In this notes, we will discuss how to numerically find the spectrum of closed Dicke model Hamiltonian in the definite symmetry sector. This is useful for the analysis of integrability and chaos aspects of the model.

Dicke model Hamiltonian is:

$$H = \omega_c a^\dagger a + \omega_z S_z + \frac{\lambda}{\sqrt{N}} (a + a^\dagger)(S_+ + S_-)$$

$a, a^\dagger$ are photon mode operators.

S_z, S_-, S_+ are "Spin-$\frac{N}{2}$" angular momentum operators.

a & $a^\dagger$ act on Hilbert Space of photon mode,

$$\mathbb{H}_p = \text{Span}\{|n\rangle \mid n \in \mathbb{N}\} \to \infty\text{-dim}$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

As in Jaynes-Cummings model case, we truncate the photon mode Hilbert space to

$$\mathbb{H}_p = \text{Span}\{|n\rangle \mid n = 0, \dots, N_{cut}\} \to (N_{cut}+1)\text{ dim}$$

then the operators $a, a^\dagger$ projected onto $\mathbb{H}_p$ act as

$$a|n\rangle = \sqrt{n-1}\,|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a^\dagger|N_{cut}\rangle = 0.$$

S_z, S_-, S_+ act on the "Spin-$\frac{N}{2}$" hilbert space

$$\mathbb{H}_S = \mathrm{Span}\left\{ \left|\tfrac{N}{2}, m\right\rangle \,\middle|\, m = -\tfrac{N}{2}, -\tfrac{N}{2}+1, \ldots, \tfrac{N}{2} \right\} \rightarrow (N+1)\text{-dim}$$

$$S_z \left|\tfrac{N}{2}, m\right\rangle = m \left|\tfrac{N}{2}, m\right\rangle$$

$$S_- \left|\tfrac{N}{2}, m\right\rangle = \sqrt{\left(\tfrac{N}{2}+m\right)\left(\tfrac{N}{2}-m+1\right)} \left|\tfrac{N}{2}, m-1\right\rangle$$

$$S_+ \left|\tfrac{N}{2}, m\right\rangle = \sqrt{\left(\tfrac{N}{2}-m\right)\left(\tfrac{N}{2}+m+1\right)} \left|\tfrac{N}{2}, m+1\right\rangle .$$

Using Holstein Primakoff representation:

$$|\tfrac{N}{2}, m\rangle \Leftrightarrow |s\rangle$$

$$S_z = b^\dagger b - \frac{N}{2}$$

$$\mathcal{H}_S = \{|s\rangle \mid s = 0, \dots, N\}$$

$$\Downarrow$$

$$S_- = \sqrt{N - b^\dagger b}\; b$$

Truncated Bosonic Hilbert Space.

$$S_+ = b^\dagger \sqrt{N - b^\dagger b}$$

Clearly S_z, S_-, S_+ doesn't take states outside $\mathcal{H}_S$.

where $b, b^\dagger$ are bosonic operators satisfying $[b, b^\dagger] = 1$

Show above representation indeed satisfies "Spin-$\frac{N}{2}$" algebra.

The Hilbert space of the model is (after truncating photon Hilbert space).

$$\mathbb{H} = \mathbb{H}_p \otimes \mathbb{H}_S$$

$$= \mathrm{Span}\left\{ |n\rangle \otimes |s\rangle \,\middle|\, \begin{matrix} n = 0, \dots, N_{cut} \\ s = 0, \dots, N \end{matrix} \right\}$$

Hamiltonian in this representation is

$$H = \omega_c a^\dagger a + \omega_z \left(b^\dagger b - \frac{N}{2}\right)$$

$$+ \frac{\lambda}{\sqrt{N}} (a + a^\dagger)\left(b^\dagger \sqrt{N - b^\dagger b} + \sqrt{N - b^\dagger b}\, b\right).$$

$\rightarrow$ Hamiltonian.

Notice H has $\mathbb{Z}_2$ symmetry (Parity).

$$\mathbb{Z}_2 = \left\{ \mathbb{I}, \hat{\pi} \equiv e^{i\pi(a^\dagger a + b^\dagger b)} \right\}.$$

$\hat{N} \equiv a^\dagger a + b^\dagger b \rightarrow$ Counts total number of excitations

<u>Ex:</u> $\hat{N} |n\rangle \otimes |s\rangle = (n+s) |n\rangle \otimes |s\rangle$

$$\Rightarrow \hat{\pi} |n\rangle \otimes |s\rangle = e^{i\pi(n+s)} |n\rangle \otimes |s\rangle.$$

$$= +|n\rangle \otimes |s\rangle \text{ if } (n+s) \text{ is even}$$

$$= -|n\rangle \otimes |s\rangle \text{ if } (n+s) \text{ is odd.}$$

As, $[\hat{\pi}, H] = 0$

and $\hat{\pi}^2 = \mathbb{I} \Rightarrow$ $\hat{\pi}$ has eigenvalues '± 1'

and $\langle \psi_+ | H | \psi_- \rangle = 0$ if $\hat{\pi}|\psi_+\rangle = +|\psi_+\rangle$

$\hat{\pi}|\psi_-\rangle = -|\psi_-\rangle$

$|\psi_+\rangle, |\psi_-\rangle \in \mathcal{H}$

So 'H' does not couple even and odd parity states.

As the symmetry is a $\mathbb{Z}_2$ symmetry, we can direct sum decompose our Composite

Hilbert space into two Hilbert subspaces with definite symmetry.

$$\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$$

$$\mathbb{H}_+ = \text{Span}\left\{ |n\rangle \otimes |s\rangle \,\middle|\, \begin{matrix} n = 0, \dots, N_{cut} \\ s = 0, \dots, N \end{matrix} \;\&\; (n+s) \text{ even} \right\}.$$

$$\mathbb{H}_- = \text{Span}\left\{ |n\rangle \otimes |s\rangle \,\middle|\, \begin{matrix} n = 0, \dots, N_{cut} \\ s = 0, \dots, N \end{matrix} \;\&\; (n+s) \text{ odd} \right\}.$$

And Hamiltonian 'H' does not couple $\mathcal{H}_+$ and $\mathcal{H}_-$ sectors

So we can construct matrix representation of H in each of these definite parity Hilbert subspaces and compute the spectrum.

* The algorithm involves:

→ Choose a parity sector and construct the basis for the Hilbert subspace with the chosen parity.

→ Construct the Hamiltonian on the chosen parity Hilbert subspace.

→ Diagonalize the Hamiltonian, obtain the spectrum and post process the spectrum.

Let us write down the pseudo code for each of these steps:

<u>Step I</u>: Basis construction step involves storing an array of pairs of labels for the basis. $\{|n\rangle \otimes |s\rangle\}$.

This can be implemented as follows:

```
Parity = "even" → Choose a parity sector.
BASIS = [];  → Initialize an empty array.
Counter = Counter + 1;
```

```
for n = 0, ..., Ncut
    for s = 0, ..., N
        if (n+s)%2 = 0 → For even sector else (n+s)%2 ≠ 0 for odd sector.
            Counter = Counter + 1;
            BASIS(counter) = [n, s]; → |n⟩⊗|s⟩.
        end
    end
end
```

<u>Step II</u>: Constructing Hamiltonian involves action of $\hat{H}$ on the basis generated above and storing corresponding matrix elements.

$$H\,|n'\rangle \otimes |s'\rangle = \underset{\text{some number}}{\#}\; |n\rangle \otimes |s\rangle.$$

so

$$\langle n| \otimes \langle s|\; H\,|n'\rangle \otimes |s'\rangle = \underset{\text{Some number.}}{\#}$$

$$\begin{array}{c|c}
 & |0\rangle\otimes|0\rangle \quad |0\rangle\otimes|1\rangle \ldots |0\rangle\otimes|N\rangle \ldots |N_{at}\rangle\otimes|0\rangle \ldots |N_{at}\rangle\otimes|N\rangle \\
\begin{array}{c} |0\rangle\otimes|0\rangle \\ \vdots \\ |N_{at}\rangle|N\rangle \end{array} & \begin{pmatrix} \# & \cdots & \cdots \\ & \ddots & \\ & & \cdots \end{pmatrix}
\end{array}$$

Fill the numbers

Looking at Hamiltonian it has two types of terms:

→ $\omega_c a^\dagger a$, $\omega_z(b^\dagger b - \frac{N}{2})$ → Doesn't change the basis label. Hence contribute to the diagonal part of the Hamiltonian.

i.e., $\omega_c a^\dagger a |n\rangle \otimes |s\rangle = \omega_c n |n\rangle \otimes |s\rangle$

$\omega_z(b^\dagger b - \frac{N}{2}) |n\rangle \otimes |s\rangle = \omega_z(s - \frac{N}{2}) |n\rangle \otimes |s\rangle$.

→ $\frac{\lambda}{\sqrt{N}} a^\dagger b^\dagger \sqrt{N - b^\dagger b}$, $\frac{\lambda}{\sqrt{N}} a \sqrt{N - b^\dagger b}\, b$, $\frac{\lambda}{\sqrt{N}} a^\dagger \sqrt{N - b^\dagger b}\, b$, $\frac{\lambda}{\sqrt{N}} a b^\dagger \sqrt{N - b^\dagger b}$

↳ Change the basis label

Ex: $\frac{\lambda}{\sqrt{N}} a^\dagger b^\dagger \sqrt{N - b^\dagger b} |n\rangle \otimes |s\rangle$

$= \frac{\lambda}{\sqrt{N}} \sqrt{n+1} \sqrt{s+1} \sqrt{N - s}\, |n+1\rangle \otimes |s+1\rangle$ Contributes to off-diagonal part.

This can be implemented as follows:

```
H = Zeros([length(BASIS), length(BASIS)]);
```

↳ Initialize H with zeros.

```
for k' = 1, ..., length(BASIS)
    [n', s'] = BASIS(k', :);
    [n, s] = [n', s'];
    find [n, s] in BASIS → gives the column index of BASIS. 'k':
    H(k, k') = ω_c n' + ω_z s';
```

% Terms originating from $\frac{\lambda}{\sqrt{N}} a^{+} b^{+} \sqrt{N-b^{+}b}$

$[n, s] = [n'+1, s'+1]$;

if $0 \le n \le N_{cut}$ && $0 \le s \le N$

find 'k' such that BASIS(k, :) = [n, s];

$H(k,k') = \frac{\lambda}{\sqrt{N}} \sqrt{n'+1} \sqrt{s'+1} \sqrt{N-s'}$;

end

% Similarly generate other terms from $\frac{\lambda}{\sqrt{N}} a \sqrt{N-b^{+}b}\, b$, $\frac{\lambda}{\sqrt{N}} a^{+} \sqrt{N-b^{+}b}\, b$ & $\frac{\lambda}{\sqrt{N}} a b^{+} \sqrt{N-b^{+}b}$.

end.

Step III

Thus obtained matrix representation of 'H' can be diagonalized using standard numerical linear algebra routines to get the Eigenvalues.

* These Eigenvalues can be used to compute the Level spacing distribution $(P(S))$ which can be compared with

$$P(S) = e^{-S} \quad \text{(poisson)}$$
$$= \frac{S\pi}{2} e^{-\pi S^2/4} \quad \text{(Wigner, GOE).}$$

We can also compute the eigenvalues of the vectorized Liouvillian for Dissipative Dicke model.

$$\rightarrow \quad \mathcal{L}* = -i[H,*] - \kappa\left[\{a^\dagger a, *\} - 2a*a^\dagger\right].$$

This is part of an ongoing project. We will be updating this notes to include it.