Numerical solution of driven-dissipative JC model

Quantum Master Equation for driven-dissipative Jaynes-Cummings model (in rotating frame):

$$\frac{\partial \rho(t)}{\partial t} = -i[H_{JC} + H_d, \rho(t)] - \frac{\kappa}{2}\left[\{a^\dagger a, \rho(t)\} - 2a\rho(t)a^\dagger\right] - \frac{\gamma}{2}\left[\{\sigma^+\sigma^-, \rho(t)\} - 2\sigma^-\rho(t)\sigma^+\right]$$

$$H_{JC} = \tilde{\omega}_c a^\dagger a + \frac{\tilde{\omega}_q}{2}\sigma^z + g(a^\dagger\sigma^- + \sigma^+ a) \qquad H_d = \mu(a^\dagger + a).$$

$$\tilde{\omega}_c = \omega_c - \omega_d \qquad \tilde{\omega}_q = \omega_q - \omega_d$$

If we define an operator $\mathcal{L}$ such that master equation is written as

$$\frac{d\rho(t)}{dt} = \mathcal{L}\,\rho(t).$$

$$\mathcal{L}* = -i[H_{JC} + H_d, *] - \frac{\kappa}{2}\left[\{a^\dagger a, *\} - 2a * a^\dagger\right]$$

$$-\frac{\gamma}{2}\left[\{\sigma^+\sigma^-, *\} - 2\sigma^- * \sigma^+\right].$$

$a, a^\dagger$ → annihilation & creation operators → $[a, a^\dagger] = 1$

$\sigma^z = [|e\rangle\langle e| - |g\rangle\langle g|]$ $\sigma^+ = |e\rangle\langle g|$ $\sigma^- = |g\rangle\langle e|$

The Hilbert space of the model is $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_q$

$\mathcal{H}_p$ → Cavity mode Hilbert space = Span$\{|n\rangle | n \in \mathbb{N}\}$.
(∞-dim)

$\mathcal{H}_q$ → Qubit Hilbert space = Span$\{|g\rangle, |e\rangle\}$.
(2-dim)

For numerical purpose, let us introduce a cutoff on the photon Hilbert space. (Fock Space)

i.e., $\mathcal{H}_p \approx \text{Span}\{|0\rangle, |1\rangle, \ldots |N_{cut}\rangle\}$.
$((N_{cut}+1)\text{-dim})$

We shall denote this truncated hilbert space by $\mathcal{H}_p$. It is to be understood that we are working in truncated space.

$a, a^\dagger$ act on $\mathcal{H}_p$ as $a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$.

Note: $a^\dagger|N_{cut}\rangle = 0$.

$\sigma^z, \sigma^-, \sigma^+$ act on $\mathcal{H}_q$ as $\sigma^z|g/e\rangle = \mp|g/e\rangle$

$\sigma^+|g\rangle = |e\rangle \quad \sigma^+|e\rangle = 0$

$\sigma^-|g\rangle = 0 \quad \sigma^-|e\rangle = |g\rangle$

When we lift these operators onto $\mathcal{H}_p \otimes \mathcal{H}_q$.

Strictly they are $a \to a \otimes \mathbb{I}_q$ $\quad \sigma^z \to \mathbb{I}_p \otimes \sigma^z$

$a^\dagger \to a^\dagger \otimes \mathbb{I}_q$ $\quad \sigma^- \to \mathbb{I}_p \otimes \sigma^-$

$\sigma^+ \to \mathbb{I}_p \otimes \sigma^+$

$\mathbb{I}_p \to$ Identity on $\mathcal{H}_p$ and $\mathbb{I}_q \to$ Identity on $\mathcal{H}_q$.

So $a, a^\dagger, \sigma^z, \sigma^-, \sigma^+$ acts on $2(N_{cut}+1)$-dim Hilbert space $\mathcal{H}_p \otimes \mathcal{H}_q$.

Similarly $\rho(t)$ - density matrix acts on $\mathcal{H}_p \otimes \mathcal{H}_q$.

Now if we choose an ordered basis

$(|0\rangle|e\rangle, |0\rangle|g\rangle, |1\rangle|e\rangle, |1\rangle|g\rangle, \ldots, |N_{cut}\rangle|e\rangle, |N_{cut}\rangle|g\rangle)$.

The operators $e^{i\phi}, a, a^\dagger, \sigma^-, \sigma^+, \sigma^z$ are represented as $2(N_{cut}+1) \times 2(N_{cut}+1)$ matrices

$$a = \begin{pmatrix} 0 & \sqrt{1} & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ & & \ddots & 1 \\ & & & \sqrt{N_{cut}-1} \\ 0 & \cdots & & 0 \end{pmatrix} \otimes \mathbb{I}_q$$

$$a^\dagger = \begin{pmatrix} 0 & \cdots & & 0 \\ \sqrt{1} & \ddots & & \vdots \\ & \ddots & & \\ 0 & \cdots & \sqrt{N_{cut}-1} & 0 \end{pmatrix} \otimes \mathbb{I}_q$$

$$\sigma^z = \mathbb{I}_p \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^- = \mathbb{I}_p \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^+ = \mathbb{I}_p \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbb{I} = \mathbb{I}_p \otimes \mathbb{I}_q$$

$\mathbb{I}_p$ = $(N_{cut}+1) \times (N_{cut}+1)$ identity matrix

$\mathbb{I}_q$ = 2×2 Identity matrix

Similarly density matrix $\rho(t)$ is a

$2(N_{cut}+1) \times 2(N_{cut}+1)$ matrix

$$\begin{pmatrix} \rho_{00}^{ee}(t) & \rho_{00}^{eg}(t) & \cdots & \rho_{0N_{cut}}^{ee}(t) & \rho_{0N_{cut}}^{eg}(t) \\ \rho_{00}^{ge}(t) & \rho_{00}^{gg}(t) & \cdots & \rho_{0N_{cut}}^{ge}(t) & \rho_{0N_{cut}}^{gg}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{N_{cut}0}^{ee}(t) & \rho_{N_{cut}0}^{eg}(t) & \cdots & \rho_{N_{cut}N_{cut}}^{ee}(t) & \rho_{N_{cut}N_{cut}}^{eg}(t) \\ \rho_{N_{cut}0}^{ge}(t) & \rho_{N_{cut}0}^{gg}(t) & \cdots & \rho_{N_{cut}N_{cut}}^{ge}(t) & \rho_{N_{cut}N_{cut}}^{gg}(t) \end{pmatrix}$$

$$\rho_{nn'}^{\sigma\sigma'}(t) = \langle n|\langle\sigma|\,\rho(t)\,|n'\rangle|\sigma'\rangle.$$

The Lindblad master equation is of same form as original, i.e.,

$$\frac{\partial \rho(t)}{\partial t} = -i[H_{JC} + H_d, \rho(t)] - \frac{\kappa}{2}\left[\{a^\dagger a, \rho(t)\} - 2a\rho(t)a^\dagger\right] - \frac{\gamma}{2}\left[\{\sigma^+\sigma^-, \rho(t)\} - 2\sigma^-\rho(t)\sigma^+\right]$$

$$H_{JC} = \tilde{\omega}_c a^\dagger a + \frac{\tilde{\omega}_{q0}}{2}\sigma^z + g(a^\dagger\sigma^- + \sigma^+ a) \qquad H_d = \mu(a^\dagger + a).$$

except that $\rho(t)$, a, $a^\dagger$, σ^z, σ^-, σ^+ should be thought of as $2(N_{cut}+1) \times 2(N_{cut}+1)$ matrices.

Although this equation is linear in $\rho(t)$, it can be inconvenient, as it has matrices multiplying $\rho(t)$ on both sides.

This equation has the structure of the LYAPNOUV EQUATION for $N \times N$ matrix $X(t)$:

$$\frac{d}{dt} X(t) = A X(t) + X(t) B + C X(t) D .$$

writing this equation in component form

$$\frac{d}{dt} X_{mn}(t) = \sum_{p=1}^{N} A_{mp} X_{pn}(t) + \sum_{p=1}^{N} X_{mp}(t) B_{pn} + \sum_{p,q=1}^{N} C_{mp} X_{pq}(t) D_{qn}$$

$$\Downarrow$$

$$\frac{d}{dt} X_{mn}(t) = \sum_{p,q=1}^{N} A_{mp} \delta_{nq} X_{pq}(t) + \sum_{p,q=1}^{N} \delta_{mp} B^{T}_{nq} X_{pq}(t) + \sum_{p,q=1}^{N} C_{mp} D^{T}_{nq} X_{pq}(t)$$

Now collect the equation by defining a vector

$$|X(t)\rangle = \begin{pmatrix} X_{11}(t) \\ \vdots \\ X_{N1}(t) \\ \vdots \\ X_{1N}(t) \\ \vdots \\ X_{NN}(t) \end{pmatrix}$$

← Column ordered VECTORIZATION of a matrix $X(t)$.

This gives,

$$\frac{d}{dt}|x(t)\rangle = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} |x(t)\rangle + \begin{bmatrix} B_{11}^T \mathbb{I} & \cdots & B_{1N}^T \mathbb{I} \\ \vdots & \ddots & \vdots \\ B_{N1}^T \mathbb{I} & \cdots & B_{NN}^T \mathbb{I} \end{bmatrix} |x(t)\rangle$$

$$+ \begin{bmatrix} D_{11}^T C & \cdots & D_{1N}^T C \\ \vdots & \ddots & \vdots \\ D_{N1}^T C & \cdots & D_{NN}^T C \end{bmatrix} |x(t)\rangle .$$

Now recalling the Kronecker product of two matrices P and Q given by

$$P \otimes Q = \begin{bmatrix} P_{11} & \cdots & P_{1N} \\ \vdots & \ddots & \vdots \\ P_{N1} & \cdots & P_{NN} \end{bmatrix} \otimes \begin{bmatrix} Q_{11} & \cdots & Q_{1N} \\ \vdots & \ddots & \vdots \\ Q_{N1} & \cdots & Q_{NN} \end{bmatrix}$$

$$P \otimes Q \equiv \begin{bmatrix} P_{11}Q & \cdots & P_{1N}Q \\ \vdots & \ddots & \vdots \\ P_{N1}Q & \cdots & P_{NN}Q \end{bmatrix}$$

we can write the LYAPNOUV equation as

$$\frac{d}{dt}|X(t)\rangle = \left[\mathbb{I} \otimes A + B^T \otimes \mathbb{I} + D^T \otimes C\right] |X(t)\rangle.$$

This process of converting the matrix equation for $X(t)$ to vector equation for $|X(t)\rangle$ is called VECTORIZATION.

Now applying this vectorization procedure to our Lindblad master equation gives

$$\frac{d}{dt}|\rho(t)\rangle = \Big\{ -i\left[\mathbb{I}\otimes(H_{JC}+H_d) - (H_{JC}+H_d)^T\otimes\mathbb{I}\right]$$

$$-\frac{\kappa}{2}\left[\mathbb{I}\otimes(a^\dagger a) + (a^\dagger a)^T\otimes\mathbb{I} - 2(a^\dagger)^T\otimes a\right]$$

$$-\frac{\gamma}{2}\left[\mathbb{I}\otimes(\sigma^+\sigma^-) + (\sigma^+\sigma^-)^T\otimes\mathbb{I} - 2(\sigma^+)^T\otimes\sigma^-\right]\Big\}|\rho(t)\rangle$$

Denote as $\check{\mathcal{L}}$

$$\therefore \quad \frac{d}{dt}|\rho(t)\rangle = \check{\mathcal{L}}\,|\rho(t)\rangle \quad \Rightarrow \quad \text{Formal solution is } |\rho(t)\rangle = e^{\check{\mathcal{L}}t}|\rho(0)\rangle.$$

Once we obtain the above solution, we can devectorize $|\rho(t)\rangle$ to get $\rho(t)$.

$\rho(t)$ can be used to compute observables as usual $\langle \hat{O}(t) \rangle = Tr\left[\hat{O}\rho(t)\right]$.

Note: One has to converge the numerics with respect to N_{cut}.

N_{cut} value needed depends on $\rho(0)$, $\tilde{\omega}_c, \tilde{\omega}_s, q, \mu, \kappa$ & ν.

Algorithm:

→ Truncate photon Hilbert space (N_{cut}).

→ Construct photon and qubit operators.

→ Construct "Vectorized" Liouvillian.

→ Solve vectorized master equation.

→ "De-vectorize" density matrix.

→ Compute observables.

"CONVERGE"