Course title: Bordism and topological field theory

Type of course: Reading course

Instructor: Pranav Pandit

Abstract

Quantum field theory has proven to be a wildly successful framework for describing a wide variety of phenomena occuring in Nature. Despite this success, we do not yet have a satisfactory understanding of what a quantum field theory is, and of the mathematical structures underpinning the physical framework.

While the general quantum field theory may be beyond the reach of current mathematics, there exist classes of quantum field theories that are amenable to rigorous mathematical study. One such class is that of *topological* quantum field theories (TQFTs, or simply TFTs for short). These are theories where observable physical quantities depend only on the topology of a 'configuration', and are insensitive to perturbations of geometric features of the configuration, such as a metric. The study of TFTs can shed light on various aspects of more realistic quantum field theories in multiple ways, some of which we may touch upon during the course. In work that led to his Fields medal, Witten poineered the use of TFTs as a framework in which to discover (and prove) deep statements in geometry and topology.

TFTs were axiomatized by Atiyah, following ideas of Segal about *conformal* field theories, as functors from a certain category of manifolds and cobordisms between them to the category of vector spaces. Within this framework, one can give a complete classification of two dimensional theories ($2d$ TFTs): the data of a $2d$ TFT is equivalent to the data of a certain type of algebraic structure called a commutative Frobenius algebra. However, such classification theorems do not exist in higher dimensions. Morally, the reason for this is that the Atiyah-Segal axiomatization does not adequately model the notion of locality in quantum field theory.

Baez-Dolan [BD95] and Freed [Fre94] advocated the idea that in order to capture the notion of a fully local quantum field theory, one should work with *higher* categories of cobordisms and higher algebraic analogues of vector spaces. Higher categories are, in a certain precise sense, generalizations of homotopy types; thus, their study belongs to the realm of algebraic topology and homotopy theory. The ideas of Baez-Dolan and Freed received a rigorous formulation in Lurie's definition of an *extended* topological field theory as a functor from a higher category of cobordisms to a fixed higher category $\mathcal{C}$. Lurie also went on to give a complete classification of extended topological field theories in all dimensions [Lur09]. This classification theorem is often called the *cobordism hypothesis*.

The central goal of this course will be to understand the statement of the cobordism hypothesis and some of its ramifications. The bulk of our time will be spent learning some of the algebraic topology underpinning this theorem, with a focus on the theory of (co)bordism.

Syllabus

The core topics for this course will be:

- Cobordism as a generalized cohomology theory, basic homotopy theory, spectra
- The Pontrjagin-Thom construction (reducing cobordism to homotopy theory)
- The Atiyah-Segal axiomatization of topological quantum field theories
- The classification of 2d TQFTs in the Atiyah-Segal framework.
- The notion of an *extended* topological field theory, and the statement of the classification theorem for such theories (*the cobordism hypothesis*).

Possible advanced topics, depending on the time available and the interests of the participants, include:

- Extended 2d TFTs appearing in topological string theory; Calabi-Yau A_∞-categories.
- Constructing 3d TFTs from modular tensor categories; examples of interest in condensed matter physics.
- Factorization algebras (algebras of observables) and factorization homology

Prerequisites

- Experience with constructing mathematical arguments and writing mathematical proofs.
- Point-set topology, the rudiments of differential topology at the level of [Mil97]
- Linear algebra, basic concepts of elementary abstract algebra [Art91].
- Basic algebraic topology (acquaintance with the key ideas of chapters 1-2 of [FF16] or chapters 1-3 of [Hat02]). Familiarity with homology/cohomology can be substituted with a willingness to work very hard: we will begin the semester with a rapid crash-course on these topics.

Evaluation

The evaluation for this course will be based on:

- *Homework assignments* (20% of the grade). There will be a weekly assignment of at most 5 problems.
- *Exams* (40% of the grade). There will be two take-home exams: one mid-term and a final exam.
- *Blackboard presentations* (20% of the grade). You will lecture on material relevant to the course (e.g., research papers), approximately once a month.
- *Expository write-up* (20% of the grade). A short (approximately 15 - 20 pages long) exposition of one of the advanced topics.

Collaboration on homework problems is encouraged, but solutions must be written up independently. The solutions to homework and exam problems should be typed up in LaTeX and emailed to the instructor before the deadline.

Texts and references

We will mainly follow the lecture notes [Fre12], aiming to cover about two lectures/chapters per week. The expository papers [Fre13, Lur09] will be our main references for the cobordism hypothesis. The following references will also be useful:

Basic references:

- For basic algebraic topology: [Hat02], [May99], [FF16], [BT82].
- For cobordism theory: [Koc96], [Sto68], [Ebe12].
- For ordinary (non-extended) TQFT: [Ati88], [Koc04].

Advanced topics

- 2d open-closed theories: [Cos07], [MS06], [KS09], [LP08].
- 3d theories, tensor categories, lattice models from condensed matter physics: [BJ10], [BJ12], [KK12], [FHLT10], [FH16], [Wit89].

References

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