Structure of the ICTS interdisciplinary mathematics Ph.D. program

Core topics

These topics would be a part of the required knowledge base for all students of the ICTS-TIFR *interdisciplinary mathematics Ph.D. program.* The list here does *not* have a one-one correspondence to courses. Rather, depending on each student's background, they will be advised to take courses, *in the first two years,* on topics they are not familiar with and only take the qualifying examination on the topics with which they are well-acquainted.

- **Analysis**: *Measure and integration*: convergence theorems; product measure and Fubini's theorem; Borel measures on locally compact Hausdorff space, and Riesz representation theorem; Lebesgue measure; regularity properties of Borel measures; Haar measures - concept and examples; complex measures, differentiation and decomposition of measures; Radon Nikodym theorem; maximal function; Lebesgue differentiation theorem; functions of bounded variation. *Functional analysis*: Topological vector spaces; Banach spaces; Hilbert spaces; Hahn Banach theorem; open mapping theorem; uniform boundedness principle; bounded linear transformation;linear functionals and dual spaces; L-p spaces; Hölder's inequality; Minkowski inequality; Spectral theorems for bounded normal operators, compact normal operators; Hilbert-Schmidt operators; *Harmonic analysis*: convolutions; approximate identity; approximation theorems; Fourier transform; Fourier inversion formula; Plancherel theorem
- **Linear algebra**: Vector spaces; Linear transformations; Linear operators; Jordan canonical form; Cayley-Hamilton theorem; Bilinear forms; Spectral theorem; fundamental subspaces of a matrix, spectral theory for symmetric matrices, Unitary operators, Gershgorin circle theorem, Fundamental theorem of linear algebra; Numerical methods in linear algebra; Condition number of a linear operator, Gaussian elimination, QR factorization, least squares, Singular value decomposition, Cholesky decomposition, Unsymmetric and Symmetric eigenvalue decomposition and problems, Diagonalization, Reduction to Hessenberg form or tridiagonal form, QR algorithm with and without shifts, Computing the SVD, Iterative methods, Arnoldi iteration, Lanczos iteration, Conjugate gradient, Generalized Minimum Residual method, Preconditioning, Fast algorithms for structured matrices, Fast algorithms for low-rank (separable) compression, Algorithms for large sparse matrices, Compressed sensing, Tensor decompositions, Multigrid, Hierarchical matrices, Fast multipole method
- **Complex analysis**: Cauchy-Riemann equation and holomorphic functions; basic properties of holomorphic functions; open mapping theorem; maximum modulus theorem; zeros of holomorphic functions, Weierstrass factorisation theorem Riemann mapping theorem; meromorphic functions; essential singularities; Picard's theorem
- **Topology and differential geometry**: *General and metric topology*: Proper maps; quotient space construction; examples of spheres, real and complex projective spaces, Grassmannians; normal and Hausdorff spaces; paracompact spaces; topological groups and continuous actions; *Homotopy theory*: Covering spaces; homotopy of maps, homotopy equivalence of spaces, contractible spaces, deformation retractions; fundamental group: universal cover and lifting problem for covering maps; Van Kampen's theorem, Galois coverings; *Smooth manifolds*: Differentiable manifolds, differentiable maps and tangent spaces, regular values

and Sard's theorem, submersions and immersions, vector fields and flows, exponential map, Frobenius theorem, Lie groups and Lie algebras, exponential map, Homogeneous spaces, tensors and differential forms, exterior algebra, Lie derivative, Orientable manifolds, Integration on manifolds and Stokes Theorem, Covariant differentiation, Riemannian metrics, Levi-Civita connection, Curvature and parallel transport, spaces of constant curvature.

- **Algebra**: *Groups*: Jordan Holder theorem; solvable groups; symmetric and alternating groups; nilpotent groups; groups acting on sets; Sylow theorems; free groups; *Rings and modules*: Polynomial rings, Zeros of polynomials, Elementary symmetric functions and Fundamental Theorem on Symmetric Functions; resultants and discriminants, Euclidean rings, Principal ideal domains and Factorial rings, Factorization in polynomial rings: *Field theory*: Finite Fields, Finite and Algebraic extensions. Algebraic closure, Algebraically closed fields. Proof of Fundamental Theorem of Algebra. Separable polynomials and Separable extensions. Splitting fields, Normal extensions, Galois extensions.
- **Differential equations**: Linear ODE with constant and periodic coefficients; nonlinear ODE existence and uniqueness; stability for fixed points and periodic orbits and Floquet theory, stable manifold theorem, bifurcations, centre manifolds, normal forms, Poincare recurrence, limit sets and attractors, Lyapunov exponents and vectors, uniformly and non-uniformly hyperbolic systems; Partial differential equations: first and second order PDE; Classification; Distribution theory; Numerical methods for ODE and PDE
- **Probability**: Probability spaces, random variables, laws of large numbers, convergence, limit theorems, conditional probability and expectation, Markov chains, martingales, large deviations, basic stochastic processes

6. Advanced topics

These would be a part of the optional knowledge base for the Ph.D. students, and the choice will be made in consultation with the student mentor / potential Ph.D. advisor. Some potential courses on these topics, to be offered by ICTS faculty, are listed at the end of this document.

- Advanced probability and stochastic processes
- Dynamical systems
- Optimization
- Advanced differential geometry and topology
- Scientific computations
- Classical mechanics and fluid dynamics
- Topics in mathematical physics

Advanced courses to be offered by ICTS-TIFR faculty (not every year), in addition to “core courses”

- **Dynamical systems**: Flow of ODE, discrete-time maps, linearization, stability for fixed points and periodic orbits and Floquet theory, stable manifold theorem, bifurcations, centre manifolds, normal forms, Poincare recurrence, limit sets and attractors, Lyapunov exponents and vectors, uniformly and non-uniformly hyperbolic systems
- **Data assimilation and filtering theory:** Martingales, stochastic processes, Ito integral, stochastic differential equations, bayes theorem, nonlinear filtering, Kushner and Zakai equations, Kalman-Bucy filter, parameter estimation and hypothesis testing, least squares and regression, particle filtering, methods for high-dimensional filtering problems
- **Geophysical fluid dynamics:** Fluid dynamical equations, rotation and stratification effects, instabilities and waves, Hamiltonian dynamics and other special topics
- **Nonlinear waves and Coherent structures:** Methods for nonlinear partial differential equations (PDEs) leading to coherent structures and patterns. Includes symmetries, conservations laws, Hamiltonian and variational methods for PDEs; interactions of structures such as waves or solitons; Lax pairs and inverse scattering; and Painleve analysis. Bifurcation theory and formation of patterns. Stability analysis of coherent structures.
- **Calculus of variations:** Necessary and sufficient conditions for a weak and strong extremum. Legendre transformation, Hamiltonian systems. Constraints and Lagrange multipliers. Space-time problems with examples from elasticity, electromagnetics, and fluid mechanics. Sturm-Liouville problems. Approximate methods. Introduction to control problems.
- **Uniform transform method and Integrable systems:** Introduction to the Uniform Transform Method (UTM), Lax pairs, boundary-value problems for linear PDEs in 1+1 variables, extensions to systems, mixed derivative and variable coefficient PDEs, moving and free boundary value problems, transform pairs, generalized Dirichlet-to-Neumann operators and inverse problems, UTM for nonlinear integrable equations.
- **Perturbation methods for differential equations and integrals:** Regular and singular points of differential equations. Asymptotic expansions for solutions of linear ordinary equations. Regular and singular perturbations. Asymptotic evaluation of integrals. Boundary layers and the WKB method. The method of multiple scales. Applications to physical systems. Derivation of reduced and approximate models.
- **Approximation theory**: Uniform approximation (Weierstrass Approximation Theorem), least squares approximation, least first power approximation, polynomial and spline interpolation, Remez algorithm, rational approximation, quadratures
- **Optimization**: Fundamentals (global, local extremum, stochastic and deterministic optimization), unconstrained optimization, line search methods, trust region methods, conjugate gradient methods, Newton methods, Quasi-Newton methods, Nonlinear least squares, nonlinear equations, constrained optimization, Linear programming: Simplex method, Interior-Point methods, Nonlinear constrained optimization, quadratic programming, Penalty, Barrier, and Augmented Lagrangian Methods, Sequential Quadratic Programming

- **Spectral methods in computational physics**: Basics, Chebyshev and Fourier series, Galerkin & Weighted Residual Methods, Polynomial interpolation and collocation, cardinal functions, Pseudospectral Methods for BVPs, Linear Eigenvalue Problems, Explicit Time-Integration Methods, Partial Summation, the FFT and MMT, Aliasing, Spectral Blocking, & Blow-Up
- **Computational statistics**: Monte Carlo Methods for Statistical Inference, Data Randomization, Partitioning, and Augmentation, Bayesian inversion, Bootstrap Methods, Estimation of Probability Density Functions Using Parametric Models, Nonparametric Estimation of Probability Density Functions, Statistical Learning and Data Mining
- **Fast algorithms in applied mathematics and statistics**: Introduction to analysis-based fast algorithms and their applications in computational science; Fast Gauss transform, Generalized Rybicki-Press algorithm, fast Fourier transform, non-uniform fast Fourier transform, hierarchical matrices, fast matrix vector products and fast direct solvers. The course will draw material from functional analysis, special function theory, asymptotics, potential theory, and matrix computations. The primary focus will be on computational statistics and the solution of the partial differential equations of electromagnetics, elasticity, and fluid dynamics using integral equation methods.
- **Stochastic Processes**: Discrete and continuous time Markov chains, poisson processes, random walks, branching processes, first passage times, recurrence and transience, stationary distribution, Renewal theory, Brownian motion, Gaussian processes, second order processes, martingales.
- **Advanced Topics in Differential Geometry**: Curves and surfaces; higher dimensional manifolds; Riemannian geometry; symplectic and Poisson geometry.
- **Topics in mathematical physics**: Moduli spaces coming from physics; instantons and monopoles; quantization of moduli spaces; integrable systems.
- **Advanced topics in Topology**: Homology (simplicial, singular and cellular); cohomology.