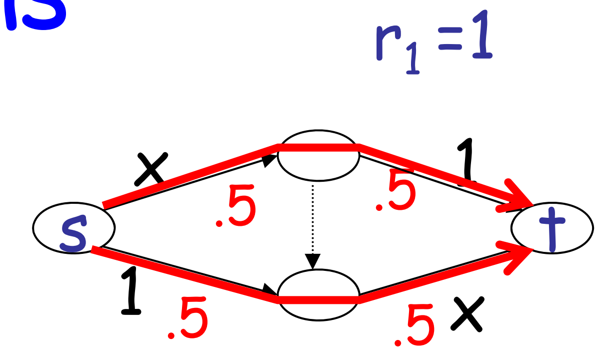


- Load-balancing jobs wanted min load
- Here want minimum delay:

  delay adds along path

  edge-delay is a function $\ell_e(\cdot)$ of the load on the edge e

Delay Functions

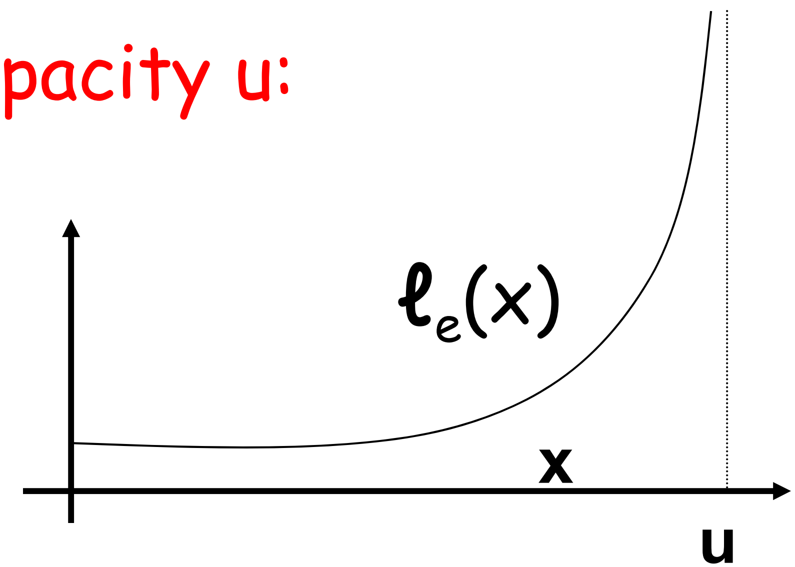


Assume $\ell_e(x)$ continuous and monotone increasing in load x on edge

No capacity of edges for now

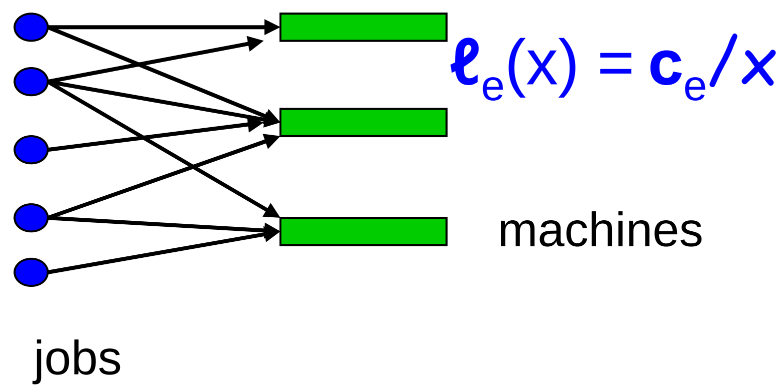
Example to model capacity u:

$\ell_e(x) = a/(u-x)$



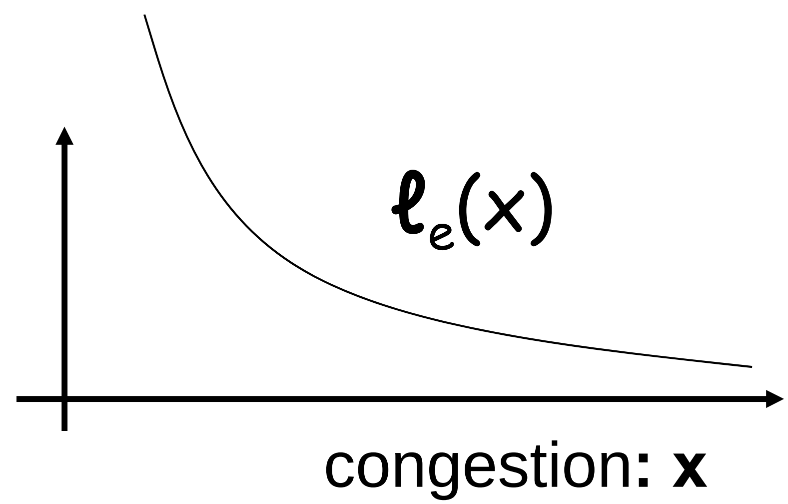
Congestion Games: Cost-sharing

- jobs $i=1,..,k$
- For each machine e a cost function $\ell_e(\cdot)$
  - E.g. cloud computing



- Cost decreasing with congestion (decreasing marginal cost)

$\ell_e(x) = c_e/x$



Goal's of the Game

Personal objective: minimize

$\ell_P(f)$ = sum of delays/cost of edges along **P**
(wrt. flow **f**)

Overall objective:

C(f) = total delay/cost of a flow **f**: $= \Sigma_P f_P \cdot \ell_P(f)$

= - social welfare
or total/average delay

What is Selfish Outcome?

Yesterday we used: Nash equilibrium

- Current strategy "best response" for all players (no incentive to deviate)

Theorem [Nash 1952]:

- Always exists if we allow randomized strategies

$$\text{Price of Anarchy} = \frac{\text{cost of worst (pure) Nash}}{\text{“socially optimum” cost}}$$

Proof technique

- bounds price of anarchy
- Tight bounds in many games

A game is (λ,μ)-smooth if, for every pair f,f* outcomes ($\lambda > 0$; $\mu < 1$):

$$\Sigma_e f^*_e \cdot \ell_e(f_e) \leq \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$$

Cost of f*

Cost of f

Proof technique

- bounds price of anarchy
- Tight bounds in many games

A game is (λ,μ)-smooth if, for every pair f, f^* outcomes ($\lambda > 0$; $\mu < 1$):

$$\Sigma_e f^*_e \cdot \ell_e(f_e) \leq \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$$

or for all $f, f^* \geq 0$

$$f^* \cdot \ell(f) \leq \lambda f^* \cdot \ell(f^*) + \mu f \cdot \ell(f)$$

Discrete version

Smooth for flows:

$$\Sigma_e f^*_e \cdot \ell_e(f_e) \le \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$$

A game is (λ,μ)-smooth if, for every pair s, s^* outcomes

$$\Sigma_i C_i(s^*_i, s_{-i}) \le \lambda \text{ cost}(s^*) + \mu \text{ cost}(s)$$

Where $\text{cost}(s) = \Sigma_i C_i(s)$

s_i strategy of user i

s_{-i} strategies of all users

Discrete version

Smooth for flows:

$\Sigma_e f^*_e \cdot \ell_e(f_e) \leq \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$

A game is (λ,μ)-smooth if, for every pair s, s^* outcomes

$\Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda \operatorname{cost}(s^*) + \mu \operatorname{cost}(s)$

Cost of s*

Cost of s

If s* has a lot smaller cost than s then
single player moves capture the improvement

assuming **$\mu<1$**

Smooth ⇒ Price of Anarchy [Roughgarden]

Use smooth for s = Nash and s* = opt

$\Sigma_i\, C_i\,(s^*_i, \mathbf{s}_{-i}) \le \lambda\ \mathbf{cost}(s^*) + \mu\ \mathbf{cost}(s)$

$$\begin{aligned} \text{cost}(s) &= \Sigma_i\, C_i\,(s_i, \mathbf{s}_{-i}) \\ &\le \Sigma_i\, C_i\,(s^*_i, \mathbf{s}_{-i}) && \text{[s a Nash eq]} \\ &\le \lambda\ \text{cost}(s^*) + \mu\ \text{cost}(s) && \text{[smooth]} \end{aligned}$$

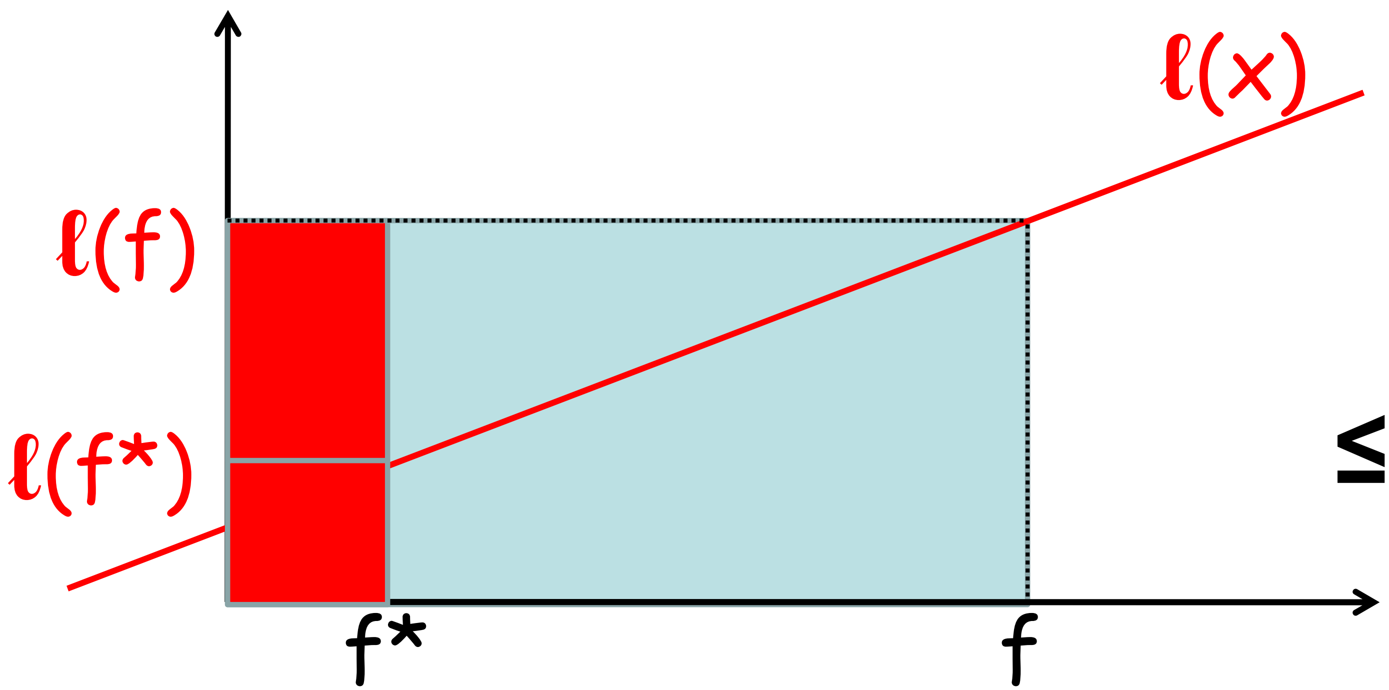
Then: $\text{cost}(s) \le \lambda/(1-\mu)\ \text{cost}(s^*)$

Note: used for s* = opt only!

Linear delay is smooth

Claim: $f^* \cdot \ell(f) \leq f^* \cdot \ell(f^*) + \frac{1}{4} f \cdot \ell(f)$

assuming $\ell(f)$ linear: $\lambda = 1$; $\mu = \frac{1}{4}$



$f^*\ell(f)$

$-f^*\ell(f^*)$

$\leq \frac{1}{4} f\, \ell(f)$

Implicit Smoothness Bounds

Examples: selfish routing, linear cost fns.

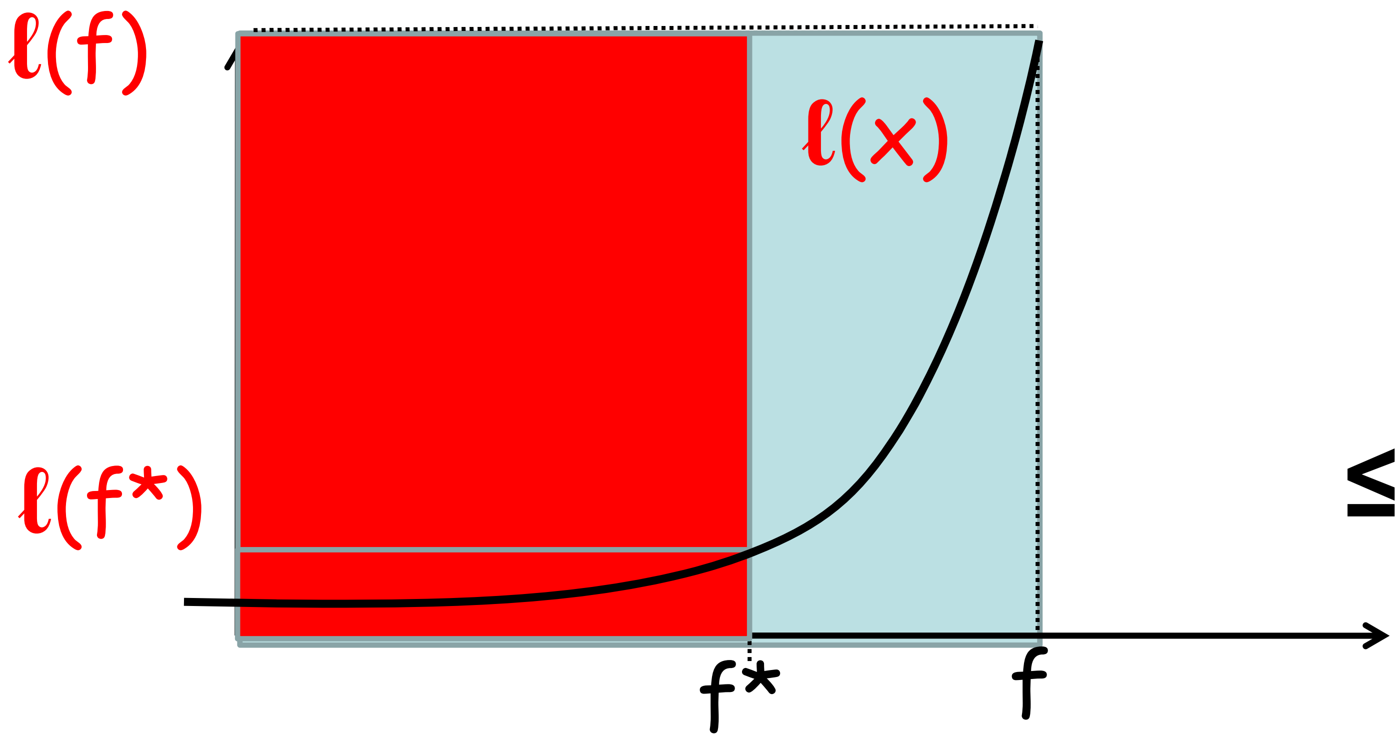
- every nonatomic game is (1,1/4)-smooth
  - follows directly from analysis in [Correa/Schulz/Stier Moses 05]
  - Implies a $\frac{3}{4}$=1/(1-$\frac{1}{4}$) bound on Price of Anarchy
- every atomic game is (5/3,1/3)-smooth
  - follows directly from analysis in [Awerbuch/Azar/Epstein 05], [Christodoulou/Koutsoupias 05]
  - Implies a 5/2 bound on Price of Anarchy

Theorem [Roughgarden 09] for congestion game the best such bound tight

General increasing delay?

Any increasing function is (1,1)-smooth

$$f^* \cdot \ell(f) \le f^* \cdot \ell(f^*) + f \cdot \ell(f)$$

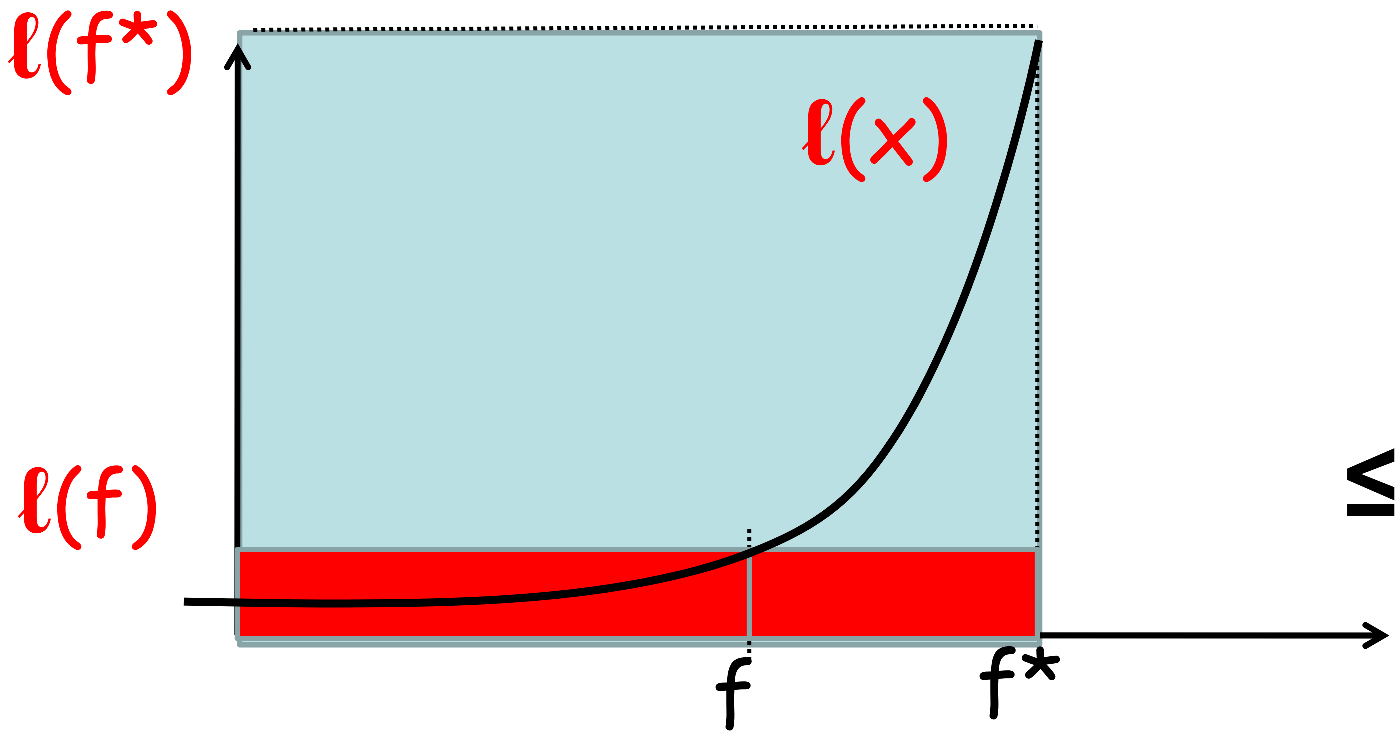


$f^* \ell(f)$
$- f^* \ell(f^*)$
$\le f\, \ell(f)$

General increasing delay?

Any increasing function is (1,1)-smooth

$$f^* \cdot \ell(f) \leq f^* \cdot \ell(f^*) + f \cdot \ell(f)$$



$$f^*\ell(f) - f\,\ell(f) \leq f^*\ell(f^*)$$

Aside: general increasing delay?

Any increasing function is (1,1)-smooth

Recall: cost(s) $\leq \lambda/(1-\mu)$ cost(s*)

(1,1)-smooth not useful

Theorem 3 (Roughgarden-Tardos):

- In any network with continuous, nondecreasing latency functions

cost of Nash with rates r_i for all i	$\leq$	cost of opt with rates $2r_i$ for all i

(1,1)-smooth $\Rightarrow$ Bicriteria bounds

cost of Nash with rates r_i for all i	$\leq$	cost of opt with rates $2r_i$ for all i

Proof: Use smooth for s = Nash players I and s* = opt with players $I \cup I'$

$$\textstyle\sum_{i \in I \cup I'} C_i(s^*_i, s_{-i}) \leq \lambda \,\mathrm{cost}(s^*) + \mu \,\mathrm{cost}(s)$$

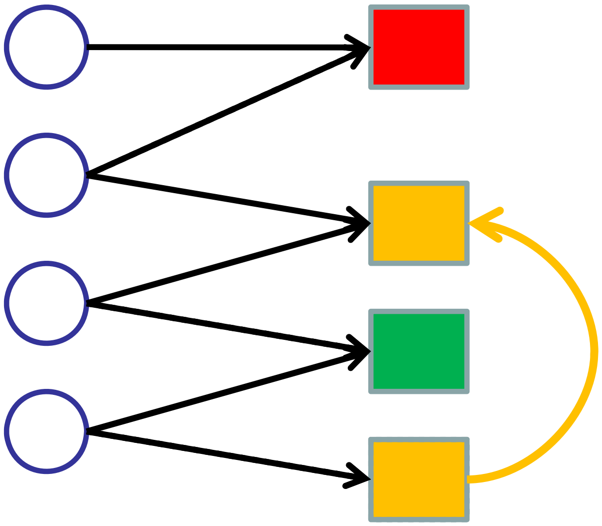
$$\mathrm{cost}(s) = \textstyle\sum_{i \in I} C_i(s_i, s_{-i}) \leq \frac{1}{2} \sum_{i \in I \cup I'} C_i(s^*_i, s_{-i})$$

[s a Nash eq: use for both i and i' in s*]

$$\leq \tfrac{1}{2} \lambda \,\mathrm{cost}(s^*) + \tfrac{1}{2} \mu \,\mathrm{cost}(s) \qquad \text{[smooth]}$$

Then: $\mathrm{cost}(s) \leq \lambda/(2-\mu)\, \mathrm{cost}(s^*)$

Smoothness for Value Problems



Vetta "competitive societies": value for facility location:

s Nash, s* Optimum

(1,-1) smooth

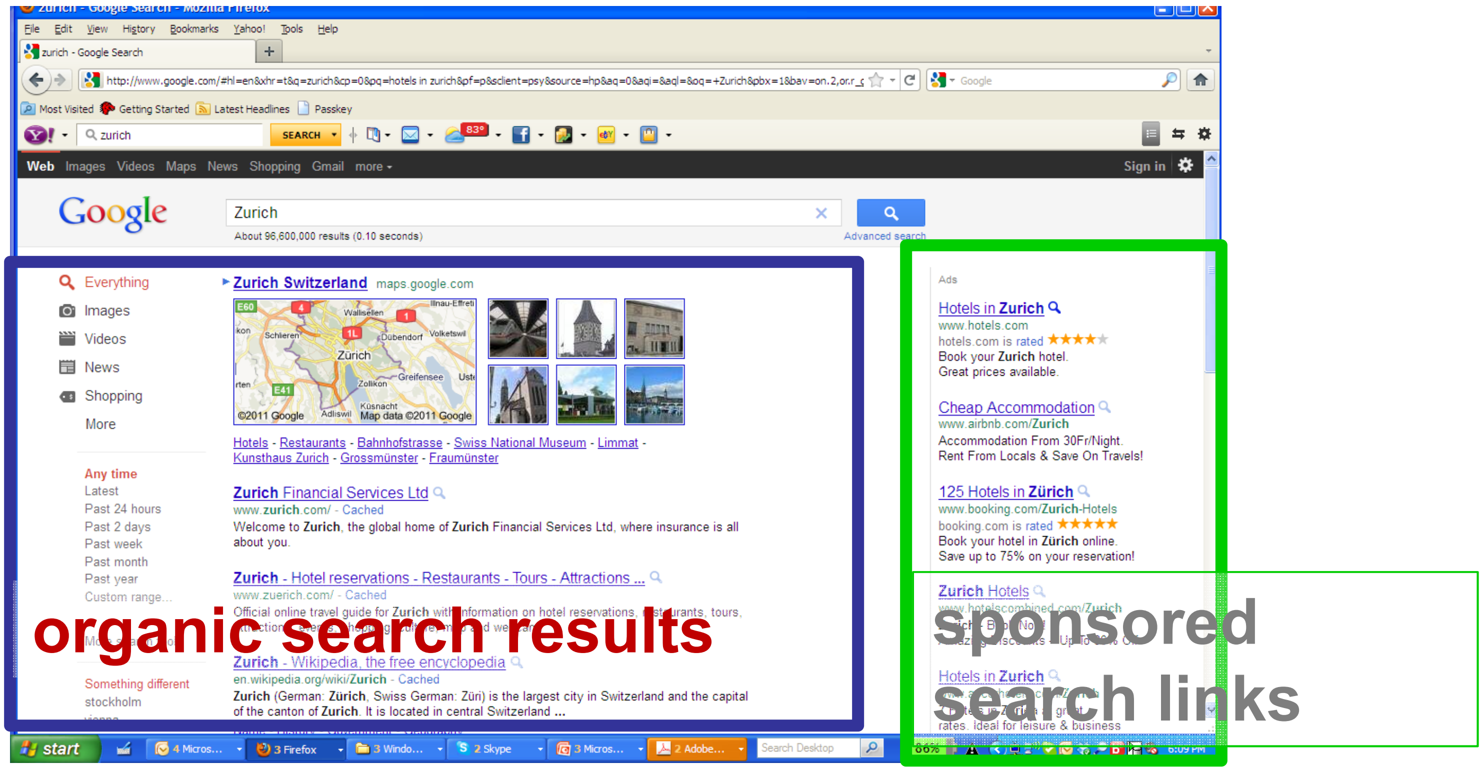
note $\lambda/(1-\mu) = \frac{1}{2}$

fyi: *Also a potential game*

$$Val(s) \geq \Sigma_i\, Val_i\,(s^*_i, s_{-i}) \geq Val(s^*) - Val(s)$$

hence $Val(S) \geq \frac{1}{2} Val(s^*)$.

Smoothness Ad Auction



Price of Anarchy using "smooth" for one s* only
Lucier & Paes Leme & T.

Selfish Outcome (2)?

Nash?

How do users coordinate on a Nash equilibrium, e.g., which do the choose?

- Does natural behavior lead no Nash?
- Which Nash?
- Finding Nash is hard in many games…
- What is natural behavior?
  - Best response?
  - Noisy Best response (e.g. logit dynamic)
  - learning?
  - Copying others?

Learning?

Iterated play where users update play based on experience

Traditional Setting: stock market

m experts   N options

Goal: can we do as well as the best expert?

Regret = long term average cost – average cost of single best strategy with hindsight.

No Regret Learning

Goal: can we do as well as the best expert?

-as the single stock in hindsight?

Idea: if there is a real expert, we should find out who it is after a while.

No regret: too hard (would need to know expert at the start)

Goal: small regret compared to range of cost/benefit

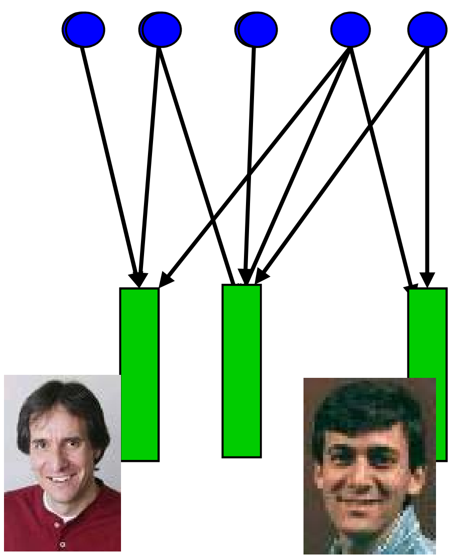
Learning in Games

Goal: can we do (almost) as well as the best expert?

Games?

Focus on a single player:

experts = strategies to play



Goal: learn to play the best strategy with hindsight

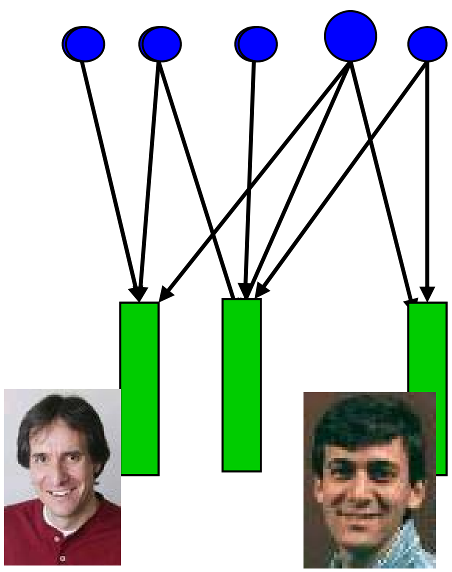
Best depends on others

Learning in Games

Focus on a single player:

experts = strategies to play

Goal: learn to play the best strategy with hindsight



Best depends on others did

Example: matching pennies

With q=(½ ,½), best value with hindsight is 0.
Regret if our value ≤ 0

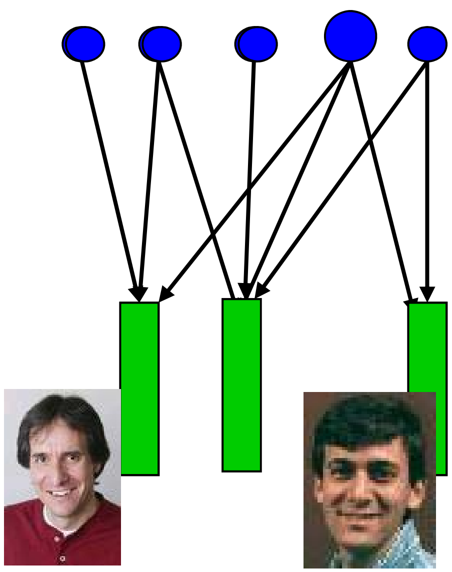
½	½
-1 1	1 -1
1 -1	-1 1

Learning in Games

Focus on a single player:

experts = strategies to play

Goal: learn to play the best strategy with hindsight



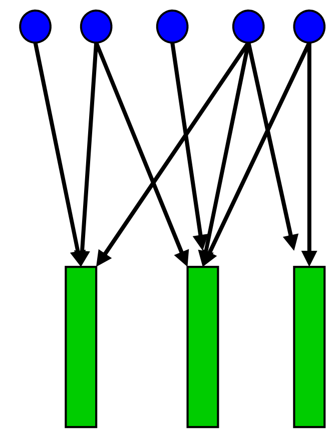
Best depends on others did

Example: matching pennies

With q=(¾ ,¼), best value with hindsight is ½ (by playing top).
Regret if our value ≤ ½

¾	¼
-1 1	1 -1
1 -1	-1 1

A Natural Learning Process



Iterated play where users update probability distributions based on experience

Example: Multiplicative update (Hedge) strategies 1,…,n

Maintain weights $w_e \geq 0$

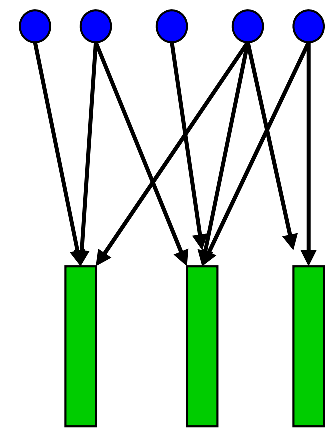
probability $p_e \sim w_e$ all e

Update w_e to $w_e (1- \varepsilon)^{cost(e)}$

α=1- ε think of ε ~ learning rate

Learning and Games

Regret = long term average cost – average cost of single best strategy with hindsight.



Nash = strategy for each player so that players have no regret

Hart & Mas-Colell: general games → Long term average play is (coarse) correlated equilibrium

Correlated? Correlate on history of play

(Coarse) correlated equilibrium

Coarse correlated equilibrium: probability distribution of outcomes such that for all players

expected cost $\leq$ exp. cost of any fixed strategy

Correlated eq. & players independent = Nash

Learning:

Players update independently, but correlate on shared history

Correlated eq.: examples

Coordination game: prob. $\frac{1}{2}$-$\frac{1}{2}$ on coordinated outcomes

0 0	1 2
2 1	0 0

Traffic light:

Nash

Correlated eq prob. $\frac{1}{2}$-$\frac{1}{2}$

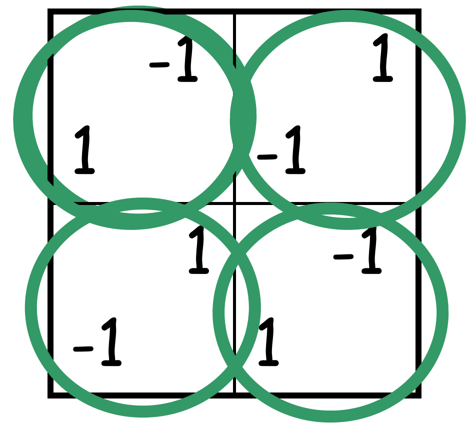
allows both to pass with no prob on crashing

0 0	0 1
1 0	−100 -100

crash

Correlated eq.: examples 2

Matching pennies:

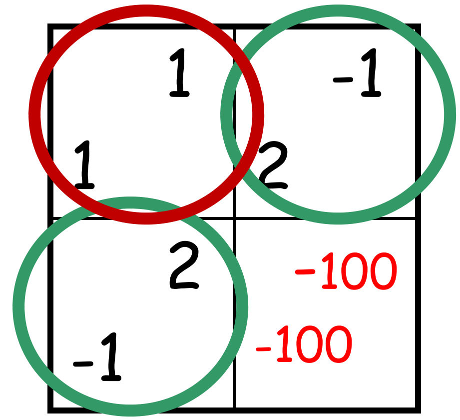


Nash: each play prob. $\frac{1}{2}$-$\frac{1}{2}$

No-regret sequence:

Correlated equilibrium

Chicken: probability $\frac{1}{2}$ and $\frac{1}{4}$, $\frac{1}{4}$ on three OK outcomes



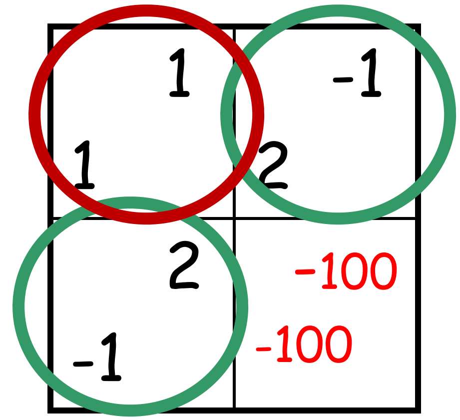
Expected value of $\frac{3}{4}$ each

Correlated equilibrium forms a convex set!

Why? No-regret is a set of linear inequalities for the probability distribution

- One ineq for each player strategy
- Player X has no regret on his strategy i

Correlated equilibrium



Correlated equilibrium forms a convex set!

Hence we can solve?

An equilibrium can be found in time polynomial in the number of variables

$= \Pi_i$ (# strategies for player i)

This is a lot!

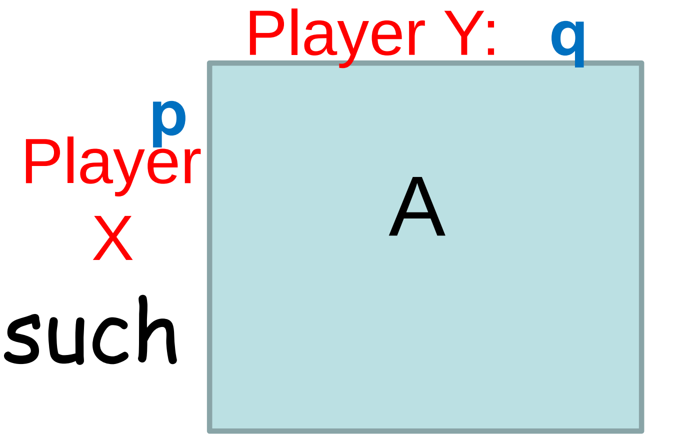
- see Papadimitriou-Roughgarden for poly time

or use learning!

Learning 0-sum games

Two-person 0-sum game:

X pays a_{ij} cost



Nash: probabilities q and p such that neither can improve.

Value:

$$v = \Sigma_{ij}\, a_{ij}\, p_i q_j = \min_i\ \Sigma_j\, a_{ij}\, q_j = \max_j\ \Sigma_i\, a_{ij}\, p_i$$

Theorem (Freund-Schapire) If both players play no regret strategies, outcome has value v, and marginal distribution the optimal p* and q*

Learning 0-sum games

Theorem (Freund-Schapire)

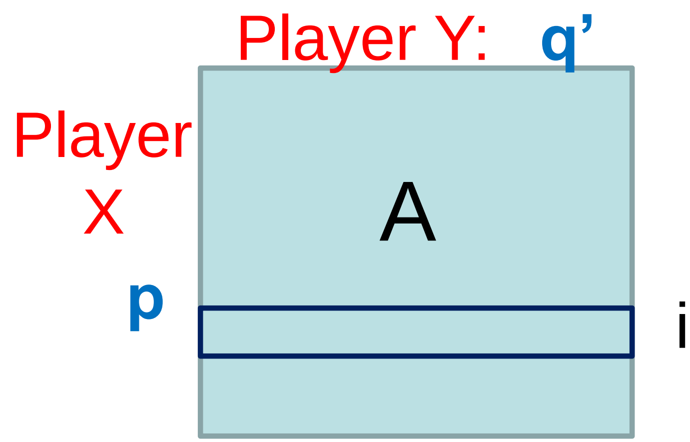
If X pays > Nash value v, it has regret

Proof: Nash value: X can always guarantee to pay no more than v.

as $v = \max_j \Sigma_i \, p_i \, a_{ij}$

So on ANY distribution q'

$$\min_i \Sigma_j \, a_{ij} \, q'_j \le \Sigma_{ij} \, p_i \, a_{ij} q'_j \le v$$

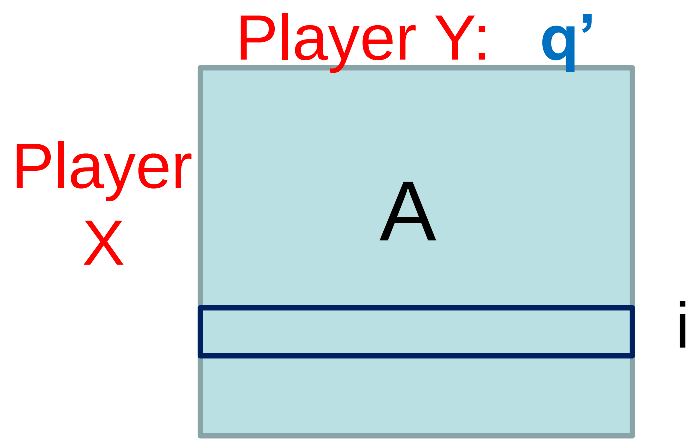


Learning 0-sum games

Theorem (Freund-Schapire)

If X pays > Nash value v, it has regret

Proof (cont): $\min_i \sum_j a_{ij} q'_j \le v$ for any q'



Apply this to the distribution of Y that results from learning:

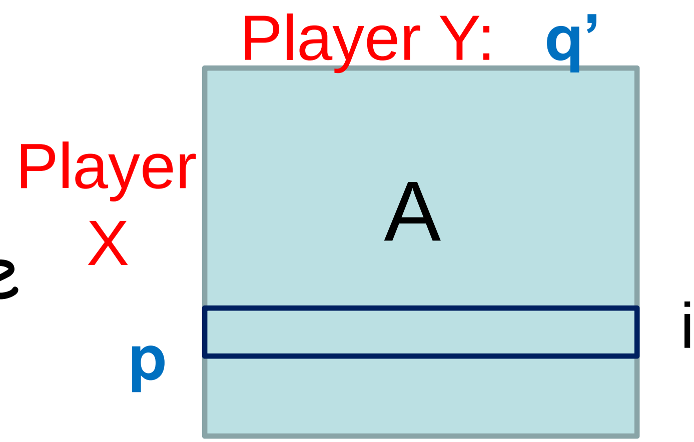
$\min_i \sum_j a_{ij} q'_j \Rightarrow$ i is best with hindsight

If $\text{cost}_X > v \ge \min_i \sum_j a_{ij} q'_j$ then X has regret!

Learning 0-sum games

Theorem (Freund-Schapire)

If both players use no-regret strategies the outcome is the Nash value v.



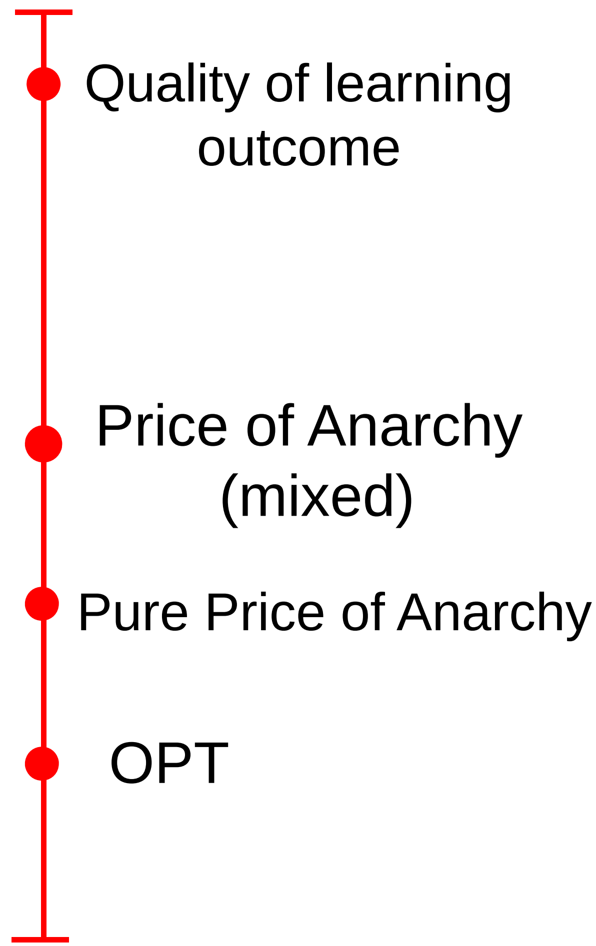
Proof: We have seen:

X has no regret than pays no more than v.

Similarly, Y has no regret than gains at least v.

Quality of learning outcomes?

Blum, Even-Dar, Ligett'06: Price of total anarchy: worst quality of a learning outcome



Hopes:

1. Learning outcomes no so bad
2. learning will not make users coordinate on bad equilibria

Recall Smooth & ⇒ Price of Anarchy [Roughgarden]

A cost-game is (λ,μ)-smooth if, for every pair s,s^* outcomes ($\lambda > 0$; $\mu < 1$):

$$\sum_i C_i(s^*_i, s_{-i}) \leq \lambda\, cost(s^*) + \mu\, cost(s)$$

Theorem: (λ,μ)-smooth ⇒

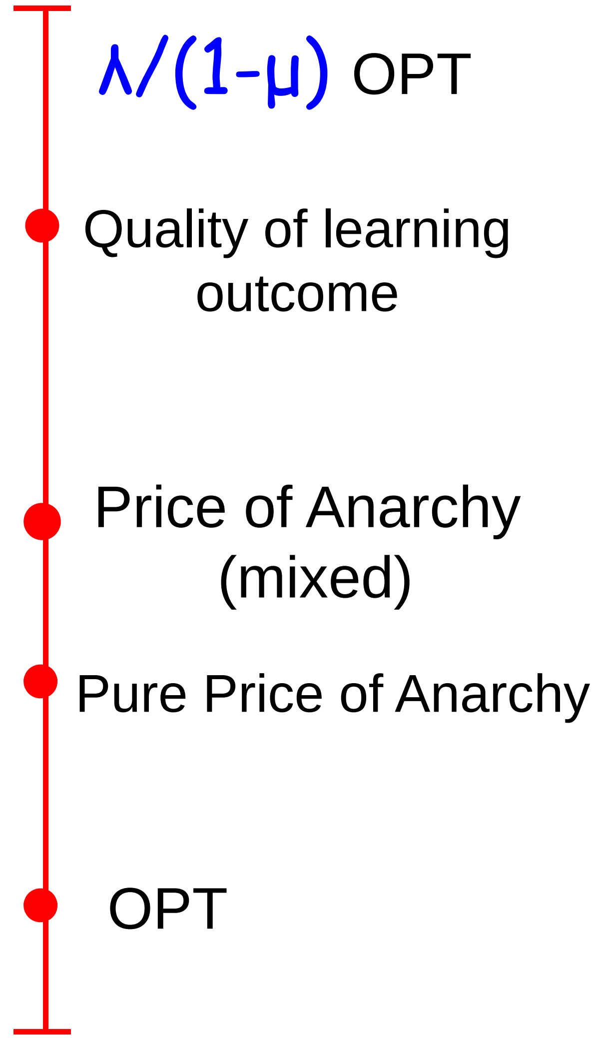
Price of Anarchy $\leq \lambda/(1-\mu)$

for congestion game the best such bound tight

Smoothness bound

Theorem [Roughgarden'09] game (λ,μ)-smooth then Price of total anarchy bounded by $\lambda/(1-\mu)$.

Recall: for congestion games $\lambda/(1-\mu)$ is tight for the best (λ,μ) for pure equilibrium.



Smooth $\Rightarrow$ Price of total Anarchy

Proof [Roughgarden]

$\Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda\, cost(s^*) + \mu\, cost(s)$ [smooth]

$s^1, \ldots, s^t, \ldots$ = no regret sequence, s^* = opt

$\Sigma_t\, cost(s^t) = \Sigma_t \Sigma_i C_i(s^t_i, s^t_{-i})$

$\leq \Sigma_i (\Sigma_t C_i(s^*_i, s^t_{-i}))$ [s^t is no regret]

$\leq \Sigma_t (\lambda\, cost(s^*) + \mu\, cost(s^t))$ [smooth s^t, s^*]

Then: $\Sigma_t\, cost(s^t)/T \leq \lambda/(1-\mu)\, cost(s^*)$

Summary so far

Example: Congestion games (which are potential games)

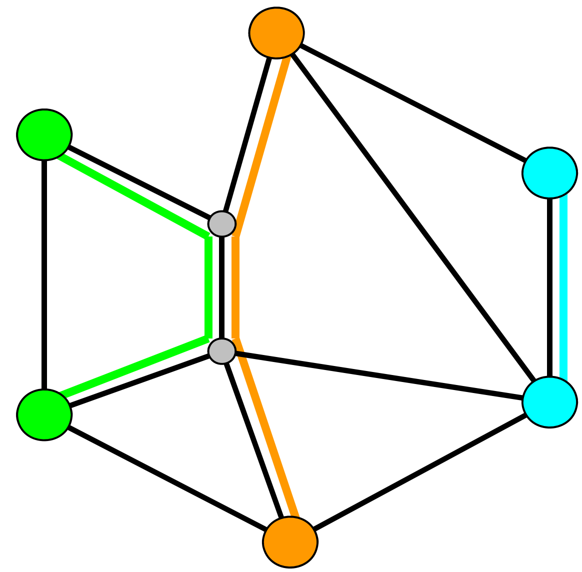
- (λ,μ)-smooth implies $\lambda/(1-\mu)$ bound on the price of anarchy.
- Bound valid when all players have “no-regret” (without reaching equilibrium).

Next: When can we expect better outcome than price of anarchy?

- As equilibria?
- As outcomes of learning?

Costs in Connection Game

Players pay for their trees,
want to minimize payments.

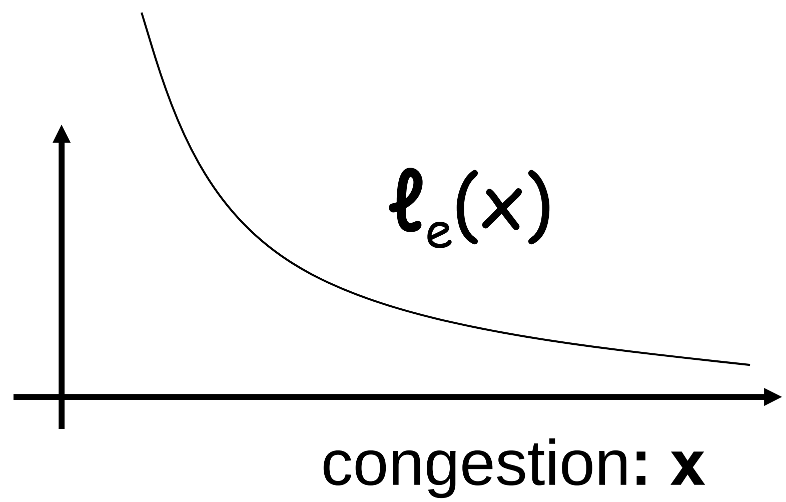


What is the cost of the edges?

$c_e(x)$ is cost of edge e for x users.

Assume economy of scale and fair sharing:

e.g.: $\ell_e(x) = c_e(x) / x$



Results for Network Design

Theorem [Anshelevich, Dasgupta, Kleinberg, Tardos, Wexler, Roughgarden FOCS'04]

Assume $c(x)$ is monotone increasing and concave function of congestion x

There exists equilibrium with cost $\leq O(\log k)$Opt for k players (bound sharp)

Using potential Φ...

- Consider the Nash with minimum value of Φ
- This Nash has,

$$\Phi(\text{Nash}) < \Phi(\text{OPT}).$$

Suppose that we also know for any solution

$$\Phi \le \text{cost} \le A\,\Phi$$

→ $\text{cost}(\text{Nash}) \le A\,\Phi(\text{Nash}) \le A\,\Phi(\text{OPT}) \le A\,\text{cost}(\text{OPT})$.

→ There is a good Nash!

Results for Network Design

proof:

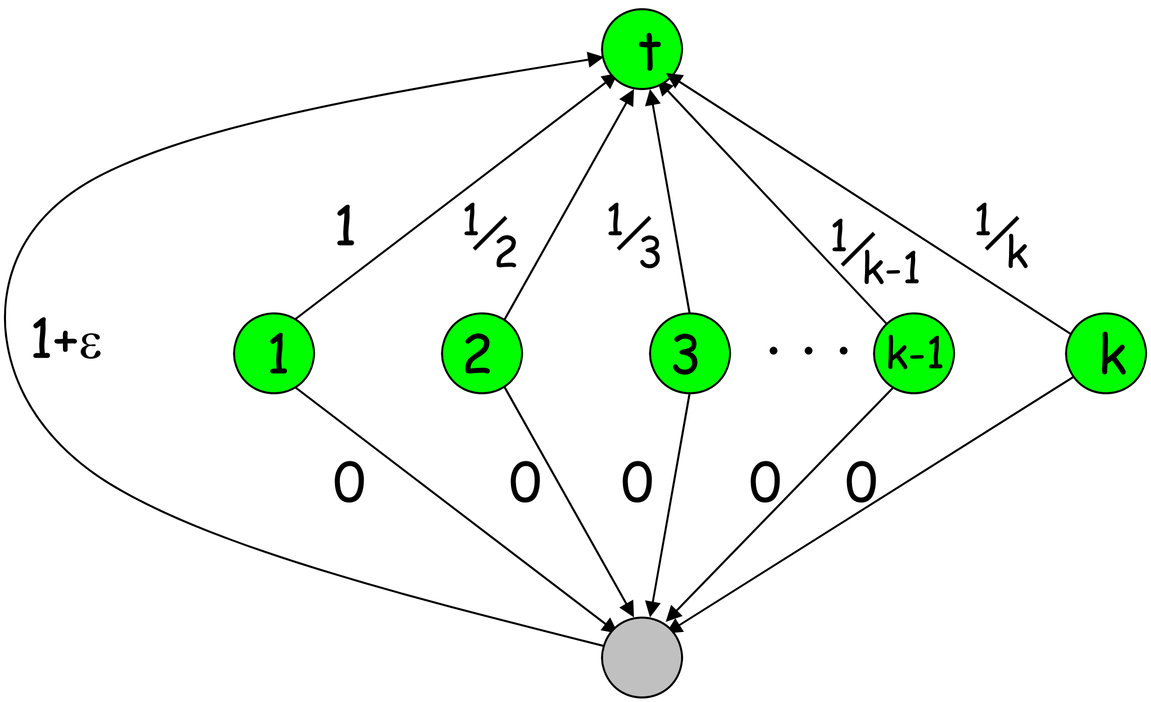
Recall: $\Phi(\mathbf{f}) = \Sigma_e (\ell_e(1)+\ldots+\ell_e(f_e)) = \Sigma_e \Phi_e$

$f_e \leq k$ users on edge e then

- true cost is $c_e(f_e) = c_e$
- Potential is $\Phi_e = c_e + c_e/2 + c_e/3+\ldots+c_e/f_e$
  $\leq c_e \cdot (1+1/2+1/3+\ldots+1/k) = c_e H_k$
- cost $\leq \Phi \leq$ cost$\cdot H_k$
- $\rightarrow$ Nash optimizing Φ cost at most H_k above the optimum

Example: Bound is Tight

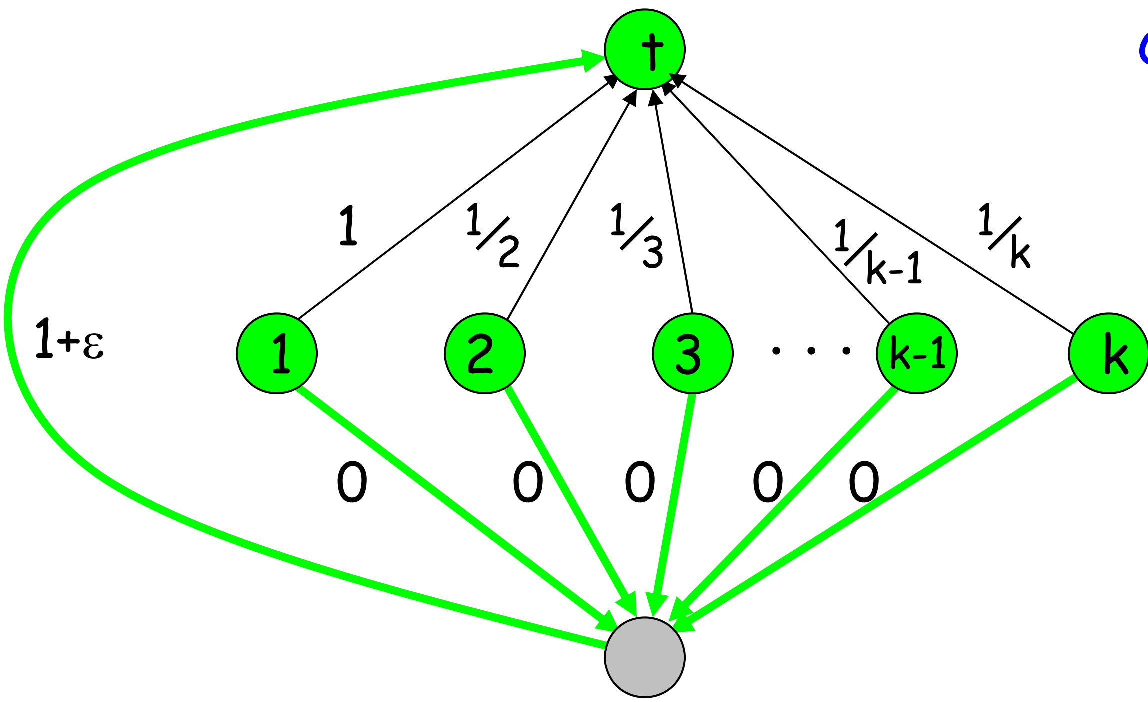
All noes 1… k want to connect to terminal t



Example: Bound is Tight

All noes 1… k want to connect to terminal t

cost(OPT) = 1+ε



Example: Bound is Tight

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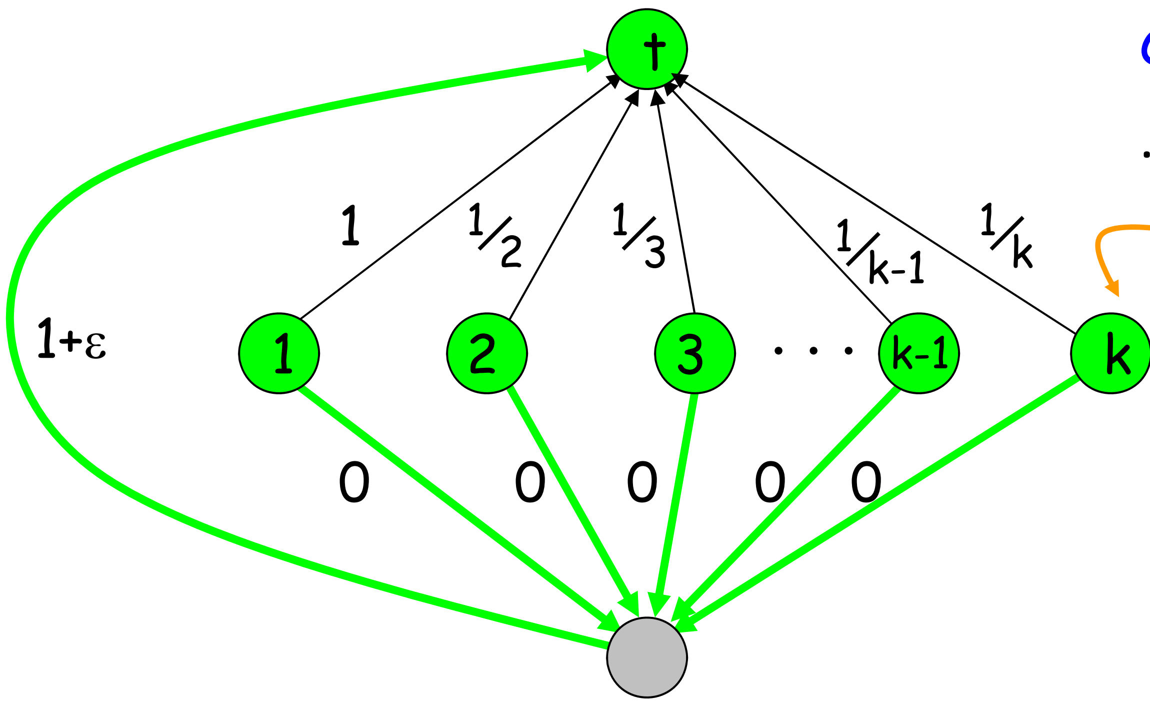


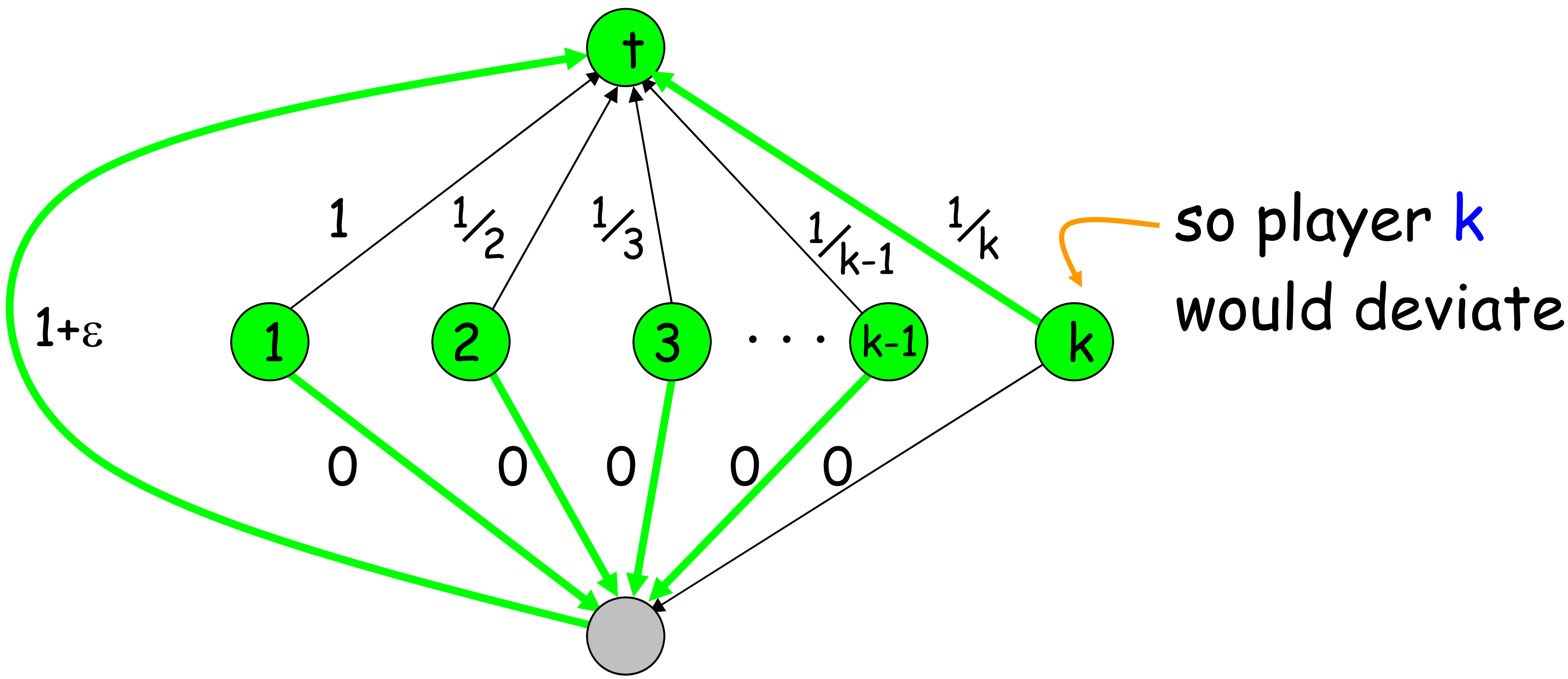
cost(OPT) = 1+ε

…but not a NE:

player k pays (1+ε)/k, could pay 1/k

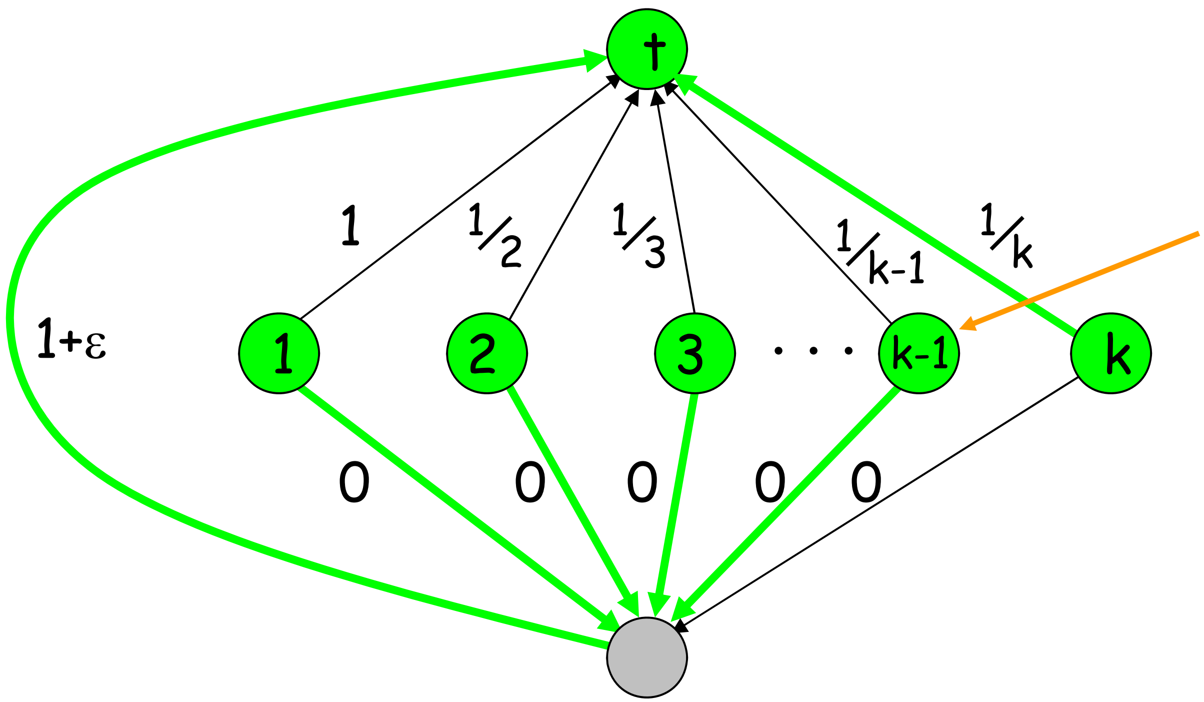
Example: Bound is Tight

All noes 1… k want to connect to terminal t



Example: Bound is Tight

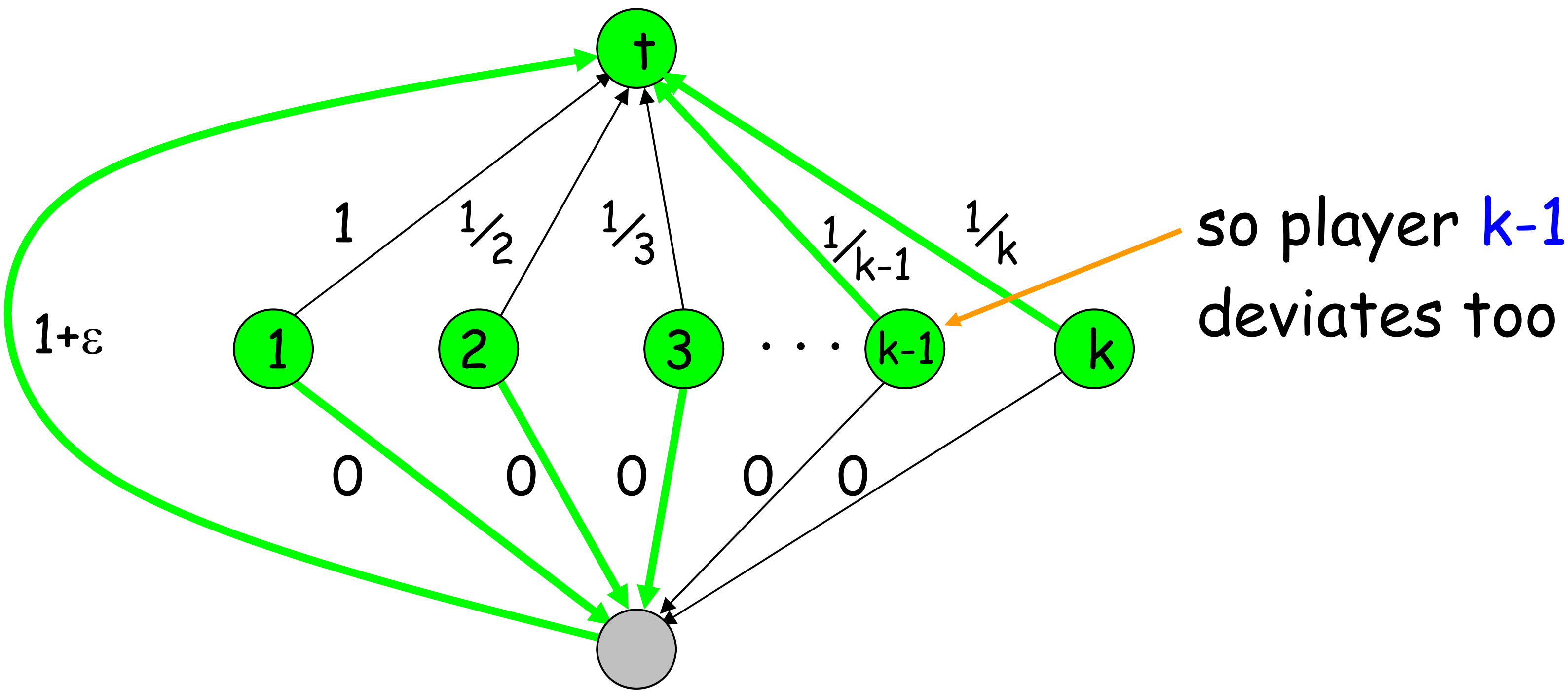
All noes 1… k want to connect to terminal t



now player k-1 pays (1+ε)/(k-1), could pay 1/(k-1)

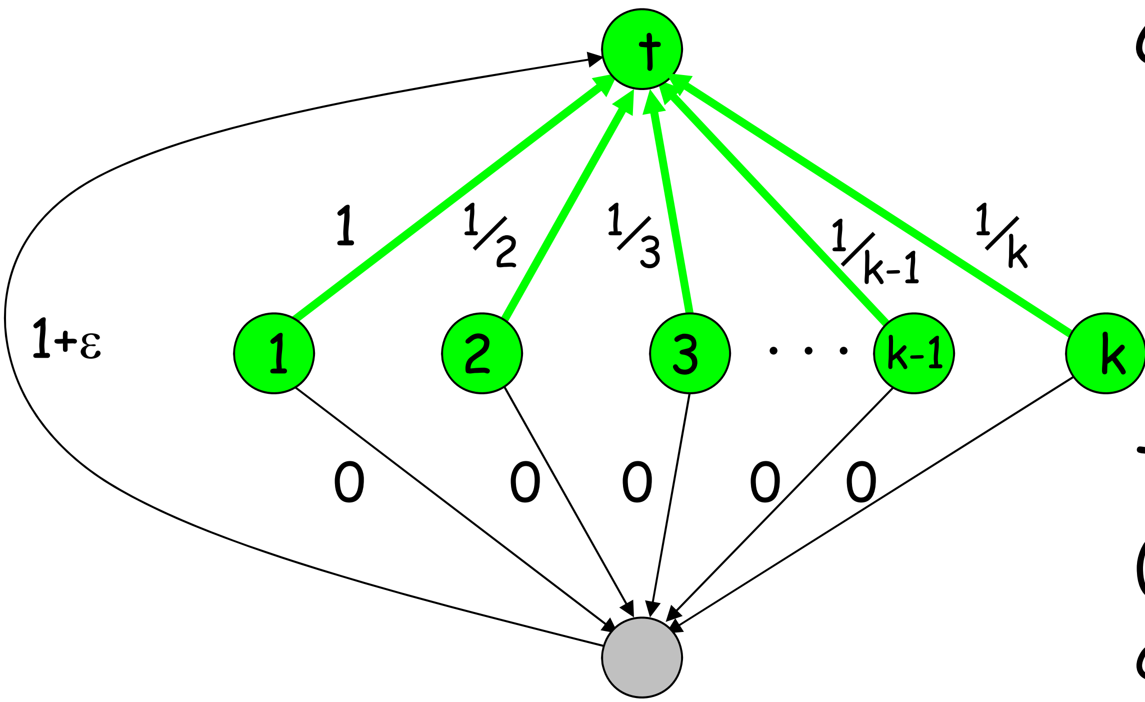
Example: Bound is Tight

All noes 1… k want to connect to terminal t



Example: Bound is Tight

All noes 1… k want to connect to terminal t



Continuing this process, all players defect.

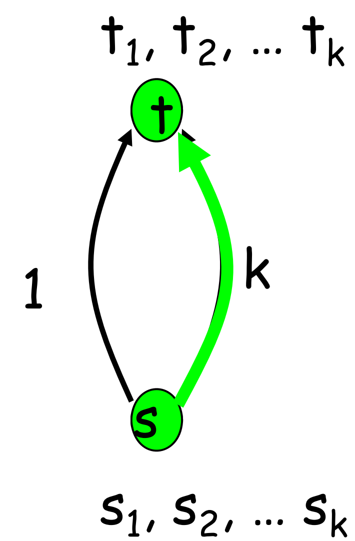
This is a NE!
(the only Nash)
cost = $1 + \frac{1}{2} + \ldots + \frac{1}{k}$

Price of Stability is $H_k = \Theta(\log k)$!

Better worst case?

Theorem worst price of anarchy for network formation a factor of k=number of players

Reason: no player will pay alone more then Opt



Question more there a reason we should expect to see this bad worst case?

- Improvements in Price of Anarchy via refined solution concepts

Selfish Outcome= non-cooperative?

Nash equilibrium: non-cooperative outcome

- Current strategy "best response" for all players
- no single user has incentive to deviate

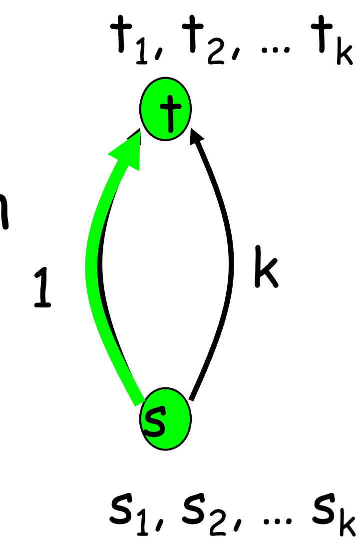
How about groups of players?

Strong Nash equilibrium: no group of players has incentive to deviate [Aumann'59]

Cooperative game?

We can use: Strong Nash equilibrium

- No subset players can coordinate a deviation and improve for every player in the set



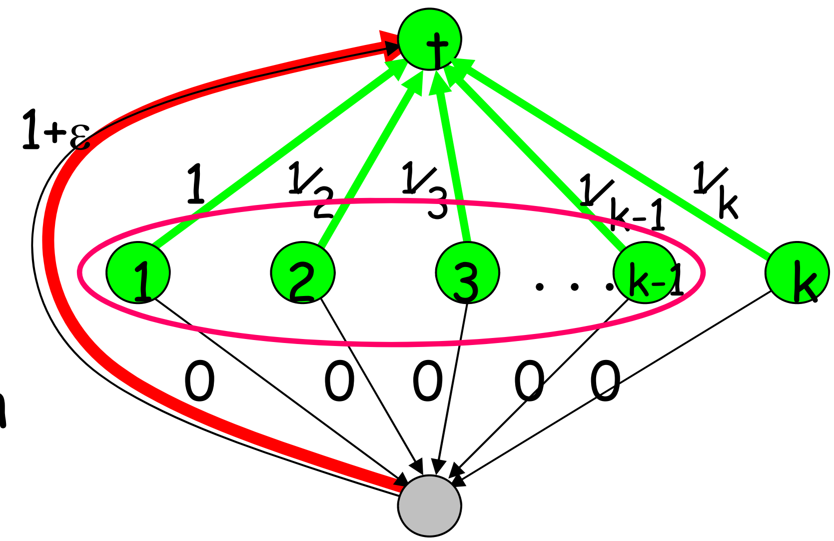
[Epstein, Feldman, Mansour EC'07]

the strong price of anarchy is O(log k)

(but strong Nash may not exists…)

Strong Price of Anarchy?

SE = strong Nash, **Opt**



As a group not all players want to move to Opt:

$\Rightarrow$ There exists player, say last player k, that is better off in current solution

$\Rightarrow$ $cost_k(SE) \leq cost_k(Opt)$

Consider remaining k-1 players.

Opt_{k-1} = Opt restricted to remaining k-1 players

As a group the remaining k-1 players also don't want to move to Opt_{k-1} $\Rightarrow$ there is a player, say k-1

$Cost_{k-1}(SE) \leq cost_{k-1}(Opt_{k-1})$

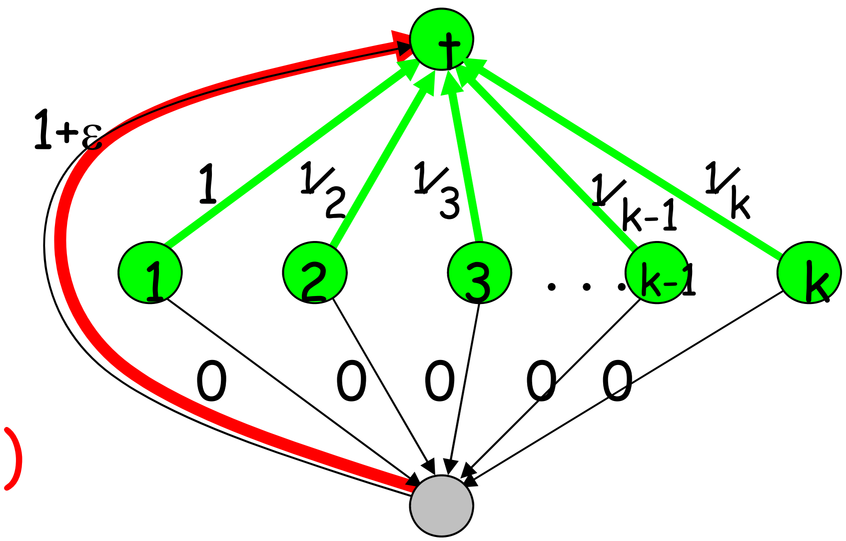
Strong Price of Anarchy

SE = strong Nash, **Opt**,

Continue…

Opt_i = Opt restricted to remaining i

We get: $cost_i(SE) \leq cost_i(Opt_i)$



Lemma: In potential games: $cost_i(Opt_i) = \Phi(Opt_i) - \Phi(Opt_{i-1})$

Proof: consider first i players only, and selfish move of player i of "not playing":

- Cost to player i: $cost_i(Opt_i)$
- potential change $\Phi(Opt_i) - \Phi(Opt_{i-1})$

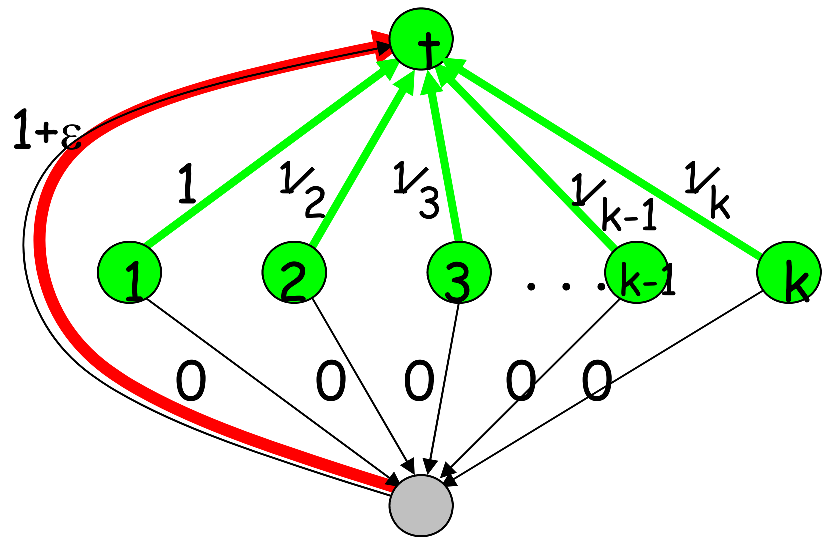
Strong Price of Anarchy

SE = strong Nash, **Opt**,

Opt_i = Opt restricted to first i

set 1…i doesn't want to move

$cost_i(SE) \leq cost_i(Opt_i)$



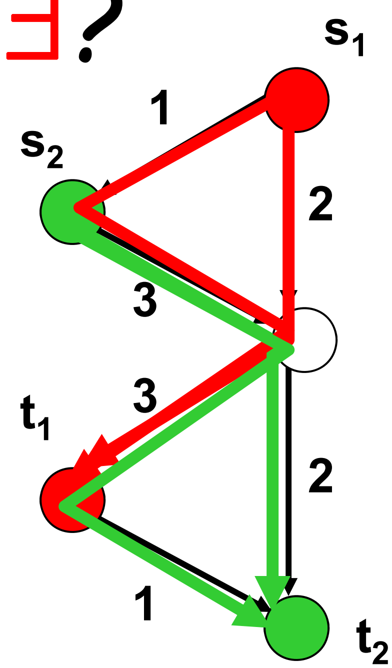
Potential game: $cost_i(Opt_i) = \Phi(Opt_i) - \Phi(Opt_{i-1})$

We get: $cost_i(SE) \leq cost_i(Opt_i) = \Phi(Opt_i) - \Phi(Opt_{i-1})$

$\sum_i cost_i(SE) \leq \sum_i \Phi(Opt_i) - \Phi(Opt_{i-1}) = \Phi(Opt)$

In cost-sharing game $\Phi(Opt) \leq H_k\, cost(Opt)$

But: strong Nash ∃?



We proved [Epstein, Feldman, Mansour EC'07] the strong price of anarchy is $O(\log k)$

But ∃?: No!!

Nash unique: cost of 5 each

It is not strong! As there is a solution better for both

cost of 4 each

It's a "prisoner dilemma"

	C	D
C	4 (row), 4 (column)	5.5 (row), 3.5 (column)
D	3.5 (row), 5.5 (column)	5 (row), 5 (column)

⇒ no strong Nash exists ∃

Outcome with collusion?

Collusion: group of users deviate together to improve their welfare

Cooperative game theory…

- No great model for outcome for most games
- Strong Nash: outcome when collusion is not useful.
- But what happens when no such outcome exists: collusion is useful?
- Bargaining: agreement when everyone colludes
  - different bargaining "games" characterized by axioms
- Different bargaining solution lead to different outcomes

Tomorrow

- Natural learning process
- better quality outcome via such learning

Next: a few examples on range of solution quality

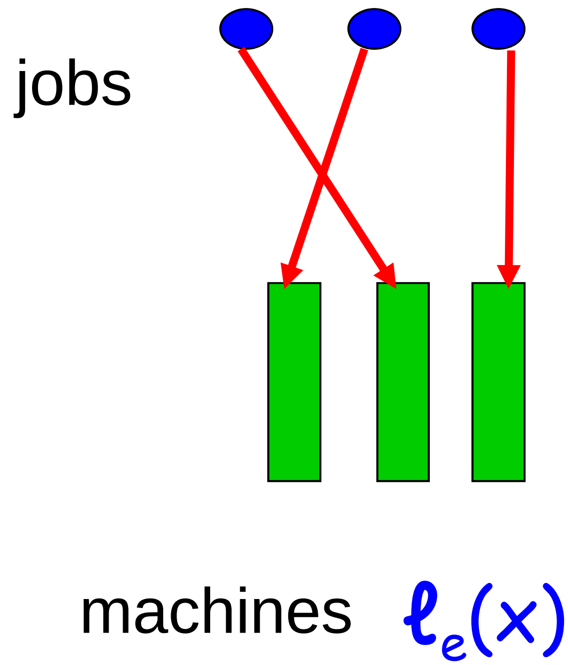
Example: Atomic Game (pure Nash)

n jobs and n machines with identical $\ell(x)$ functions

Pure Nash: each job selects a different machine, load = $\ell(1)$: optimal

More generally: if all jobs are identical, and all jobs can go to all machines $\Rightarrow$ pure Nash: minimizes max load

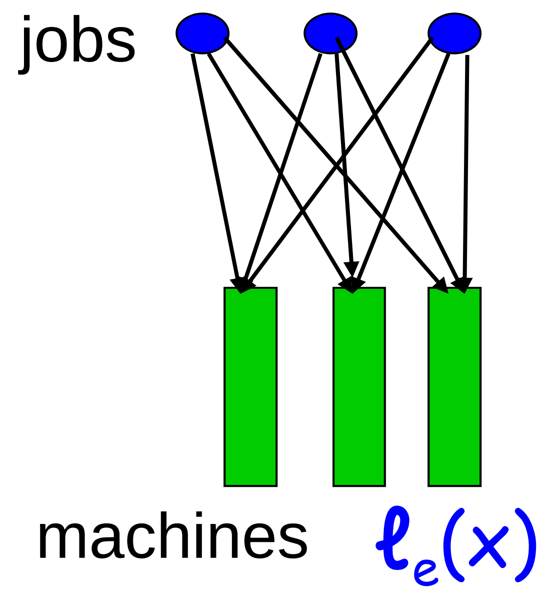
Load balancing:



Example: Atomic Game (mixed Nash)

n jobs and n machines with identical $\ell_e(x)$ functions

Load balancing:



Mixed Nash: e.g. each job selects uniformly random:

With high prob.
max load ~ log n/loglog n

⇒expected load is

> ~ $\ell_e(1) + \ell_e(\log n)/n$

a lot more when $\ell_e(x)$ grows fast

Example: Cost-sharing (mixed vs pure)

n jobs and n machines with identical costs c_e/x functions

Pure Nash: select one machine to use. Total cost c_e

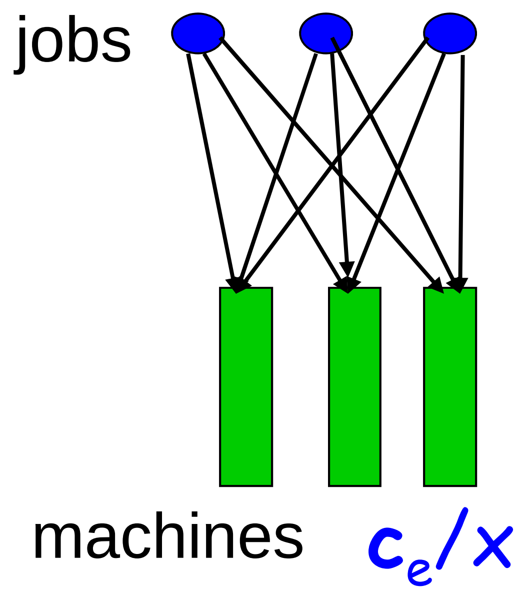
Mixed Nash: e.g. each job selects uniformly random:

With high prob.

expected cost $\sim \Omega(n\, c_e)$

$\Omega(n)$ times more than pure Nash

Cost-sharing:

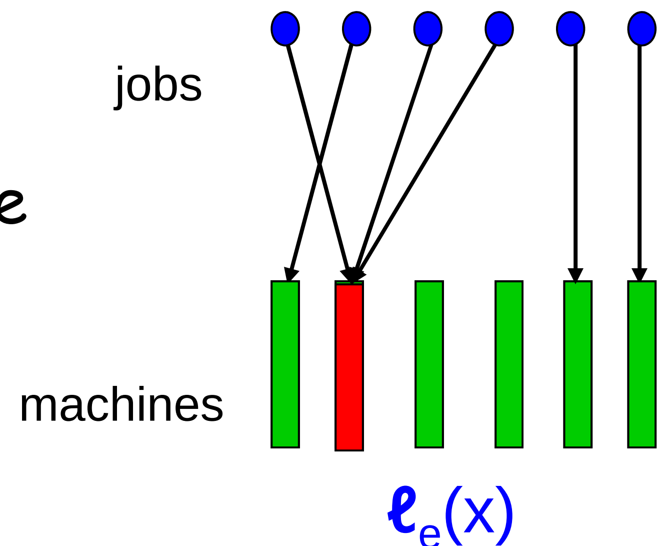


Bad Correlated Equilibrium: Load Balancing

n jobs and n machines with identical $\ell_e(x)$ functions

- Select a k jobs and 1 machine at random and send all k jobs to the one machine.
- Send all remaining jobs to different machines

Load balancing:



Correlated equilibrium if two costs same

- Correlated play cost: $\sim \ell_e(1) + k/n\ \ell_e(k)$
- Fixed other strategy cost $\sim \ell_e(2)$

When $\ell_e(x)=x$ costs balance when $k=\sqrt{n}$: bad congestion