CLUSTERING OF HEAVY PARTICLES IN VORTICAL FLOWS: A SELECTIVE REVIEW

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Abstract: Heavy particles in a turbulent flow tend to leave regions of high vorticity and cluster into regions of high strain. The consequences of such clustering have been studied in a variety of situations over the past few decades, and this problem has seen several review papers already. Our objectives in this paper are three-fold. (i) We introduce the reader to the basic ideas, and explain why the problem is interesting. (ii) Using an N-vortex system we present an interesting case where particles are attracted to the vicinity of vortices. A new scaling for the critical Stokes number of attraction is obtained. (iii) We review a number of papers which are related to cloud physics in this context.

1. Introduction and Outline

When we add milk to tea and stir, we get a homogeneous distribution of milk in tea. But when particles of larger density than the fluid are added into a turbulent flow, they cluster preferentially into some regions and their number density in the fluid is not uniform. This is a well-known phenomenon seen in both experiments (see [18] for the classical experiment, and [33, 2, 32] for more recent experiments) and numerical simulations (see [45] for the earliest simulation and [44, 48, 19, 4, 5, 42, 22, 13] for more recent work). This feature is of great interest since turbulent flows laden with particles are ubiquitous in nature and in industrial applications. Dust storms, sand dunes, snow avalanches, ocean spray, sediment flows and plankton blooms (see e.g. [39, 15]) are just some of the situations of interest.

Decades of research has therefore gone into this area, and several review articles have been written on it (in particular [50, 29, 43, 27, 54, 14, 21]). We do not therefore attempt an exhaustive review of the vast literature on particle clustering. We will focus on the dynamics of small heavy inertial particles (in which the density of the particle ρ_p is much greater than the density of the fluid ρ_f). We are motivated primarily by the dynamics of inertial particles and their relationship to rain formation in clouds; the studies we review will, therefore, be primarily of interest to researchers interested in cloud physics. We have tried to make the paper self-contained.

We restrict ourselves to dilute particle suspensions in incompressible flow. Also the particles of our interest are small enough that the Reynolds number based on the particle radius and relative velocity with the fluid are very small, so the fluid applies Stokes drag on the particle. A one-way coupling, in which the fluid flow determines the particle dynamics, but the particles do not affect the flow, is a fair assumption in this case. The main parameter in the problem, therefore, apart from the flow Reynolds number, is the particle Stokes number $St \equiv \tau/\tau_f$, where $\tau = \frac{2}{9}\frac{a^2}{\nu}\frac{\rho_p}{\rho_f}$ is the Stokes time-lag associated with the particle, τ_f is a typical flow time-scale, a is the particle radius, and ν is the kinematic viscosity of the surrounding fluid. Defining U and L as typical flow velocity and length scales respectively, we have $\tau_f = L/U$ and the flow Reynolds number $Re = UL/\nu$. The Stokes number may be rewritten as

$$St = \frac{2}{9}\frac{\epsilon^2 Re}{R}, \tag{1.1}$$

where we define $\epsilon \equiv a/L$ as the ratio of particle size to flow length scale, and $R \equiv \rho_f/\rho_p$, the ratio of fluid and particle densities. For the flows of our interest, $\epsilon \ll 1$, $R \ll 1$ and $Re \gg 1$.

Droplets of one liquid in another can also be treated as particles in flow, so long as the droplet Reynolds number is much smaller than unity and the Bond number (which characterises the strength of surface tension) is low. Such droplets will remain spherical. Further, if the droplet viscosity is much higher than the viscosity of the surroundings, then Stokes drag is applicable. In this article we will use the terms 'particle' and 'droplet' interchangeably, but what distinguishes droplets from particles is that when droplets collide with each other, they can coalesce.

In section 2, we describe the basic methods used to analyse and study the dynamics of small heavy particles suspended in turbulent flows; and in section 3, we describe recent studies from a variety of sub-fields that use the formulation we have described in order to understand physical phenomena.

2. Heavy Particles in Turbulence

The equation of motion for a small sphere in fluid flow was first proposed by Maxey and Riley [30]. If we include the lift force due to Saffman [40], we have the following equation:

$$\begin{aligned}\rho_p \dot{\mathbf{v}} \;=\;& \rho_f \frac{D\mathbf{u}}{Dt} + (\rho_p - \rho_f)\,\mathbf{g} - \frac{9\nu\rho_f}{2a^2}\left(\mathbf{v} - \mathbf{u} - \frac{a^2}{6}\Delta\mathbf{u}\right) - \frac{\rho_f}{2}\left[\dot{\mathbf{v}} - \frac{D}{Dt}\left(\mathbf{u} + \frac{a^2}{10}\Delta\mathbf{u}\right)\right] \\ &- \frac{9\rho_f}{2a}\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{ds}{\sqrt{t-s}}\left[\dot{\mathbf{v}}(s) - \frac{d}{ds}\left(\mathbf{u} + \frac{a^2}{6}\Delta u\right)_{\mathbf{x}=\mathbf{x}(s)}\right] \\ &+ \frac{6.46\rho_f}{\frac{4}{3}\pi a}\sqrt{\frac{\nu}{|\omega|}}\left((\mathbf{v} - \mathbf{u}) \times \omega\right),\end{aligned} \tag{2.1}$$

where $\mathbf{v}$ and $\mathbf{u}$ are the particle and fluid velocities respectively, ω is the local vorticity, Δ is the Laplacian operator, $\mathbf{g}$ is the acceleration due to gravity, D/Dt is the material derivative following a fluid streamline, and an overdot represents a derivative in time following a particle's trajectory. In order, the right hand side of equation (2.1) has the following terms: inertial force, buoyancy, Stokes drag, the added mass which accounts for the fact that the particle imparts kinetic energy to the surrounding fluid, the Basset history which account for the wake left by the particle along its path, and the Saffman lift term which is analogous to a Magnus force. The terms with a factor $a^2\Delta u$ are called Faxen corrections, and account for changes in the flow over lengthscales of the particle size.

Equation (2.1) in nondimensional form reads

$$\begin{aligned}
\dot{\mathbf{v}} &= R\frac{D\mathbf{u}}{Dt} + \frac{1-R}{Fr} - \frac{1}{St}\left(\mathbf{v} - \mathbf{u} - \frac{\epsilon^2}{6}\Delta\mathbf{u}\right) - \frac{R}{2}\left[\dot{\mathbf{v}} - \frac{D}{Dt}\left(\mathbf{u} + \frac{\epsilon^2}{10}\Delta\mathbf{u}\right)\right] \\
&- \sqrt{\frac{9R}{2\pi St}}\int_0^t \frac{ds}{\sqrt{t-s}}\left[\dot{\mathbf{v}}(s) - \frac{d}{ds}\left(\mathbf{u} + \frac{\epsilon^2}{6}\Delta u\right)_{\mathbf{x}=\mathbf{x}(s)}\right] \\
&+ \frac{10.28}{\pi}\sqrt{\frac{R}{St\,|\omega|}}\left((\mathbf{v}-\mathbf{u})\times\omega\right), \qquad (2.2)
\end{aligned}$$

The Froude number $Fr \equiv U^2/gL$ gives the ratio of inertial to gravitation forces in the flow. Therefore, in the small heavy particle ($\epsilon \ll 1$, $R \ll 1$) limit, if we additionally impose $R \ll St$ and $Fr \gg 1$ the equation governing the motion of the particle reduces to

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \mathbf{v} \\
\frac{d\mathbf{v}}{dt} &= \frac{\mathbf{u}-\mathbf{v}}{St}. \qquad (2.3)
\end{aligned}$$

Several fundamental properties of the dynamics of such particles are known. In particular, the dissipative nature of the particle dynamics results in the particles occupying an ever shrinking region of phase space. This can be seen by noting that the rate of contraction of the volume V of phase space accessible to the dynamical system given by eqns. (2.3) is $\frac{\partial V}{\partial t} = \nabla \cdot J = -N_d/St$ in N_d-dimensional space. Here, J is the current given by the 4 component vector $[\mathbf{v}, (\mathbf{u}-\mathbf{v})/St]^T$, and the operator ∇ consists of partial derivatives with respect to the phase space coordinates, i.e., components of $\mathbf{x}$ and $\mathbf{v}$. Even the region of physical space occupied by the particles shrinks. Bec [4] showed for randomly stirred flows that the space occupied by the particles is a fractal attractor.

Thus we may obtain complex particle dynamics with even a simple linear Stokes drag model, where the drag is proportional to the relative velocity between particle and fluid (2.3). Many simplifications are adopted to derive this model as mentioned above. As long as the particle Reynolds number is $\lesssim 0.01$ [58] , and the particle radius is much smaller than the Kolmogorov scale, this is a good approximation. Direct Numerical Simulations (DNS) of turbulence-particle interaction based on the linear Stokes drag model are shown to accurately capture the experimentally observed spatial clustering of inertial particles in isotropic turbulence [42]. The assumption of linear drag fails for particles with large $St(\gtrsim 3)$ settling in gravity, where nonlinear effects are required to reproduce the experimentally observed behaviour that large particles settle slower in turbulent flow than in a quiescent fluid [20]. However, since most studies assume a linear drag, will limit our discussion to the linear drag model of equation (2.3).

It is obvious that the timescale on which the velocity of the particle adjusts to the flow timescale is very different from the timescale associated with the motion of the particle itself. Haller and Sapsis [24] show that equation (2.3) can be treated as a singular perturbation problem in the Stokes time scale, and that the leading term in the perturbation expansion may be found by rearranging (2.3) to give

$$\mathbf{v} = \mathbf{u} - St\frac{d\mathbf{v}}{dt}, \qquad (2.4)$$

$$\text{i.e.,}\quad \mathbf{v} = \mathbf{u} - St\frac{d\{\mathbf{u} - St\ d\mathbf{v}/dt]\}}{dt}. \tag{2.5}$$

It is standard practice to neglect higher powers of the Stokes number at small Stokes number. Thus, for small enough Stokes number, the motion of the particles is approximated by the so-called inertial equation

$$\mathbf{v} = \mathbf{u} - St\frac{D\mathbf{u}}{Dt}. \tag{2.6}$$

Note that, crucially, equation (2.6) describes the motion of particles in terms of a velocity field, whereas equation (2.3) does not. The consequence of this is that equation (2.6) fails to account for caustics (see section 2.3). This approximation, however, is useful in the small Stokes limit to explain the dynamics, as done below.

2.1. **Heavy particles are expelled from vortical regions .** A direct consequence of equation (2.3) is that heavy particles are expelled out of regions of high vorticity and settle in regions of high strain. To see this clearly, we write equation (2.3) in cylindrical polar ($r-\theta$) coordinates, and model the region of high vorticity as a point vortex at the origin (so that the fluid velocity is $\mathbf{u}_f = \frac{\Gamma}{2\pi}\frac{\mathbf{e}_\theta}{r}$). The location of a particle is its radius vector, $\mathbf{r} = r\mathbf{e}_r$. The velocity and acceleration of the particle are, therefore,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{e}_r\frac{dr}{dt} + \mathbf{e}_\theta r\frac{d\theta}{dt}$$
$$\frac{d\mathbf{v}}{dt} = \mathbf{e}_r\frac{d^2r}{dt^2} + 2\mathbf{e}_\theta\frac{dr}{dt}\frac{d\theta}{dt} - \mathbf{e}_r r\left(\frac{d\theta}{dt}\right)^2 + \mathbf{e}_\theta r\frac{d^2\theta}{dt^2}.$$

Writing the above equations along the coordinate directions, we get

$$\begin{aligned}\frac{dr}{dt} &= v_r\\ \frac{d\theta}{dt} &= \omega\\ \frac{dv_r}{dt} &= -\frac{v_r}{\tau} + r\omega^2\\ \frac{d\omega}{dt} &= \frac{\Omega-\omega}{\tau} - 2\frac{v_r}{r}\omega.\end{aligned} \tag{2.7}$$

Here Ω is the vorticity in the flow, r and θ are the radial and azimuthal position of the particle measured from a given origin, v_r the radial velocity and ω the angular velocity of the particle.

If we assume an axisymmetric velocity field, such as that of a vortex, and that the particle has an initial velocity equal to that of the fluid, equations (2.7) can be used to show that the particle a) maintains the same angular velocity as the fluid; and b) drifts outwards with a velocity v_r that obeys the third of equations (2.7) (see [36]). It follows that all heavy particles are expelled outwards from the centre of a vortex. Another way to see this is as follows.

The flows we consider are incompressible, i.e. $\nabla\cdot\mathbf{u} = 0$. However, since particles cluster into regions of high number density, the particle velocity field $\mathbf{v}$ will not be divergence-free. In regions from where particles are expelled, we must have $\nabla\cdot\mathbf{v} > 0$, and in regions where they collect, we must have $\nabla\cdot\mathbf{v} < 0$. We may take the divergence of the inertial equation (2.6), and use the fact that $\nabla \cdot \mathbf{u} = 0$ to get the divergence of the particle velocity as

$$\nabla \cdot \mathbf{v} = -\tau \nabla \cdot \frac{D\mathbf{u}}{Dt}.$$

This gives, since $D\mathbf{u}/Dt = \partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u}$, and $\nabla \cdot \partial \mathbf{u}/\partial t = 0$,

$$\nabla \cdot \mathbf{v} = -\tau \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}).$$

This can be expanded to (in index notation; note that we've switched to Cartesian coordinates here, but the identity will not change)

$$\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = \frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

We then note that

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij},$$

where $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and $\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ are the symmetric and anti-symmetric parts of the velocity gradient tensor, and this gives

$$\nabla \cdot \mathbf{v} = -\tau\left(S^2 - \Omega^2\right) = \tau Q \tag{2.8}$$

The quantity $Q = \left(\Omega^2 - S^2\right)$ is called the Okubo-Weiss parameter, and predicts where particles in a flow will end up. Using the Okubo-Weiss condition, several things can be predicted about particle clustering. It is evident from equation 2.8 that particles leave regions where $\Omega > S$ and cluster into regions where $S > \Omega$, i.e., a negative value for the Q is a necessary condition for particles to cluster. In fact, Haller and Sapsis [24] show, for a fixed frame of reference, that the Okubo-Weiss parameter cannot be negative in a region with closed streamlines. This can be pictured as saying that heavy particles cannot cluster near a vortex, as shown schematically in figure 2.1.

The simple picture given above no longer holds in rotating frames (as shown in [37]), leading to the following interesting consequences.

2.2. **Attracting fixed points in rotating frames.** In a rotating frame, there can be attracting fixed points in the vicinity of vortices, where a large number of particles can cluster. In particular, these fixed points are within closed streamlines in the rotating frame of reference. In two-dimensional turbulence, vortices often group themselves into twos and threes and these interactions can result in "non-traditional" clustering of particles [37], as shown below.

Consider two identical like-signed point vortices rotating about each other. Given the strength of the vortices, the angular velocity with which they will rotate about a point midway between them is a known constant. In a frame of reference centred at this point and rotating at this angular velocity, therefore, the two vortices are stationary. While we may expect that what is true of one vortex (that heavy particles leave the vortex) is doubly true of a pair of vortices, we find that there exist, in fact, stable attracting fixed points at which the particles can aggregate as long as the vortices exist. The fixed points are found by solving the pair of

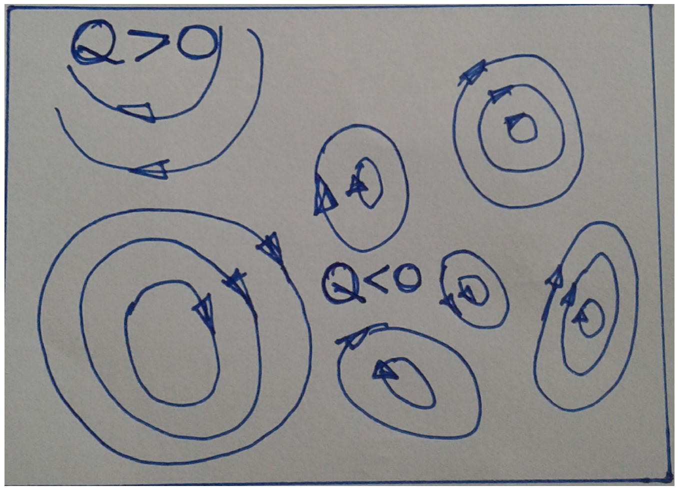

FIGURE 2.1. Vortical regions with closed streamlines, where $Q > 0$ and strain-dominated regions where $Q < 0$. In steady flow heavy particles are always centrifuged out of vortical regions and collect in strain-dominated regions.

equations (where Ω is the angular velocity with which the frame rotates)

$$\frac{d\hat{\mathbf{x}}}{d\hat{t}} = \hat{\mathbf{v}}$$

$$\frac{d\hat{\mathbf{v}}}{d\hat{t}} = \frac{\hat{\mathbf{u}} - \hat{\mathbf{v}}}{St} - 2\vec{\Omega} \times \hat{\mathbf{v}} + \Omega^2 \hat{\mathbf{r}}$$

Note that these are ‘fixed’ points only in the rotating frame of reference. In the lab-fixed frame, these points execute limit cycles. These fixed points are attractive to small heavy particles of Stokes number below a critical value. The location of these fixed points is dependent on Stokes number as well. A very high number of particles arrives at these fixed points, and co-rotates with them. Fixed points for heavy inertial particles of this kind can also be found for regular arrangements of $N > 2$ vortices. Figure 2.2 shows the basin boundaries for the elliptic fixed points that occur next to four like-signed vortices. In this figure there are four fixed points close to the tips of the spiral seen. All particles of Stokes number $1/300$ that start in the red regions settle into one of the four attracting fixed points. Given that the basin of attraction extends over a significant area, it is easy to imagine that the number density of particles is extremely high at the fixed points.

Similarly, in an N-vortex configuration, when N identical point vortices are placed at the vertices of a regular polygon, we have N fixed points. The behaviour of N vortices started out thus is interesting in itself, and the reader is referred to [47]. In the present article we ask how stable the fixed points are, i.e., up to what Stokes number particles can they attract to themselves. Figure 2.3 shows that the critical Stokes number falls exponentially with the number of vortices. As mentioned above, normally vortices interact in smaller groups, so these results suggest that moving fixed points need to be studied further as a mechanism for producing enormous particle clustering into extremely small neighbourhoods.

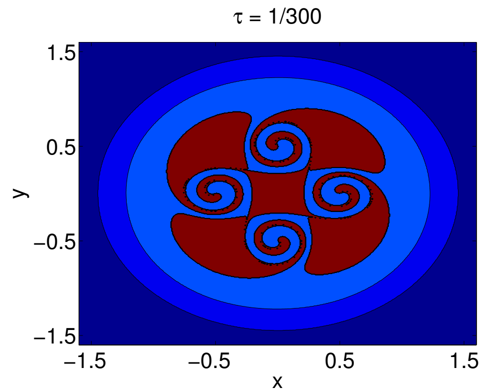

FIGURE 2.2. The basins of attraction of the elliptic fixed points for a system of four identical like-signed vortices are shown in red.

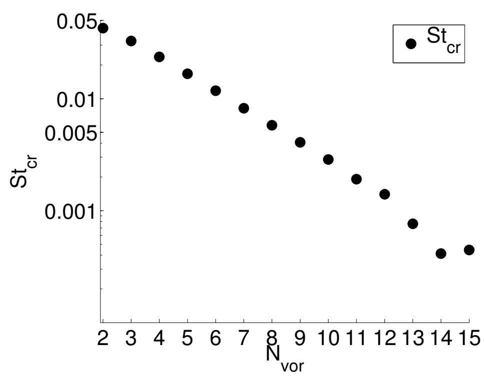

FIGURE 2.3. The critical Stokes number for any number of vortices. Particles below the critical Stokes number cluster into fixed points in the vicinity of the vortices.

2.3. **Caustics.** We noted that expressing the particle velocity as a field and expressing its divergence in terms of the Okubo-Weiss parameter is an approximation at $\mathcal{O}(St)$. In general, however, the velocities of inertial particles in fluid flow do not form a field, and can be multi-valued. That is, two particles can exist at the same location at a given time with (very) different velocities. This phenomenon is called caustics, and has been proposed independently by Mehlig and Wilkinson and coworkers [55] and by Falkovich and coworkers [16] and is beginning to be observed in experiments (see [8]). Caustics are known to play an important role in the clustering of inertial particles for larger Stokes numbers. This is usually understood as being because particles of small Stokes numbers are much more susceptible to preferential concentration by coherent structures while, as the inertia increases, the role of 'ergodic' or non-coherent flow structures becomes ever more important (see [9]) and the clustering of particles depends on particle histories. We now discuss the literature on caustics and other relevant aspects of particle clustering, with most of our attention given to the initiation of rain in clouds.

3. Clustering, collisions and droplet growth in a cloud

As mentioned, our interest is mainly in the applications of the theory of Stokesian particles to water droplets in clouds. We are interested, in particular, in warm cumulus clouds ('warm' because these clouds have no ice, and 'cumulus' because they appear like heaps and have significant turbulence). Cumulus clouds that are capable of bearing rain have relatively short lifetimes–they go from being essentially large parcels of moist air to clouds that rain in half an hour or less. No complete explanation has been found for this rapidity of onset of rain.

Clouds start with condensation nuclei (salt or dust particles) that are fractions of a micron in diameter. These nuclei initially grow by vapour diffusion in the supersaturated environment afforded them by the cloud. This rate of growth by vapour diffusion is inversely proportional to the particle radius; for particles larger than $10-20\mu$, the rate of growth is much too small. At the other end of the droplet size range, droplets that are about 50μ or larger have significant rates of collision with other droplets (either of their own size or with smaller droplets), and can grow relatively quickly. The problem of explaining how droplets grow from about 10μ to about 50μ on timescales short enough to explain rapid rain formation is a long-unsolved problem of much interest.

The first attempts at explaining the 'droplet growth bottleneck' invoked only the fact that inertial particles preferentially sample regions of the flow with low vorticity or high strain. Shaw etal's paper [44] argued that the regions of the flow with high vorticity would also end up with greater supersaturations than high-strain regions, and therefore be able to nucleate and grow particles much faster. Several objections were raised against this prediction, including by Vaillancourt, Yau, Grabowski and co-workers [50, 49, 48] who point out in the latter two papers that the decoherence times fall as the Reynolds numbers increase, meaning that particles that may have sampled a high supersaturation region at one instant will soon sample a region with lower supersaturation (see also the review by Grabowski [21]). In short, preferential concentration by itself is not enough to explain the droplet size broadening observed in clouds. And, therefore, it cannot, by itself, explain why rain initiation times are short. However, preferentially concentrated droplets are, it may be imagined, prone to larger rates of collision; and if enough collisions occur, particles can quickly grow in size. Preferential concentration remains of indirect interest, therefore.

In addition to clustering, inertial particles in turbulent flows also have higher rates of collision than that for particles that follow fluid streamlines ('fluid particles'), first derived by Saffman and Turner [41] (see equation 3.3). The first numerical study of collision rates was done by Sundaram and Collins [46], followed by a large number of other studies. The phenomenon of caustics (section 2.3) is thought to account for the increased rate of collisions, is the focus of much research on turbulent particle-laden flows (see, e.g. [57, 31, 17, 51, 52]). The line of reasoning followed by such studies is the following.

Consider a reference particle relative to which the rest of the flow and the particles move. If all n particles per unit volume in the flow are of the same radius a, the collision zone of the reference particle is a sphere of radius $2a$ centered on it. Hence $n4\pi(2a)^2\langle|w_r|\rangle$ is the number of particles coming into its collision zone per unit time. The number of particles colliding with our reference particle per unit time is, then,

$$\mathcal{R} = \frac{1}{2} 4\pi n (2a)^2 \langle |w_r| \rangle = n\Gamma, \tag{3.1}$$

where $\langle |w_r| \rangle$ denotes the average magnitude of the radial component of the velocity of particles relative to the reference particle. The factor $\frac{1}{2}$ accounts for the fact that, in an incompressible particulate flow, only half of the particles on an average are approaching the target particle, and the other half is moving away. This way of finding the collision rate–using a sphere centred on the reference particle–is called the spherical volume formulation, and, in turbulent flows, is found to predict collision rates better than the more familiar cylindrical formulation in which the reference particle is assumed to trace out a cylinder of radius $2a$. For molecules of a gas, the two formulations are identical. For details, see [53].

Here Γ is called the collision kernel and its definition is clear from the above equation. Since we are interested in the growth of droplet size, we should look at the rate of coalescence which can be obtained by multiplying the collision rate by a coalescence efficiency, E to get

$$\mathcal{R} = nE\Gamma = n\kappa, \tag{3.2}$$

where κ is termed as the collection kernel. The coalescence efficiency is the fraction of collision events resulting in droplet coalescence. Due to the lubricating layer trapped between the approaching particles, among other reasons [28], all collisions need not result in coalescence. But in the case of colliding cloud droplets, it is shown that coalescence efficiency is ≈ 1 [3] and in the following discussion, we consider 100% coalescence efficiency. Under this approximation, collision rate is same as coalescence rate.

In the limit of $St \to 0$, the particles follow the fluid streamlines exactly and the collision occurs only due to shearing action; hence, $\langle |w_r| \rangle \sim 2a/\tau_K$, where τ_K is the collision time-scale. Saffman and Turner [41] derived the following exact expression for the collision rate of tracers suspended in isotropic turbulence and approximating the velocity field around the reference particle as hyperbolic and the velocity gradients to be Gaussian-distributed,

$$\mathcal{R}_{ST} = \sqrt{\frac{8\pi}{15}} \frac{n(2a)^3}{\tau_K}. \tag{3.3}$$

Note that this formula does not take into account the effect of the concentration inhomogeneity due to preferential clustering because the velocity field of suspended tracers is divergence free. The effect of preferential concentration at higher St is incorporated by changing the particle density at a distance r from the reference particle to $ng(r)$, where $g(r)$ is a radial correlation function and is defined as the ratio of probability density of finding a pair of particle at distance r in the real system to that in a system with uniform distribution of particles. Hence, for uniform concentration, $g = 1$. For a detailed comparison of different theoretical models describing the spatial distribution and the relative velocities of inertial particles in isotropic turbulence, see refs. [9, 10]. If it is assumed that particle clustering does not significantly change the relative velocity, the rate of collision becomes

$$\mathcal{R}_{adv} = \sqrt{\frac{8\pi}{15}} \frac{n(2a)^3}{\tau_K} g(2a). \tag{3.4}$$

At a given St, $g(r)$ exhibits a power-law dependence on r: $g(r) \sim r^{-\varsigma}$ [38] and $0 < \varsigma < 0.7$ [5]. However, Sundaram and Collins [46] reported an abrupt increase in collision rate in their numerical experiments, as the Stokes number exceeds a threshold value. Preferential clustering cannot explain this observation, as it is thought to be effective only when the particle relaxation time matches with the correlation time of the fluid flow field, or, $St \sim 1$. The phenomenon of caustics (also called "sling caustics" or "fold caustics"), briefly described in section 2.3, was proposed as the explanation. Caustics occur when there are large negative velocity gradients in the flow, causing the momentum-position manifold in the phase space of the particles to fold over itself. As a result the particle velocity field becomes multi-valued leading to the existence of particles with large variation in velocity within a small separation. In addition to this effect, caustics also leads to a non-local clustering mechanism at high St, whereby the particle exhibits a memory of its interaction with the fluid velocity in its path history [11, 12].

At very large Stokes number, the particles become completely uncorrelated with the fluid motion and hence their motion can be described by gas-kinetic theory [1]. Subsequently, $\langle |w_r| \rangle \sim \frac{\eta}{\tau_K} F(St, Re)$, where the function $F(St, Re)$ represents the rate of caustic formation [23]. Hence the rate of collision due to caustics is given by

$$\mathcal{R}_{caust} = \frac{na^2\eta}{\tau_K} F(St, Re). \tag{3.5}$$

The total collision rate is approximated by the sum of the contributions due to preferential concentration ($\mathcal{R}_{adv}$) and caustics ($\mathcal{R}_{caust}$) [57].

$$\mathcal{R} = \mathcal{R}_{adv} + \mathcal{R}_{caust}. \tag{3.6}$$

In the limit of $St \to 0$, the function F has the the asymptotic form of $F(St, Re) \sim \exp(-C/St)$. This is obtained from a 1-D white noise model and is motivated by the notion that caustics formation has an activated form similar to the Arrhenius term determining the rate of an activated chemical process [55]. Numerical simulations of Navier-Stokes equations [17] show that the action $C \approx 2$. Note that as $St \to 0$, $F \to 0$; thus reducing Eq. (3.6) to Eq. (3.3). In the opposite limit of large St, $F(St, \infty) \sim K\sqrt{St}$ because at large Re, the relative velocity can depend only on the turbulent energy dissipation rate ϵ and the particle response time τ_P. Dimensional analysis gives $\langle |w_r| \rangle \sim \sqrt{\epsilon \tau_P}$ [31]. Flows modelled using kinematic simulations (in which the fluid velocity field is described by a random field tailored in such a way that its statistics follow the Kolmogorov theory for fully-developed turbulence.) show that $K \approx 50$ [52].

The two central questions to ask, therefore, are the following: first, do caustics explain the enhanced collision rates observed in turbulent flows? and second, does the increased rate of collisions explain the rapid initiation of rain in warm cumulus clouds? There does not seem to be a clear concensus on either question, although we lean towards answering yes to both questions.

Initial work on caustics suggested that the drastic increase in collision rate can explain the rapid onset of rain. However, as we have seen above, caustics were thought to be dominant only for $St \gtrsim 1$ ([51]), and the typical Stokes numbers of $St \sim 0.01$ for water droplets in a cloud could mean that the influence of caustics is negligible in warm rain initiation [35]. In fact, Wilkinson even suggests [56] that gravity alone (in the light of large deviation theory and "lucky" droplets) is enough to explain rapid rain initiation, based on the fact that the fraction of cloud droplet that has to undergo a runaway growth is less than one in a trillion. However, as estimated by [35], this theory can explain runaway droplet growth only after the droplet size already reaches 50μ.

In addition, Khain *et al.* [27] point out that analytical and numerical studies tend to overestimate the increase in the rate of collisions on account of ignoring important effects like gravitational settling. Recent studies are divided on this question. Bec *et al.* [6] find that gravity has competing effects on the two factors–clustering of particles and relative velocities between particles–that determine the collision rate. Gravity increases the amount of clustering, but decreases the relative velocities between particles. For small St, they report that the increase in clustering dominates the decrease in relative velocities, leading to a net increase in the collision rates. Other studies by Park & Lee [34] find that for $St > 1$ and strong gravity, the clustering of particles increases. Comprehensive studies by Ireland *et al.* [25, 26] find that gravitational settling reduces the clustering for small Stokes numbers, while increasing the degree of clustering for larger Stokes numbers. The latter two studies also suggest that the rates of collisions between like-sized particles are significantly lower because of gravity for Stokes numbers $St \gtrsim 0.1$.

In spite of these findings, the central idea that particle collisions induced by turbulence is an essential part of the explanation of rapid rain initiation holds. This is partly because the Stokes numbers in clouds are, at $St = \mathcal{O}(0.01)$, smaller than the value at which gravity seems to become a factor. In particular, the fact that the interactions of such small particles are likely most effective with vortices in the dissipative range is reason to believe that the centrifugal effect is still important.

The intermittent nature of turbulent flows leads to the development of localized high vorticity regions. Recent work by Bec *et al.* [7] shows that intermittency of turbulent mixing leads to an enhanced growth rate of coagulating aggregates in dilute suspensions as in a warm cloud. Intermittency has also been invoked in the experimental study by Bewley *et al.* [8] to argue that caustics will be crucial in clouds. Additionally, in ongoing work by our group, we find that droplets in the size range appropriate to clouds interacting with dissipative range vortices are capable of achieving very high collision rates, and hence that the increase of collision rates due to caustics can be expected to play a major role in droplet growth. However, due to the difficulties associated with experimental and numerical studies simulating the actual conditions prevalent in clouds, conclusive evidence–one way or the other–is yet to come by.

4. Conclusions

We have discussed the dynamics of small Stokes number heavy particles in steady flow, and explained why they leave regions of high vorticity and cluster into regions of high strain. We then discuss how this analysis misses "moving fixed points", i.e., points near a vortex where a very large number of particles can cluster, but which are moving in the lab frame of reference. We have shown here that the critical Stokes number for particle aggregation in these fixed points goes down exponentially with increasing number of interacting vortices. Finally, we have discussed a few important studies on droplet growth in clouds to delineate present knowledge of how droplets are thought to grow.

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