Networks Games:
the price of anarchy, stability, and learning

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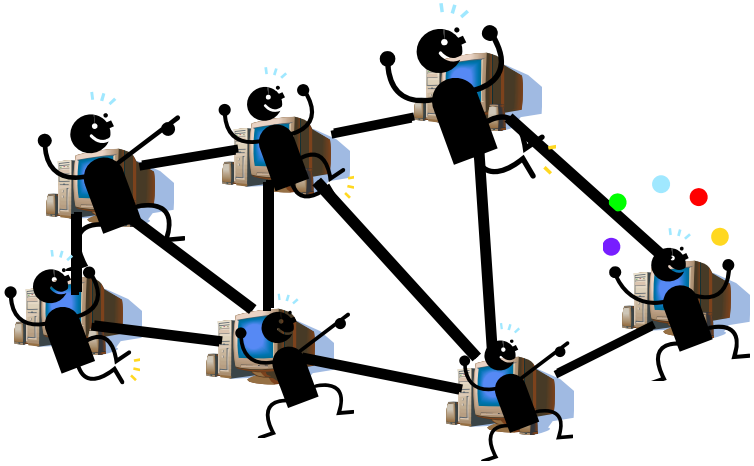
Why care about Games?

Users with a multitude of diverse economic interests sharing a Network (Internet)

- browsers
- routers
- servers

Complex Networks in all aspects of life :

- social networks,
- financial networks,
- buyer-seller networks



Selfishness:

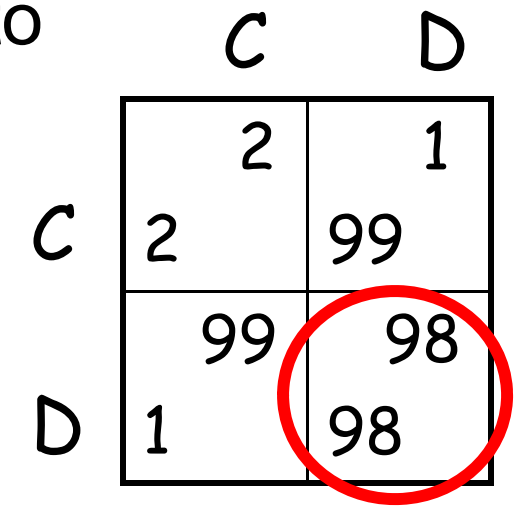
Parties deviate from their protocol if it is in their interest

Model Resulting Issues as Games on Networks

Main question:
Quality of Selfish outcome

Well known: Central design can lead to better outcome than selfishness.

e.g.: Prisoner Dilemma



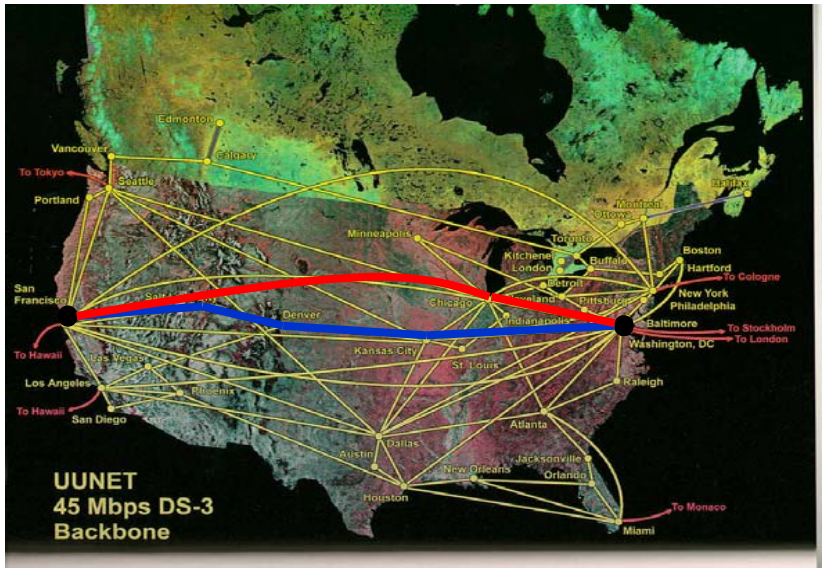
Question: how much better?

Our Games

- Routing, Network formation: Users select paths that connects their terminals to minimize their own delay or cost
- Selling of goods: implicit network
- Contagion of financial risk

Congestion games

Example: Routing Game

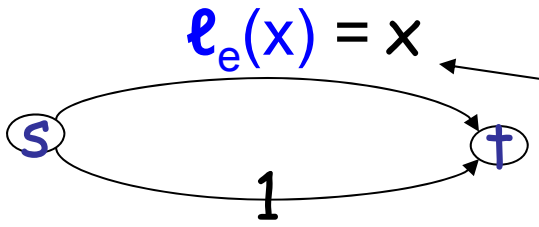


- Traffic subject to congestion delays
- cars and packets follow shortest path

Large number of participants!!

Congestion sensitive load balancing

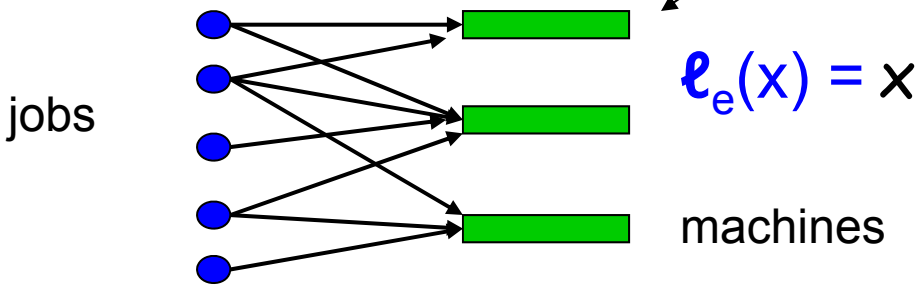
Routing network:



Cost/Delay/Response time as a fn of load:

x unit of load → causes delay $\ell_e(x)$

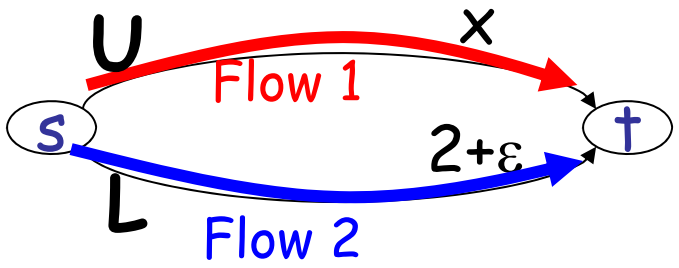
Load balancing:



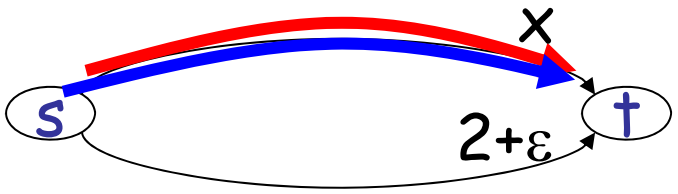
A congestion game

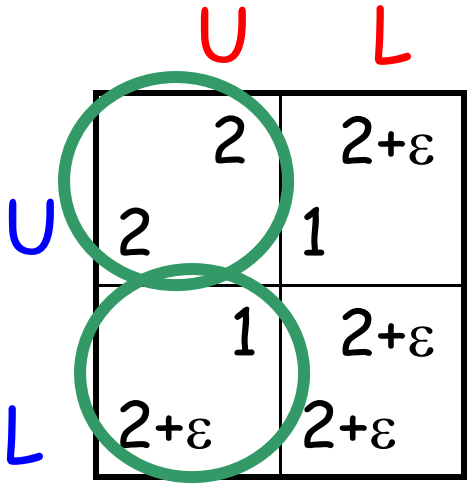
Examples on two links: load balancing

Two players each have one unit of flow to sent



An envy free solution:

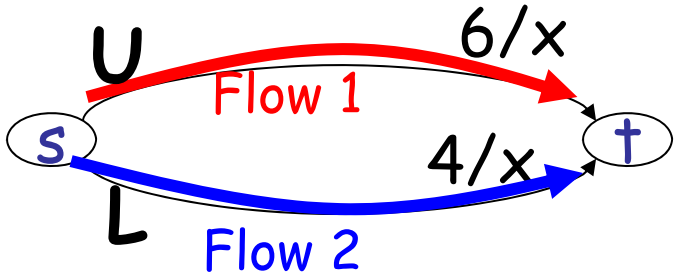




Cost increasing with congestion

Examples on two links: coordination game

Two players each have one unit of flow to sent



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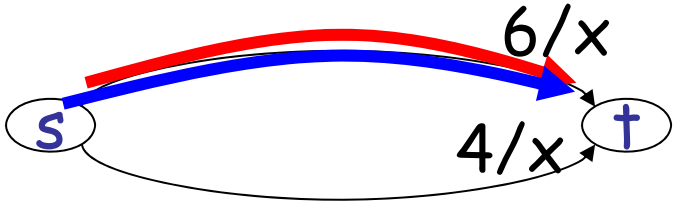
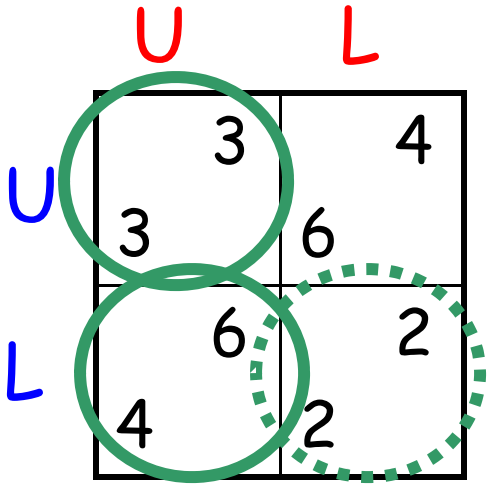


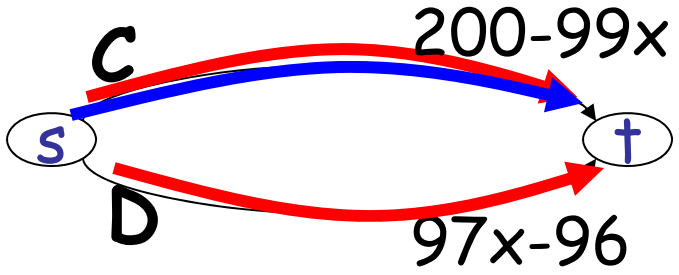
Two solutions of different value



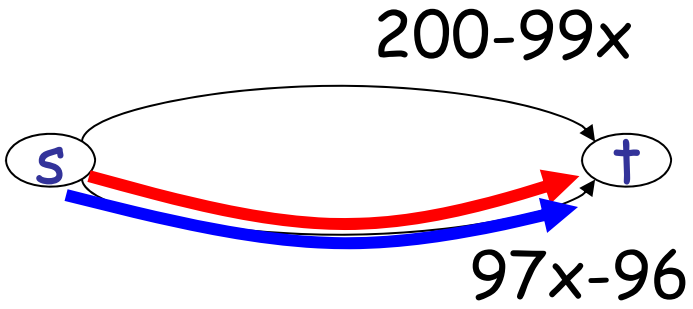
Cost-sharing: cost decreasing with congestion

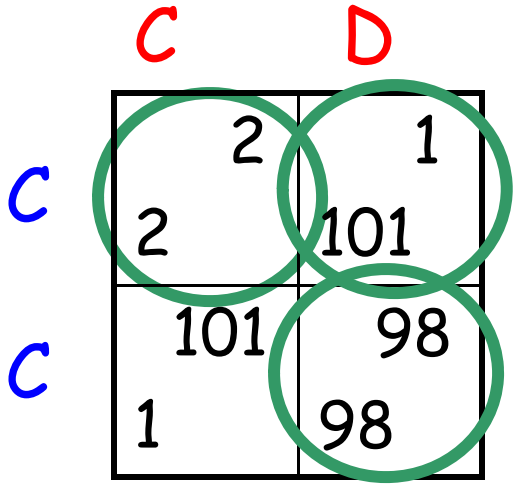
Examples on two links: Prisoner's Dilemma

Two players each have one unit of flow to sent



An envy free solution:





C has decreasing, D has increasing congestion cost

Today:

focus on: pure equilibria
very small and same sized jobs
social welfare

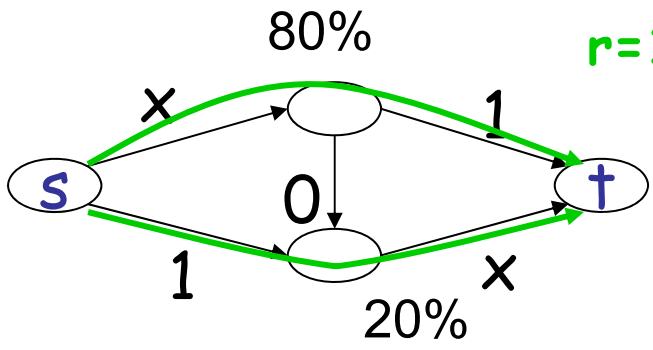
- Does a pure equilibria exists?
- Does a high quality equilibria exists?
- Are all equilibria high quality?

some of the results extend to expected social welfare

Atomic vs. Non-atomic Game

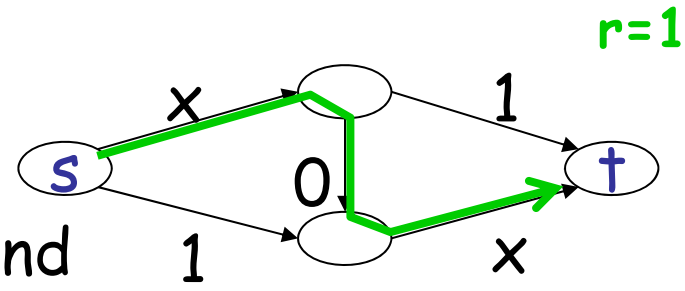
Non-atomic game:

- Users control an infinitesimally small amount of flow
- equilibrium: all flow path carrying flow are minimum total delay



Atomic Game:

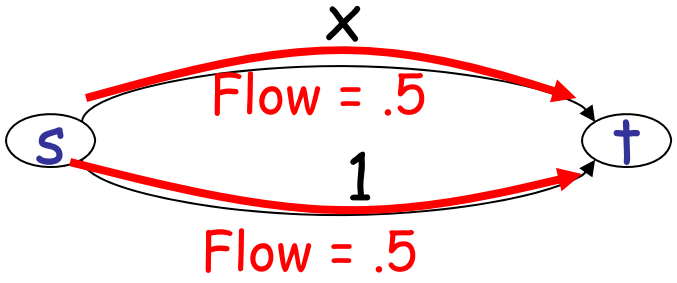
- Each user controls a unit of flow, and
- selects a single path or machine



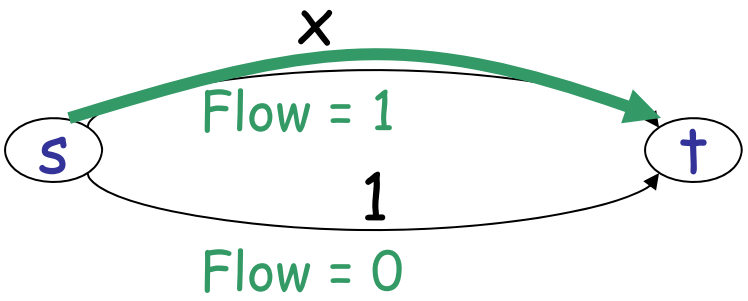
Both congestion games: cost on edge e depends on the congestion (number of users)

Example of nonatomic flow on two links

- One unit of flow sent from s to t



Traffic on lower edge is envious.



An envy free solution:

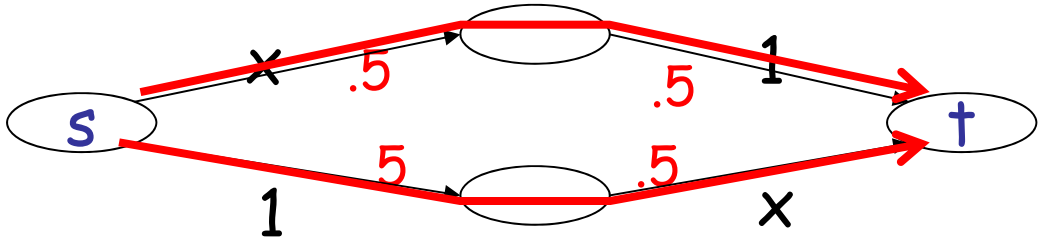
No-one is better off

Infinite number of players

- will make analysis cleaner by continuous math

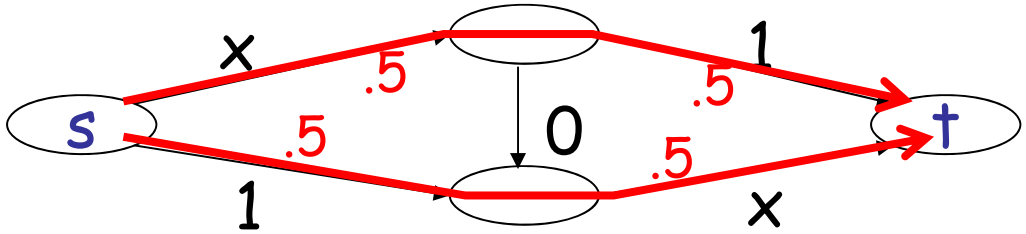
Braess's Paradox

Original Network



Cost of Nash flow = 1.5

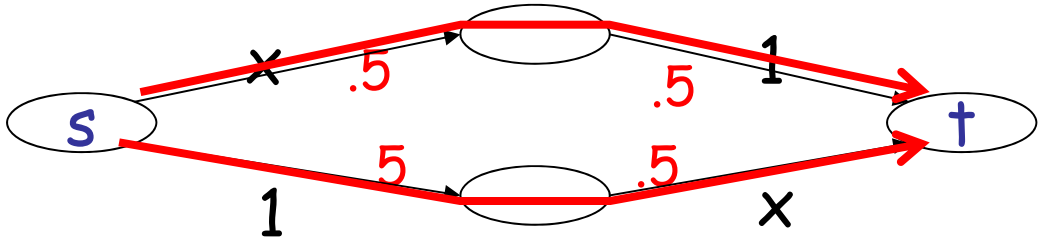
Added edge:



Effect?

Braess's Paradox

Original Network



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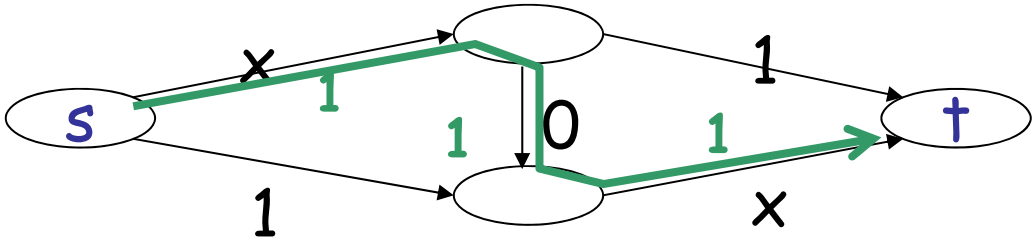
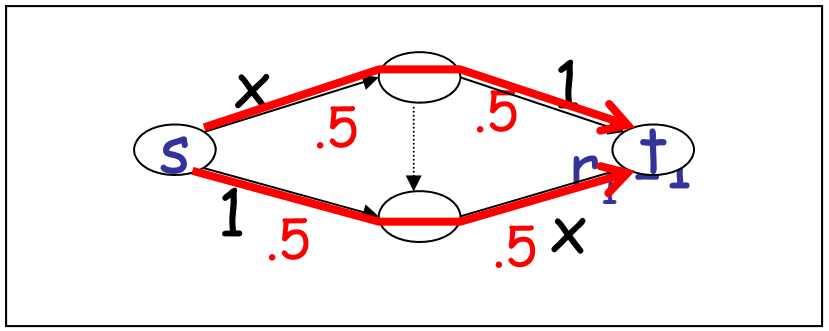


Cost of Nash flow = 2

All the flow has increased delay!

Model of Routing Game

- A directed graph $G = (V,E)$
- source-sink pairs s_i, t_i for $i=1,..,k$
- rate $r_i \geq 0$ of traffic between s_i and t_i for each $i=1,..,k$

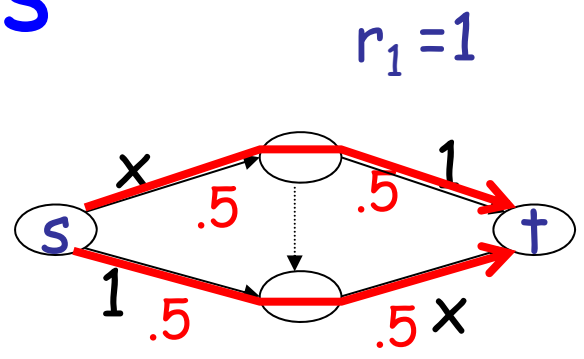


- Load-balancing jobs wanted min load
- Here want minimum delay:

  delay adds along path

  edge-delay is a function $\ell_e(\cdot)$ of the load on the edge e

Delay Functions

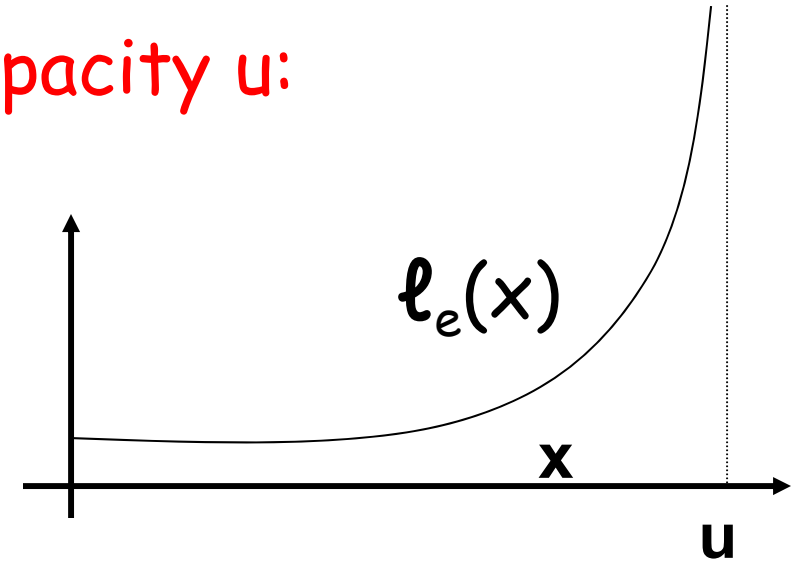


Assume $\ell_e(x)$ continuous and monotone increasing in load x on edge

No capacity of edges for now

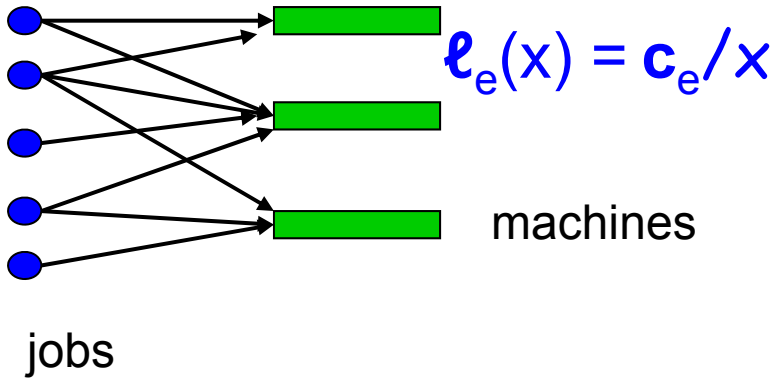
Example to model capacity u:

$\ell_e(x) = a/(u-x)$



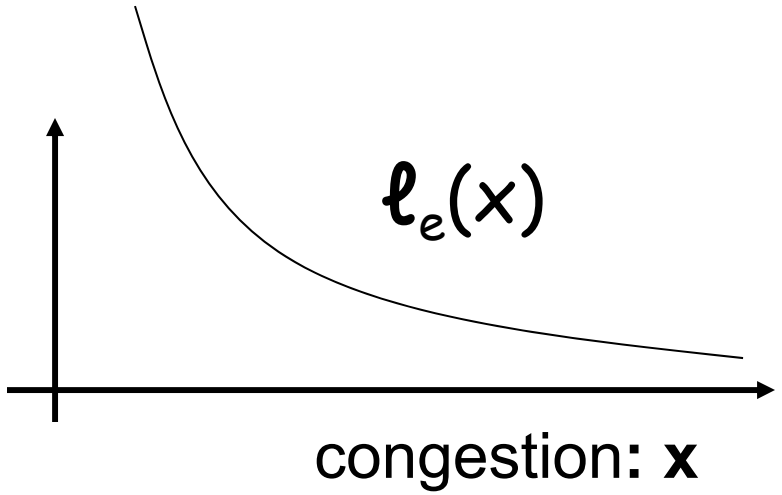
Congestion Games: Cost-sharing

- jobs $i=1,..,k$
- For each machine e a cost function $\ell_e(\cdot)$
  - E.g. cloud computing



- Cost decreasing with congestion (decreasing marginal cost)

$\ell_e(x) = c_e/x$



Goal's of the Game: min delay

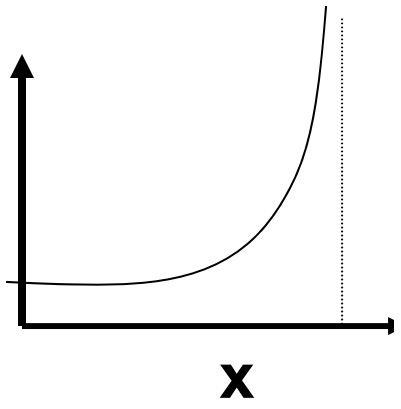
Personal objective: minimize

$\ell_P(\mathbf{f})$ = sum of delays of edges along **P** (wrt. flow **f**)

Overall objective:

C(f) = total delay of a flow **f**: $= \Sigma_P f_P \cdot \ell_P(\mathbf{f})$

= - social welfare
or total/average delay



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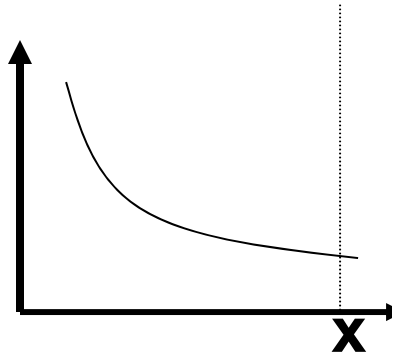


What is Selfish Outcome?

We will use: Nash equilibrium

- Current strategy "best response" for all players (no incentive to deviate)

Theorem [Nash 1952]:

- Always exists if we allow randomized strategies

$$\text{Price of Anarchy} = \frac{\text{cost of worst (pure) Nash}}{\text{"socially optimum" cost}}$$

Connecting Nash and Opt

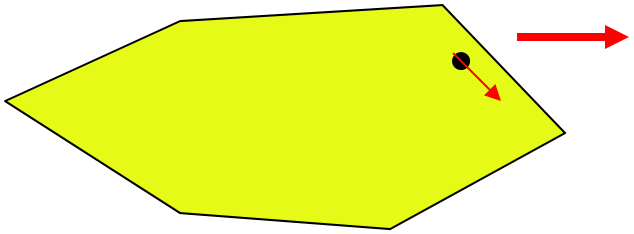
- Min-latency flow
  - for one s-t pair for simplicity

- minimize $C(f) = \Sigma_P f_P \bullet \ell_P(f) = \Sigma_e f_e \bullet \ell_e(f_e)$
- subject to: f is an s-t flow
- carrying r units

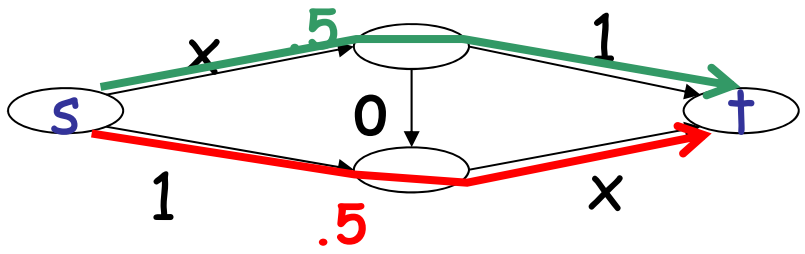
- By summing over edges rather than paths where f_e = amount of flow on edge e

Characterizing the Optimal Flow

Optimality condition (if objective is convex): small change doesn't improve cost



For flows: all flow travels along minimum-gradient paths

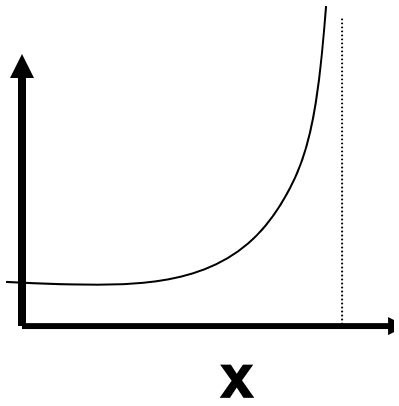


gradient is: $(x\ \ell(x))' = \ell(x) + x\ \ell'(x)$

selfish part

altruistic term

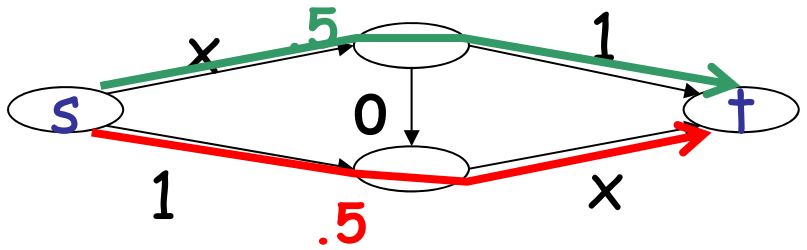
$(x\ \ell(x))$ convex if $\ell(x)$ convex



Optimal versus Nash Flow (convex)

Optimality condition: flow f is at minimum cost iff all flow travels along minimum-gradient paths: gradient is $\ell(x)+x\,\ell'(x)$

selfish part altruistic term



Nash: flow f is at Nash equilibrium iff all flow travels along minimum-latency paths: $\ell(x)$

selfish part only

Nash $\leftrightarrow$ Min-Cost

Corollary 1: min cost is "Nash" with "delay" $\ell(x)+x\,\ell'(x)$

Use of Corollary: If $x\,\ell'(x)$ is changed as tax, selfish users follow optimal paths

Nash $\leftrightarrow$ Min-Cost

Corollary 2: Nash is "min cost" with "cost"

$$\Phi(f) = \Sigma_e \int_0^{f_e} \ell_e(x)\, dx$$

Why?

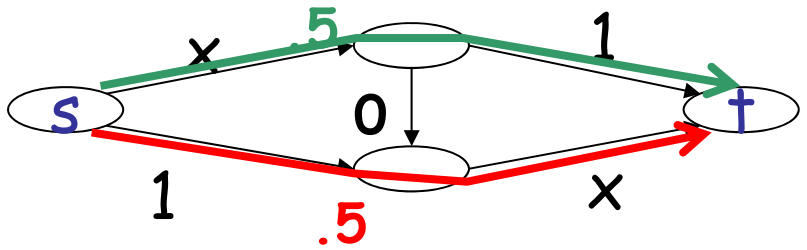
gradient of $\Phi(f)$ is delay:

$$\left(\int_0^{f_e} \ell_e(x)\, dx\right)' = \ell(x)$$

Optimal versus Nash Flow (not-convex)

Local optimality condition: flow f is at minimum cost iff all flow travels along minimum-gradient paths: gradient is $\ell(x)+x\,\ell'(x)$

selfish part altruistic term



Nash: flow f is at Nash equilibrium iff all flow travels along minimum-latency paths: $\ell(x)$

selfish part only

Using function Φ

- The solution minimizing Φ is Nash

Theorem (Beckmann'56)

- In a network latency functions $\ell_e(x)$ that are continuous,
- a deterministic Nash equilibrium exists,
- If f is monotone increasing then it is essentially unique

Using potential Φ...

- Nash minimizes the function Φ
- Hence,

$$\Phi(\text{Nash}) \leq \Phi(\text{OPT}).$$

Suppose that we also know for any solution

$$\Phi \leq \text{cost} \leq \mathbf{A}\ \Phi$$

$\rightarrow$ cost(Nash) $\leq$ ***A*** Φ(Nash) $\leq$ ***A*** Φ(OPT) $\leq$ ***A*** cost(OPT).

$\rightarrow$ the Nash solution has good quality

Example: $\Phi \le \text{cost} \le A\,\Phi$

Example: $\ell_e(x) = x^d$ then

- total delay is $x \cdot \ell_e(x) = x^{d+1}$
- potential is $\int \ell_e(\xi)\, d\xi = x^{d+1}/(d+1)$

More generally: delay $\ell_e(x)$ degree d polynomial:

- ratio at most d+1

Sharp bound: price of anarchy for degree d polynomials is $O(d/\log d)$.

Sharper results for non-atomic games

Theorem 1 (Roughgarden-Tardos)

- In a network with linear latency functions
  - i.e., of the form $\ell_e(x)=a_e x+b_e$
- the cost of a Nash flow is at most 4/3 times that of the minimum-latency flow

Sharper results for non-atomic games

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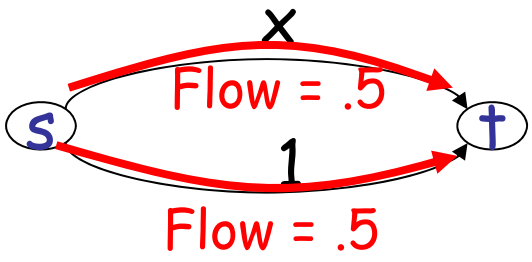
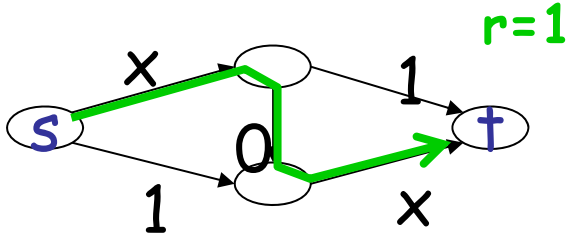
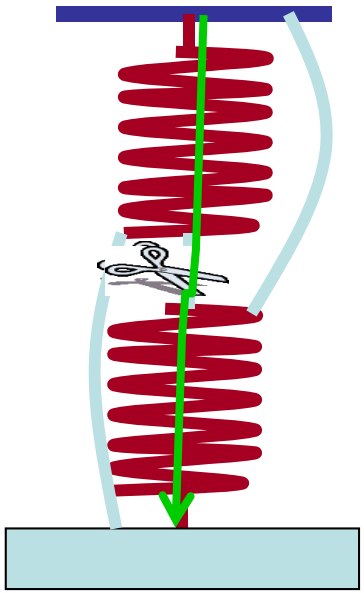




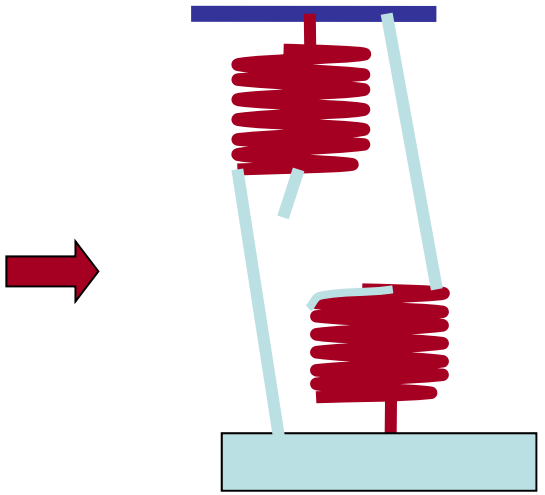
Nash cost 1 optimum 3/4

Nash cost 2 optimum 1.5

Braess paradox in springs (aside)



Cutting middle string

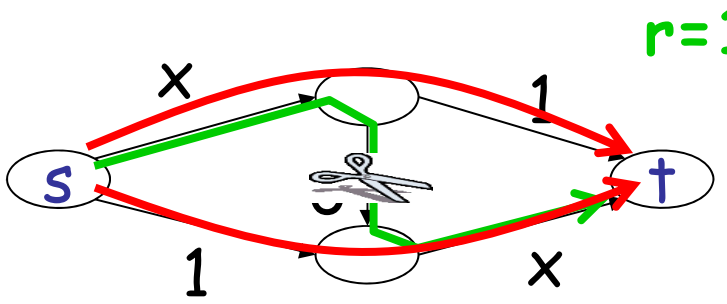


makes the weight rise

power flow along springs

Flow=power; delay=distance

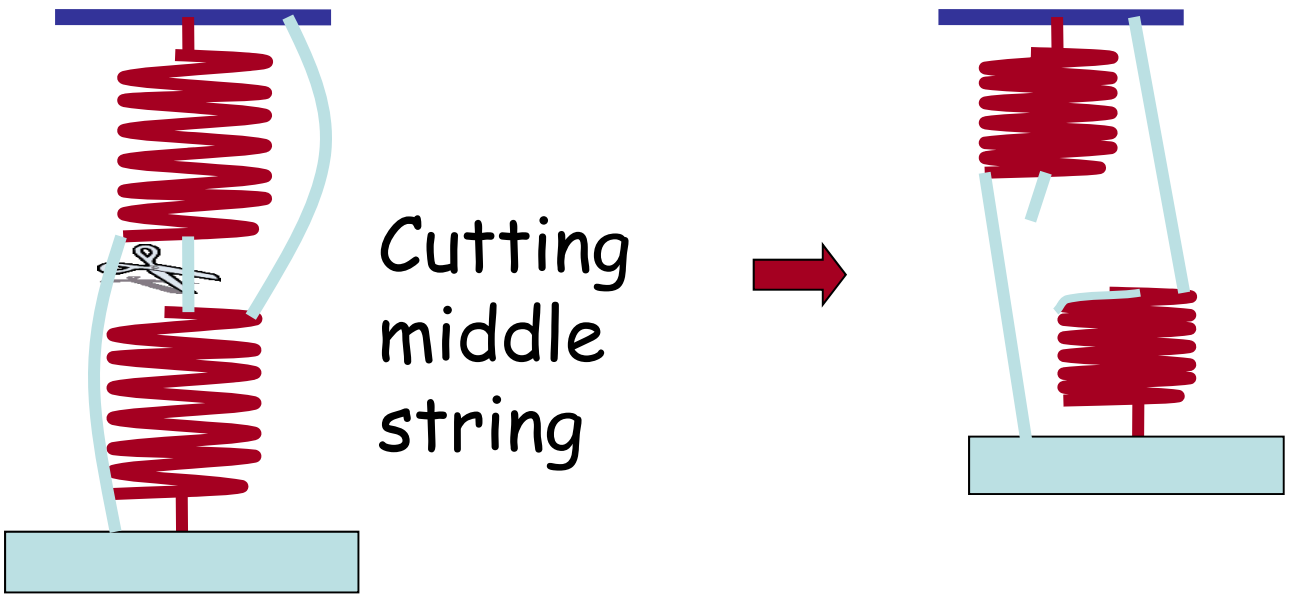
Power conserves,
transfers only along min path



Bounds for spring paradox

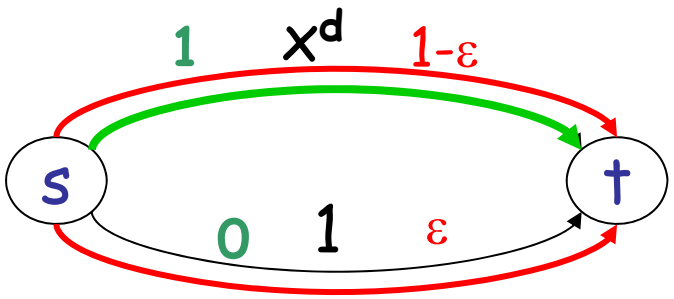
Theorem 1′ (Roughgarden-Tardos)

In a network with springs and strings cutting some strings can decrease the stretch by at most a factor of 4/3.



General Latency Functions

- **Question:** what about more general edge latency functions?
- **Bad Example:** (r = 1, d large)



A Nash flow can cost arbitrarily more than the optimal (min-cost) flow

For non-atomic games

Theorem 3 (Roughgarden-Tardos):

- In any network with continuous, nondecreasing latency functions

cost of Nash with rates r_i for all i	$\leq$	cost of opt with rates $2r_i$ for all i

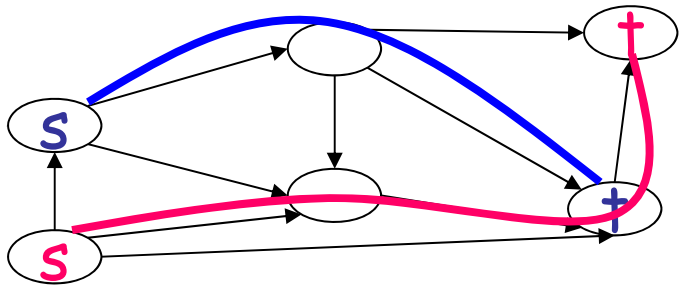
Proof idea:

Opt may cost very little, but marginal cost is as high as latency in Nash

$\rightarrow$ Augmenting to double rate costs at least as much as Nash

Atomic (discrete) Analog

- Each user controls one unit of flow, and
- selects a single path



Theorem Change in potential is same as function change perceived by one user

[Rosenthal'73, Monderer Shapley'96,]

$\Phi(\mathbf{f}) = \Sigma_e (\ell_e(1)+\ldots+ \ell_e(f_e)) = \Sigma_e \Phi_e$

Even though moving player ignores all other users

[Recall continuous potential: $\Phi(f) = \Sigma_e \int_0^{f_e} \ell_e(x)\, dx$]

Corollary: Nash equilibria are local min. of $\Phi(\mathbf{f})$

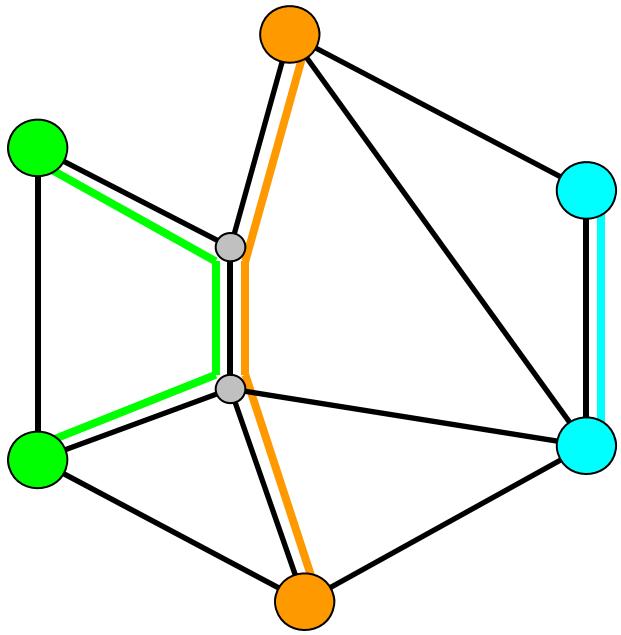
Network Design as Potential Game

Given: $G = (V,E)$,
costs $c_e(x)$ for all $e \in E$,
k terminal sets (colors)

Have a player for each color.

Each player wants to build a network in which his nodes are connected.

Player strategy: select a tree connecting his set.

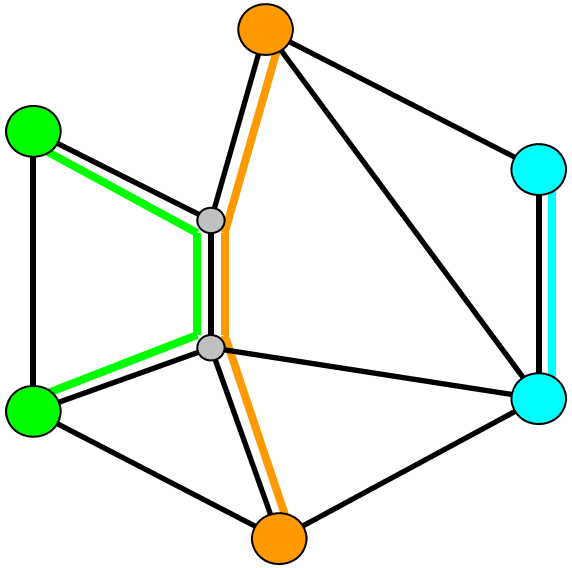


Costs in Connection Game

Players pay for their trees,
want to minimize payments.

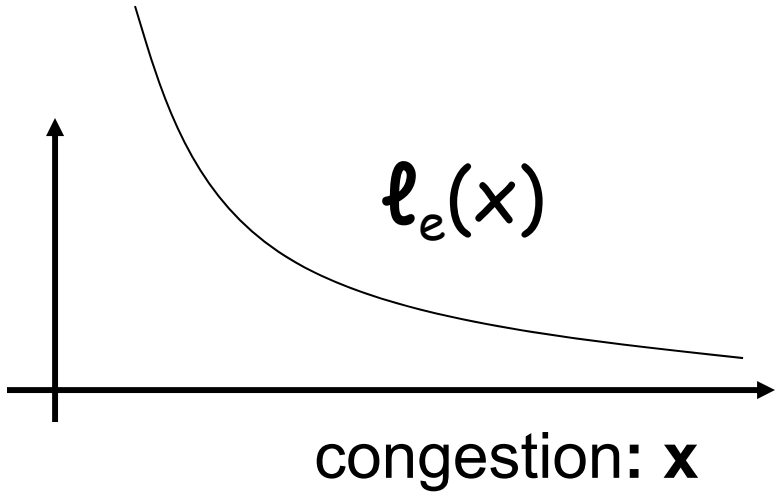
What is the cost of the edges?

$c_e(x)$ is cost of edge e for x users.

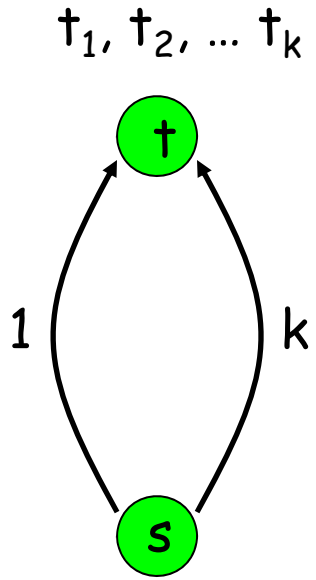


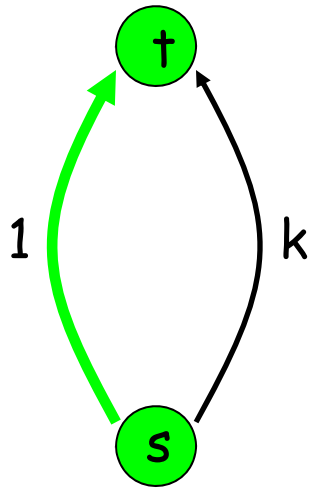
Assume economy of scale and fair sharing:

e.g.: $\ell_e(x) = c_e(x)/x$

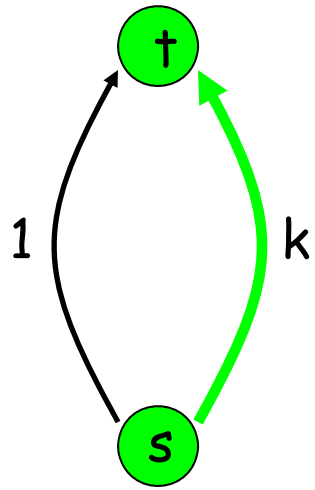


A Simple Example





One NE:
each player
pays 1/k



Another NE:
each player
pays 1

Results for Network Design

Theorem [Anshelevich, Dasgupta, Kleinberg, Tardos, Wexler, Roughgarden FOCS'04]

There exists equilibrium with cost $\leq O(\log k)$Opt for k players (bound sharp)

Proof

$$\text{cost} \leq \Phi \leq \text{cost} \cdot O(\log k)$$

Why? cost c while $\Phi = c+c/2+c/3+\ldots=c\, H_k$

$$\text{Price of Stability} = \frac{\text{cost of best selfish outcome}}{\text{“socially optimum” cost}}$$

Summary: Potential games and the Price of Anarchy

Potential games: Player best response improves a potential function Φ

$\Rightarrow$ Nash are local optima of the potential function Φ

$\Rightarrow$potential function's true optima is high quality Nash.

But why care about Best Nash/Opt ratio?

Best Nash is good quality…

$$\text{Price of Stability} = \frac{\text{cost of best selfish outcome}}{\text{“socially optimum” cost}}$$

$$\text{Price of Anarchy} = \frac{\text{cost of worst selfish outcome}}{\text{“socially optimum” cost}}$$

Potential argument $\Rightarrow$ Low price of stability

But do we care?

Atomic Game: Price of Anarchy?

Non-atomic game: Nash is unique…

Atomic Nash not unique! Design with constraint for stability

What about other Nash equilbria?

- Discrete routing
- Cost sharing

Theorem: Can be bounded for some classes of delay functions

e.g., polynomials of degree at most d at most exponential in d: $O(2^d\, d^{d+1})$ Awerbuch-Azar-Epstein, Christodoulou-Koutsoupias STOC'05

Proof technique

- bounds price of anarchy (not stability)
- Tight bounds in many games

A game is (λ,μ)-smooth if, for every pair f, f^* outcomes ($\lambda > 0$; $\mu < 1$):

$$\Sigma_e f^*_e \cdot \ell_e(f_e) \leq \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$$

Cost of f^* | Cost of f

Proof technique

- bounds price of anarchy (not stability)
- Tight bounds in many games

A game is (λ,μ)-smooth if, for every pair f, f^* outcomes ($\lambda > 0$; $\mu < 1$):

$$\Sigma_e f^*_e \cdot \ell_e(f_e) \leq \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$$

or for all $f, f^* \geq 0$

$$f^* \cdot \ell(f) \leq \lambda f^* \cdot \ell(f^*) + \mu f \cdot \ell(f)$$

Discrete version

Smooth for flows:

$\Sigma_e f^*_e \cdot \ell_e(f_e) \leq \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$

A game is (λ,μ)-smooth if, for every pair s,s^* outcomes

$\Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda \text{ cost}(s^*) + \mu \text{ cost}(s)$

Where $\text{cost}(s) = \Sigma_i C_i(s)$

s_i strategy of user i

s_{-i} strategies of all users

Discrete version

Smooth for flows:

$\Sigma_e f^*_e \cdot \ell_e(f_e) \le \lambda \Sigma_e f^*_e \cdot \ell_e(f^*_e) + \mu \Sigma_e f_e \cdot \ell_e(f_e)$

A game is (λ,μ)-smooth if, for every pair s,s^* outcomes

$\Sigma_i C_i(s^*_i, s_{-i}) \le \lambda\ \text{cost}(s^*) + \mu\ \text{cost}(s)$

Cost of s^* | Cost of s

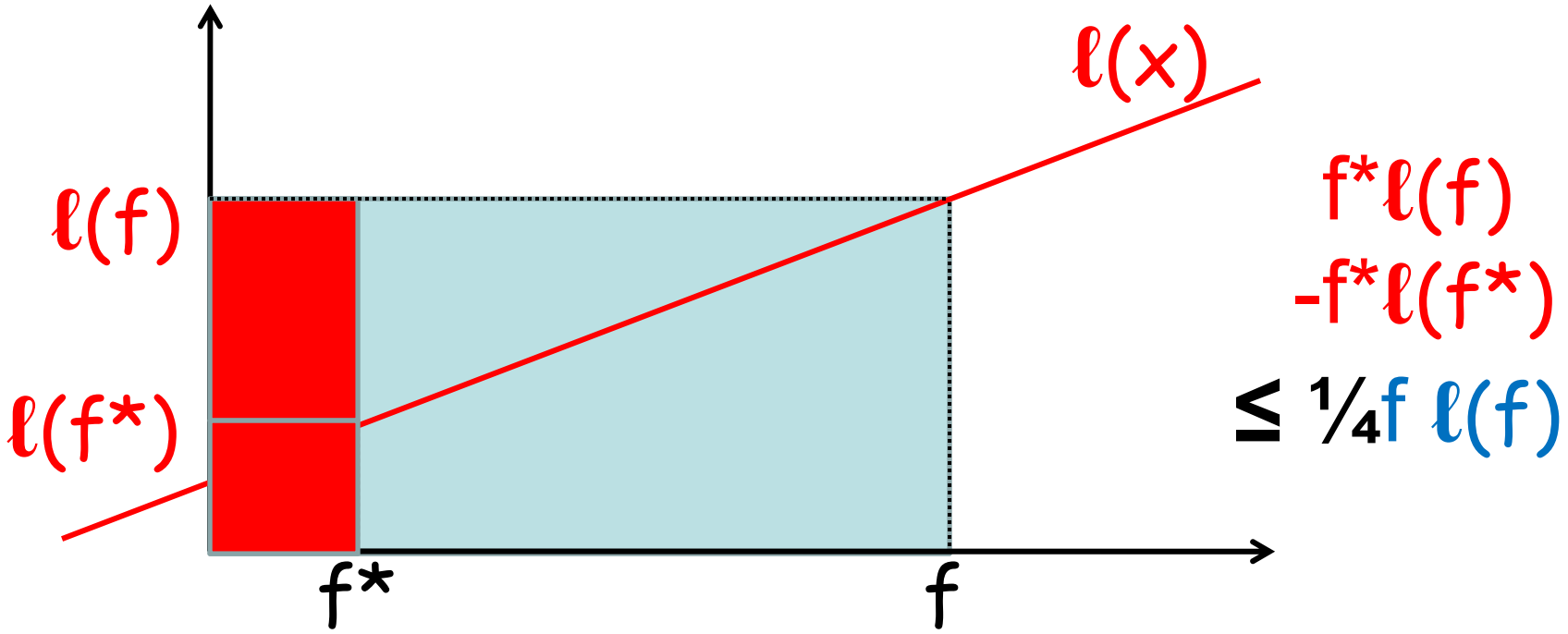
If s^* has a lot smaller cost than s then single player moves capture the improvement

assuming $\mu<1$

Linear delay is smooth

Claim: $f^* \cdot \ell(f) \le f^* \cdot \ell(f^*) + \frac{1}{4} f \cdot \ell(f)$

assuming $\ell(f)$ linear: $\lambda = 1$; $\mu = \frac{1}{4}$



Smooth ⇒ Price of Anarchy

[Roughgarden]

Use smooth for s = Nash and s^* = opt

$\Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda\ \mathbf{cost(s^*)} + \mu\ \mathbf{cost(s)}$

$$\begin{aligned} \text{cost}(s) &= \Sigma_i C_i(s_i, s_{-i}) \\ &\leq \Sigma_i C_i(s^*_i, s_{-i}) \qquad [s \text{ a Nash eq}] \\ &\leq \lambda\ \text{cost}(s^*) + \mu\ \text{cost}(s) \qquad [\text{smooth}] \end{aligned}$$

Then: $\text{cost}(s) \leq \lambda/(1-\mu)\ \text{cost}(s^*)$

Note: used for s^* = opt only!

Atomic Smoothness Bound

atomic linear delay smooth

$\Sigma_i C_i(f^*_i, f_{-i}) \le \lambda\ \mathbf{cost(f^*)} + \mu\ \mathbf{cost(f)}$

Consider edge by edge:

(nonatomic version):

$f^* \cdot \ell(f) \le \lambda\ f^* \cdot \ell(f^*) + \mu\ f \cdot \ell(f)$

Atomic version

$f^* \cdot \ell(f+1) \le \lambda\ f^* \cdot \ell(f^*) + \mu\ f \cdot \ell(f)$

basic inequality: $y(z+1) \le (5/3)y^2 + (1/3)z^2$

Implicit Smoothness Bounds

Examples: selfish routing, linear cost fns.

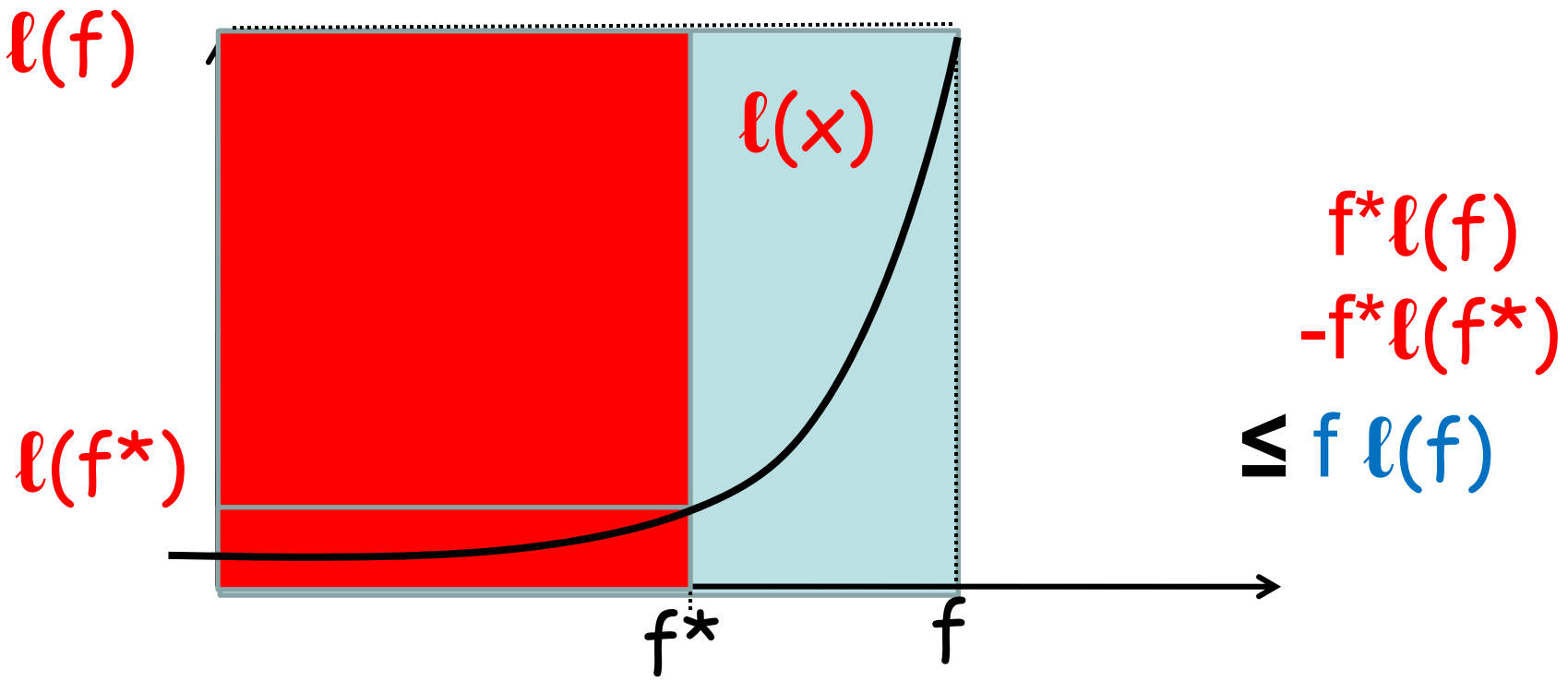
- every nonatomic game is (1,1/4)-smooth
  - follows directly from analysis in [Correa/Schulz/Stier Moses 05]
  - Implies a $\frac{3}{4}=1/(1-\frac{1}{4})$ bound on Price of Anarchy
- every atomic game is (5/3,1/3)-smooth
  - follows directly from analysis in [Awerbuch/Azar/Epstein 05], [Christodoulou/Koutsoupias 05]
  - Implies a 5/2 bound on Price of Anarchy

Theorem [Roughgarden 09] for congestion game the best such bound tight

General increasing delay?

Any increasing function is (1,1)-smooth

$$f^* \cdot \ell(f) \le f^* \cdot \ell(f^*) + f \cdot \ell(f)$$



Summary

Congestion games are potential games

- $\exists$ Pure equilibria (min of potential)
- Min of potential has OK quality
- Price of stability (or anarchy when unique)
- Smoothness and stronger Price of anarchy bounds
  - Applies to some other games also

Tomorrow:

- Learning in games (why and how?)
- solutions reached via learning