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# EXCITATIONS AND INTERACTIONS IN $d = 1$ STRING THEORY

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We discuss the singlet sector of the $d = 1$ matrix model in terms of a Dirac fermion formalism. The leading order two- and three-point functions of the density fluctuations are obtained by this method. This allows us to construct the effective action to that order and hence provide the equation of motion. This equation is compared with the one obtained from the continuum approach. We also compare continuum results for correlation functions with the matrix model ones and discuss the nature of gravitational dressing for this regularization. Finally, we address the question of boundary conditions within the framework of the $d = 1$ unitary matrix model, considered as a regularized version of the Hermitian model, and study the implications of a generalized action with an additional parameter (analogous to the $\theta$ parameter) which give rise to quasi-periodic wave functions.

## 1. Introduction and Summary of Results

In a series of recent works[1–3] we have studied the problem of two-dimensional quantum field theories coupled to gravity. Our original aim was to arrive at a natural setting for the theory space formulation[4,5] of string theory, where (1) there is no restriction on the central charge of the matter sector, and (2) the theory has, within it, the ingredients to describe trajectories which join special points in the theory space, namely the classical vacua which correspond to conformally invariant theories. One of our main results has been that the matter + gravity system can be regarded as a field theory of the Liouville mode and as matter fields in the background of the fiducial metric. Generic couplings, or backgrounds, now depend both on the Liouville mode and on the matter degrees of freedom and satisfy equations of motion in $d + 1$ variables ($d$ = matter, 1 = Liouville). This is because reparametrization invariance of the theory implies that all objects in which the conformal mode has been integrated should be Weyl-invariant in its dependence upon the fiducial metric. This condition, stated as the vanishing of the "$B$ function," gives rise to the equations of motion.[3] Other related works are those of J. Polchinski[6] and T. Banks and J. Lykken.[7]

These ideas were illustrated in various situations:

(a) For $d$ scalar fields coupled to gravity, at $d = 25$, the spectrum and the tree level $S$ matrix were shown to be identical to that of the "$d = 26$ critical string."[2]

(b) In the case of $d < 1$, we considered the $(m, m + 1)$ minimal models coupled to gravity, and could effectively describe the interpolation between two minimal models, for

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$m$ very large, by means of a "string field" that depends only on the Liouville mode, a function $\kappa(\eta)$ which satisfies the field equation[3]

$$(\partial_\eta^2 + Q\partial_\eta + h)\kappa(\eta) = b\kappa^2(\eta) + o(\kappa^3)\,. \tag{1.1}$$

(c) In the case of $d$ scalar fields coupled to gravity, perturbed by a "tachyon" background, the tachyon coupling $T$, which depends on $d$ coordinates $\phi_i$ and the Liouville mode $\eta$ satisfies the $(d+1)$-dimensional field equation[1,3]

$$(\partial_\eta^2 + Q\partial_\eta + \partial_i\partial_i + 2)T(\phi, \eta) + T^2(\phi, \eta) + \ldots = 0\,, \tag{1.2}$$

where

$$Q = \sqrt{\frac{25 - d}{3}}\,. \tag{1.3}$$

To see the spectrum from this equation, we have to eliminate the linear derivative piece. By defining $T = e^{-(1/2)Q\eta}\tilde{T}$ we get

$$\left(\partial_\eta^2 + \partial_i\partial_i + \frac{1}{4}(8 - Q^2)\right)\tilde{T}(\phi, \eta) + e^{-(1/2)Q\eta}\tilde{T}^2(\phi, \eta) + \ldots = 0\,. \tag{1.4}$$

This equation tells us that the spectrum at $d = 1$ (i.e. $Q^2 = 8$) is that of a massless particle. For $d > 1$, there is a tachyon in the spectrum and hence for much the same reasons as in 26-dimensional critical string theories, where it ruins the perturbation expansion, these theories may not exist. It is likely that the tachyon perturbation drives $d > 1$ theories to a stable point which is $d = 1$. It would also be interesting to understand how one can reach models with $d < 1$ by appropriate perturbations of the $d = 1$ model.

The main purpose of this paper is to understand, in some detail, the cutoff string field theory at $d = 1$,[8–10] formulated as the quantum mechanics of the matrix Hamiltonian which was originally discussed by Brézin, Itzykson, Parisi and Zuber[11]:

$$\mathrm{H} = -\frac{1}{2N}\nabla_M^2 + N\,\mathrm{tr}\,V(M)\,, \tag{1.5}$$

where

$$\nabla_M^2 = \sum_{i<j}\left[\left(\frac{\partial}{\partial\,\mathrm{Re}\,M_{ij}}\right)^2 + \left(\frac{\partial}{\partial\,\mathrm{Im}\,M_{ij}}\right)^2\right] + \sum_i \frac{\partial^2}{\partial M_{ii}^2}\,, \tag{1.6}$$

and $V(M)$ is a polynomial. We can expect the results of the continuum theory and that from the matrix model approach to agree in the low momentum region only.

Since this Hamiltonian is invariant under $U(N)$ transformations, $M \to UMU^\dagger$, there would be wave functions transforming according to various different representations of

$U(N)$. (To be more precise, these consist of the trivial representation and the representations that can be generated by taking products of the adjoint.) It is not yet clear whether states which transform nontrivially under $U(N)$ are related to the string degrees of freedom. Hence we want to analyze the singlet sector of the model. We use the fermionic representation of this sector as explained below. This representation has two major advantages:

(a) The model is well-defined even for finite $N$ and for noncritical values of the coupling. Hence the nature of the various regularizations are most clearly recognized in this picture.

(b) It is easier to see various approximate and exact symmetries of the system from this point of view.

As is well known, $\nabla_M^2$, acting on the singlet sector wave function $\psi(\Lambda)$, has the form

$$\nabla_M^2\psi(\Lambda) = \frac{1}{\Delta(\Lambda)}\left(\sum_i \frac{\partial^2}{\partial\lambda_i^2}\right)\Delta(\Lambda)\psi(\Lambda) , \tag{1.7}$$

where $\Lambda = (\lambda_1, \ldots, \lambda_N)$, $\lambda_i$ being the eigenvalues of $M$. $\Delta(\Lambda)$ is the Vandermonde determinant:

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) . \tag{1.8}$$

If we use $\chi(\Lambda) = \Delta(\Lambda)\psi(\Lambda)$, instead of $\psi(\Lambda)$, as the wave function, the effective Hamiltonian becomes

$$\mathrm{H}_f = -\frac{1}{2N}\sum_i \frac{\partial^2}{\partial\lambda_i^2} + N\sum_i V(\lambda_i) . \tag{1.9}$$

This is the Hamiltonian for noninteracting particles. However, since $\psi(\Lambda)$ is symmetric, $\chi(\Lambda)$ is antisymmetric. Hence the problem reduces to that of noninteracting fermions moving in an external potential.

The ground state is obtained by filling the $N$ states which are lowermost in energy. The corresponding total energy becomes singular when the Fermi energy of the system approaches the value of the potential at a stationary point. This is related to the fact that the time period of the classical orbits, corresponding to the states near the Fermi surface, starts diverging when they can approach the stationary point. In the semiclassical analysis, one can obtain the nature of the singularities. Also, one can take the double scaling limit by keeping fixed the energy difference between the value of the potential at the stationary point and that of the Fermi energy as $N$ goes to infinity. The inverse of this energy difference can be identified as the string coupling constant, $g_{\rm str}$.[9]

In this paper, we develop a Dirac fermion formalism for the states near the Fermi surface which works well for the leading contributions and lends valuable insight into many results.

To obtain the equations of motion we write this theory as a second-quantized theory with the action

$$S = \int dt d\lambda \chi^\dagger(\lambda, t)\left(i\partial_t + \frac{1}{2N}\partial_\lambda^2 - NV(\lambda)\right)\chi(\lambda, t)$$

$$+ \int dt d\lambda N a_0(\lambda, t)\chi^\dagger(\lambda, t)\chi(\lambda, t) , \tag{1.10}$$

where $\chi(\lambda, t)$ is a second-quantized fermion field in two dimensions and $a_0(\lambda, t)$ is the source function conjugate to density. The corresponding vacuum to vacuum amplitude $Z[a_0]$ contains the information of all correlation functions of density. Let

$$F[a_0] = \ln Z[a_0] . \tag{1.11}$$

By taking the Legendre transformation of $F[a_0]$ we obtain the effective action $\Gamma[\rho]$ where

$$\rho(\lambda, t) = \frac{\delta F[a_0]}{\delta a_0(\lambda, t)} , \tag{1.12}$$

and

$$\Gamma[\rho] = \int d\lambda dt \rho(\lambda, t)\, a_0(\lambda, t) - F[a_0] . \tag{1.13}$$

When $a_0 = 0$ we have

$$\frac{\delta\Gamma[\rho]}{\delta\rho(\lambda, t)} = 0 , \tag{1.14}$$

which is the quantum equation of motion.

Since we are looking at an effective bosonic theory we define the field variable that is used for bosonizing relativisitic fermion theories,

$$\phi(\lambda, t) = \int^\lambda [\rho(\lambda, t) - \langle\rho(\lambda, t)\rangle] . \tag{1.15}$$

We will find, perturbatively, the leading order terms in the equation of motion in terms of this variable.

This paper is organized as follows. In the second section, after a set of transformations, we write the nonrelativistic Hamiltonian as a Dirac Hamiltonian plus a perturbation which is small near the Fermi surface. The transformations involve changing over from $\lambda$ to $\tau = \tau(\lambda)$, which is proportional to the probability of being to the left of $\lambda$ for a particle in a certain state at the Fermi level. They also involve a "chiral gauge transformation" of the fermion field. In the next two sections we utilize this Dirac formalism to calculate the two-point and the three-point function of the density in the leading order and indicate the connections with bosonization. Then, in the fifth section, we obtain the leading order effective action and the corresponding equation of motion. We keep only the terms first or lower order in $g_{\rm str}$. In terms of the light-cone variables $t^{\pm} = t \pm \tau$, we have the equation

$$\partial_+\partial_-\phi = \frac{\pi g_{\rm str}}{2}\{\partial_+[f(\tau)(\partial_+\phi)^2] - \partial_-[f(\tau)(\partial_-\phi)^2]\}\ , \tag{1.16}$$

which describe a massless particle with translationally noninvariant interaction. This is very similar to Eq. (1.4). The function $f(\tau)$ is proportional to the square of the probability density (in terms of $\lambda$) for a particle in a specific state at the Fermi level.

The next section compares the correlators of the matrix model with the continuum predictions. It seems that more than one kind of gravitational dressings (corresponding to the different roots of the KPZ equation[12,13]) are being provided by the matrix model although there seems to be no clear pattern.

In the last section, we discuss the well-defined one-dimensional string theory by considering the $d = 1$ unitary matrix model,[14] which is a specific regularization of the bottomless inverted harmonic oscillator. In the singlet sector, this problem is the same as that of $N$ fermions moving on a circle under the influence of a potential. Usually the boundary condition of the single particle wave function is periodic or antiperiodic, depending upon whether $N$ is odd or even respectively. One can, however, consider quasi-periodic boundary conditions also. We indicate that this is the case if we modify the Lagrangian by a certain total derivative term. It is not yet clear whether the coefficient of the total derivative term represents any important ambiguous parameter of the $d = 1$ string theory.

A preliminary version of this paper was reported at the Cargese meeting, 1990.[15] While this work was in progress, we became aware of the work of J. Polchinski,[16] and S. Das and A. Jevicki.[17]

## 2. The Dirac Fermion Representation

Let us take the Schrödinger equation,

$$\mathrm{H}\phi_n(\lambda) = E_n\phi_n(\lambda)\ , \tag{2.1}$$

where

$$\mathrm{H} = -\frac{1}{2N}\frac{d^2}{d\lambda^2} + NV(\lambda)\ . \tag{2.2}$$

If we change over from $\lambda$ to $\tau$ defined by $\rho_0(\lambda)d\lambda = d\tau$, then the new Hamiltonian should be

$$\mathrm{H}_{\rm new} = \rho_0^{-1/2}\mathrm{H}\rho_0^{1/2}\ , \tag{2.3}$$

and the new wave function is given by

$$\phi_{\rm new} = \rho_0^{-1/2}\phi\ . \tag{2.4}$$

We use $f'$ for $df/d\lambda$ and $\dot{f}$ for $df/d\tau$. We have $f' = \rho_0 \cdot \dot{f}$. Thus

$$\begin{aligned} \mathrm{H}_{\rm new} &= -\frac{1}{2N}\left(\rho_0^{-1/2}(\lambda)\frac{d}{d\lambda}\rho_0^{1/2}(\lambda)\right) + NV(\lambda) \\ &= -\frac{1}{2N}\left(\frac{d}{d\tau}\rho_0 - \frac{1}{2}\dot{\rho}_0\right)\left(\rho_0\frac{d}{d\tau} + \frac{1}{2}\dot{\rho}_0\right) + NV\ . \end{aligned} \tag{2.5}$$

Now suppose we give the following unitary transformation to the wave functions:

$$\phi(\lambda) \to e^{\mp iN\Theta_0(\lambda)}\phi(\lambda)\ . \tag{2.6}$$

Then the effective Hamiltonian will be

$$\begin{aligned} \mathrm{H}_{\rm eff} &= e^{\mp iN\Theta_0}\mathrm{H}_{\rm new}e^{\pm iN\Theta_0} \\ &= \frac{1}{2N}\left(i\frac{d}{d\tau}\rho_0 - \frac{i}{2}\dot{\rho}_0 \mp N\Theta_0'\right)\left(i\rho_0\frac{d}{d\tau} + \frac{i}{2}\dot{\rho}_0 \mp N\Theta_0'\right) + NV\ . \end{aligned} \tag{2.7}$$

Shifting all the energies by $E_0$, we have

$$\begin{aligned} \widehat{\mathrm{H}} &= \frac{1}{2N}\left(i\frac{d}{d\tau}\rho_0 - \frac{i}{2}\dot{\rho}_0 \mp N\Theta_0'\right)\left(i\rho_0\frac{d}{d\tau} + \frac{i}{2}\dot{\rho}_0 \mp N\Theta_0'\right) + NV - E_0 \\ &= \mp\frac{i}{2}\left(\rho_0\Theta_0'\frac{d}{d\tau} + \frac{d}{d\tau}\rho_0\Theta_0'\right) - \frac{1}{2N}\frac{d}{d\tau}\rho_0^2\frac{d}{d\tau} \\ &\quad + \left\{\frac{1}{2N}\left(\frac{\rho_0'}{2\rho_0}\right)^2 - \frac{1}{4N}\frac{\rho_0''}{\rho_0} + \frac{1}{2}N\Theta_0'^2 + NV - E_0\right\}\ . \end{aligned} \tag{2.8}$$

If we require

$$\mathrm{H}(\rho_0^{1/2}e^{\pm iN\Theta_0}) = E_0(\rho_0^{1/2}e^{\pm iN\Theta_0})\ , \tag{2.9}$$

then

$$\rho_0\Theta_0' = \text{const}\ , \tag{2.10}$$

and

$$\frac{1}{2N}\left(\frac{\rho_0'}{2\rho_0}\right)^2 - \frac{1}{4N}\frac{\rho_0''}{\rho_0} + \frac{1}{2}N\Theta_0'^2 + NV - E_0 = 0\ . \tag{2.11}$$

If $\rho_0\Theta_0' \neq 0$, by choosing suitable normalization for $\rho_0$ we can make

$$\rho_0\Theta_0' = 1 \,. \tag{2.12}$$

Then

$$\hat{\mathrm{H}} = \mp i\frac{d}{d\tau} - \frac{1}{2N}\frac{d}{d\tau}\rho_0^2\frac{d}{d\tau} \,. \tag{2.13}$$

Note that, had we not used a genuinely complex solution (i.e. a solution whose real and imaginary parts are linearly independent), $\rho_0\Theta_0'$ would be zero and we would not get the term linear in $d/d\tau$.

To get the scales right, let us make some estimates. The leading order solution for large $N$ of the equation

$$\mathrm{H}\phi = E_0\phi \tag{2.14}$$

is

$$\phi = \text{const}\cdot\frac{1}{\sqrt[4]{2[\epsilon_0 - V(\lambda)]}}\exp\left(\pm iN\int^{\lambda} d\lambda'\sqrt{2[\epsilon_0 - V(\lambda')]}\right), \tag{2.15}$$

where

$$\epsilon_0 = \frac{E_0}{N} \,. \tag{2.16}$$

If we choose the constant to be 1, we have

$$\rho_0(\lambda) = \frac{1}{\sqrt{2[\epsilon_0 - V(\lambda)]}} \,,$$
$$\Theta_0(\lambda) = \int^{\lambda} d\lambda'\sqrt{2[\epsilon_0 - V(\lambda)]} \,, \tag{2.17}$$

and

$$\rho_0\Theta_0' = 1 \,. \tag{2.18}$$

Let the potential have a maximum at $\lambda_0$ with $V''(\lambda_0) \neq 0$. If we take a solution for $\epsilon_0$ very near $V(\lambda_0)$ then most of the probability is concentrated near that tip. Classically this is manifested by the particle spending a lot of time near the turning point, which is very close to the rather flat region around the potential maximum.

In this region $V(\lambda) \approx V(\lambda_0) - \frac{1}{2}|V''(\lambda_0)|(\lambda - \lambda_0)^2$,

$$\tau - a \sim \int_{\lambda_1}^{\lambda} d\lambda' \frac{1}{\sqrt{|V''(\lambda_0)|(\lambda' - \lambda_0)^2 - 2[V(\lambda_0) - \epsilon_0]}}$$

$$\sim \frac{1}{[|V''(\lambda_0)|]^{1/2}} \int_{\lambda_0 + \sqrt{2\mu}}^{\lambda} d\lambda' \frac{1}{[(\lambda' - \lambda_0)^2 - 2\mu]^{1/2}} . \tag{2.19}$$

By convention we make $V''(\lambda_0) = -1$ and define $\mu$ to be $V(\lambda_0) - \epsilon_0$.

Upon integration we get

$$\tau - a = \cosh^{-1}\left(\frac{\lambda - \lambda_0}{\sqrt{2\mu}}\right) , \tag{2.20}$$

or

$$\lambda = \lambda_0 + \sqrt{2\mu} \cosh(\tau - a) , \tag{2.21}$$

where $a$ is the value of $\tau$ at the turning point.

Now

$$\rho_0^2 \sim \frac{1}{4\mu \sinh^2(\tau - a)} , \tag{2.22}$$

$$\hat{\mathrm{H}} \sim \mp i \frac{\partial}{\partial \tau} - \frac{1}{8N\mu} \frac{\partial}{\partial \tau} \frac{1}{\sinh^2(\tau - a)} \frac{\partial}{\partial \tau} . \tag{2.23}$$

This estimate can be trusted, when $\tau$ is not too near $a$.

To recover an approximate relativistic fermion picture from a nonrelativistic one, the most natural thing to do is to take the reference energy level $E_0$ to be the Fermi level $E_f$. If we now want the expression of $\hat{\mathrm{H}}$ in terms of $\tau$ to be a scaled expression, that is, if we want to keep $\tau - a$ as a scaled variable, we need to have $N\mu$ = fixed. (This is true irrespective of the semiclassical approximation that we made to reach this expression of $\hat{\mathrm{H}}$.)

Strictly speaking, for this problem the wave functions are not exactly like $\rho^{1/2}e^{+iN\Theta}$ and $\rho^{1/2}e^{-iN\Theta}$, but a specific linear combination which depends upon the energy and the boundary conditions. In terms of $\tau$ variables, $\rho^{1/2}e^{\pm iN\Theta}$ (after the relevant transformations) looks like a plane wave in the leading order. The extent of classically allowed $\lambda - \lambda_0$ is roughly from 0 to, say, 1. The corresponding range of $\tau - a$ is from 0 to $\ln 1/\sqrt{\mu}$. The level spacing goes as inverse of this range. Hence the boundary condition can give rise to mixing of left- and right-moving plane waves which can change the energy almost by $1/(\ln 1/\sqrt{\mu})$. This vanishes in the scaling limit.

Thus we are allowed, in the scaling limit, to deal with chiral states which are almost exact eigenstates. The Hamiltonian which makes the right-moving states near the Fermi surface look like plane waves is

$$\hat{\mathrm{H}}_{\mathrm{R}} = -i\partial_\tau - \frac{1}{2N}\partial_\tau \rho_f^2 \partial_\tau \,. \tag{2.24}$$

The Hamiltonian which does the same for the left-moving states is

$$\hat{\mathrm{H}}_{\mathrm{L}} = i\partial_\tau - \frac{1}{2N}\partial_\tau \rho_f^2 \partial_\tau \,. \tag{2.25}$$

Both the Hamiltonians have information about all the states. However, for the left-moving states, the second term in $\hat{\mathrm{H}}_{\mathrm{R}}$ cannot be considered as a small perturbation. Similar problems arise for right-moving states and $\hat{\mathrm{H}}_{\mathrm{L}}$.

Thus, for the calculations where only states near the Fermi surface matter, one can describe the left-moving states by $\hat{\mathrm{H}}_{\mathrm{L}}$ and the right-moving by $\hat{\mathrm{H}}_{\mathrm{R}}$. This gives a Dirac-like Hamiltonian. In the second-quantized notation the Hamiltonian is

$$\mathrm{H} = \int d\tau (\psi_+^\dagger \hat{\mathrm{H}}_{\mathrm{L}} \psi_+ + \psi_-^\dagger \hat{\mathrm{H}}_{\mathrm{R}} \psi_-) \,. \tag{2.16}$$

To be honest, one should discard half the solutions of each of the Hamiltonians, not to overcount the states. This would be an ultraviolet cutoff in the theory. This cutoff would refer to the value of the momenta where the second term starts dominating over the first. For calculations involving processes near the Fermi surface, this cutoff is not important.

In many of the leading order calculations, this problem does not show up. This effective ultraviolet cutoff parameter, in a certain region of $\tau$, is finite in the scaled picture (as opposed to the semiclassical case). Hence one has to be careful about it.

## 3. The Two-Point Function of Density Fluctuations

We have

$$G^{(2)}(1, 2) = \langle 0 | T\rho(\lambda_1, t_1)\rho(\lambda_2, t_2) | 0 \rangle_c \,, \tag{3.1}$$

where $\rho(\lambda, t)$ is the eigenvalue/fermion density. If we look only at the connected part, we can see the correlation of density fluctuation, $\rho(\lambda, t) - \langle \rho(\lambda, t) \rangle$, only. This density fluctuation can also be represented by $\chi^\dagger \chi$, normal-ordered with respect to the Fermi sea.

If we change over to $\tau$ variables we have

$$\begin{aligned} \bar{G}^{(2)}(1, 2) &= \langle 0 | T\bar{\rho}(\tau_1, t_1)\bar{\rho}(\tau_2, t_2) | 0 \rangle_c \\ &= \frac{d\lambda_1}{d\tau_1}\frac{d\lambda_2}{d\tau_2} \langle 0 | T\rho(\lambda_1, t_1)\rho(\lambda_2, t_2) | 0 \rangle \,. \end{aligned} \tag{3.2}$$

We call $\tilde{\chi}_L = \psi_+$ and $\tilde{\chi}_R = \psi_-$

$$:\chi^\dagger\chi: \to :\psi_+^\dagger\psi_+: + :\psi_-^\dagger\psi_-: , \tag{3.3}$$

$$\tilde{G}^{(2)}(1,2) = \langle 0|T:\psi_+^\dagger(1)\psi_+(1)::\psi_+^\dagger(2)\psi_+(2):|0\rangle + (+\to -) . \tag{3.4}$$

Take $t_1 > t_2$ and consider

$$\langle 0|:\psi_-^\dagger(1)\psi_-(1)::\psi_-^\dagger(2)\psi_-(2):|0\rangle = S_p^{(-)}(1,2)S_h^{(-)}(1,2) . \tag{3.5}$$

$S_p^{(-)}$ and $S_h^{(-)}$ are the particle and the hole propagators respectively for a given chirality. The wave functions for the states around the Fermi sea after the transformation look like

$$\langle \tau|\alpha\rangle = \left(\frac{\omega}{2\pi}\right)^{1/2} e^{i(\alpha-N)\omega\tau} , \tag{3.6}$$

where

$$\omega = \frac{2\pi}{L} \qquad L = \int_{-\infty}^{\infty} d\lambda\rho_0(\lambda) , \tag{3.7}$$

and $\alpha$ numbers the single-particle levels from the bottom of the Fermi sea.

$$S_p^{(-)}(1,2) = \sum_{\alpha>n} \langle\tau_1|\alpha\rangle\langle\alpha|\tau_2\rangle e^{-iN(\epsilon_\alpha-\epsilon_N)(t_1-t_2)} . \tag{3.8}$$

In the leading order

$$S_p^{(-)}(1,2) = \sum_{m=1}^{\infty} \frac{\omega}{2\pi} e^{-im\omega[(t_1-t_2)-(\tau_1-\tau_2)]} . \tag{3.9}$$

We define $t^\pm = t \pm \tau$. Hence

$$S_p^{(-)}(1,2) = \frac{\omega}{2\pi}\sum_{m=1}^{\infty} e^{-im\omega(t_1^- - t_2^-)} = \frac{\omega}{2\pi}\sum_{m=1}^{\infty} e^{-im\omega t_{12}^-} . \tag{3.10}$$

Similarly

$$S_h^{(-)}(1,\,2) = \sum_{\alpha \leq N} \langle \tau_2 | \alpha \rangle \langle \alpha | \tau_1 \rangle \, e^{-iN(\epsilon_N - \epsilon_\alpha)(t_1 - t_2)} \,, \tag{3.11}$$

which, in the leading order, becomes

$$S_h^{(-)}(1,\,2) = \frac{\omega}{2\pi} \sum_{n=0}^{\infty} e^{-in\omega t_{12}^-} \,. \tag{3.12}$$

Note that $S_p^{(-)} \approx S_h^{(-)}$. In fact, if we used $E_0$ not as the Fermi energy but the energy which is the average of the Fermi energy and the energy of the next higher level, then, in the leading order, we would have

$$S_p^{(-)} = S_h^{(-)} = \frac{\omega}{2\pi} \sum_{n=0}^{\infty} \exp\left[-i\left(n + \frac{1}{2}\right) \omega t_{12}^-\right] , \tag{3.13}$$

where the charge conjugation symmetry is more obvious. From here onward we use these propagators with explicit symmetry.

Now we take

$$\begin{aligned} G^{(2)}(1,\,2) &= \left(\frac{\omega}{2\pi}\right)^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-i(m+n+1)\omega t_{12}^+} + (- \rightarrow +) \\ &= \left(\frac{\omega}{2\pi}\right)^2 \sum_{j=1}^{\infty} j(e^{-ij\omega t_{12}^-} + e^{-ij\omega t_{12}^+}) \\ &= -\left(\frac{\omega}{2\pi}\right)^2 \sum_{j=1}^{\infty} j \int \frac{dE}{2\pi i} \frac{1}{E - j\omega + i\epsilon} \{e^{-i(Et_{12} - j\omega\tau_{12})} + e^{-i(Et_{12} + j\omega\tau_{12})}\} \\ &= -\frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E - |j\omega| + i\epsilon} e^{-i(Et_{12} - j\omega\tau)} \,. \end{aligned} \tag{3.14}$$

For $t_2 > t_1$

$$\begin{aligned} G^{(2)}(1,\,2) &= \left(\frac{\omega}{2\pi}\right)^2 \sum_{j=1}^{\infty} j(e^{-ij\omega t_{12}^-} + e^{-ij\omega t_{12}^+}) \\ &= \frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E + |j\omega| - i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \,. \end{aligned} \tag{3.15}$$

Since

$$\int dE \frac{1}{E - |j\omega| + i\epsilon} e^{-iEt_{12}} = 0 \qquad \text{for } t_2 > t_1 \,, \tag{3.16}$$

and

$$\int dE \frac{1}{E + |j\omega| - i\epsilon} e^{-iEt_{12}} = 0 \qquad \text{for } t_1 > t_2 , \tag{3.17}$$

in general

$$\begin{aligned} G^{(2)}(1, 2) &= -\frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E - |j\omega| + i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \\ &\quad + \frac{\omega}{8\pi^3 i} \sum_{j=-\infty}^{\infty} |j\omega| \int dE \frac{1}{E + |j\omega| - i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \\ &= \frac{i\omega}{4\pi^3} \sum_{j=-\infty}^{\infty} \int dE \frac{(j\omega)^2}{E^2 - (j\omega)^2 + i\epsilon} e^{-i(Et_{12} - j\omega\tau_{12})} \\ &= \frac{\partial}{\partial\tau_1} \frac{\partial}{\partial\tau_2} \left[ \sum_{j=-\infty}^{\infty} \frac{i\omega}{4\pi^3} \int_{-\infty}^{\infty} dE \frac{e^{-i(Et_{12} - j\omega\tau_{12})}}{E^2 - (j\omega)^2 + i\epsilon} \right] . \end{aligned} \tag{3.18}$$

The expression inside the square brackets is the correlator of a free Bose field. This is not surprising since what we have done is to bosonize the noninteracting fermions in a finite volume. We identify the free Bose field through the well-known relation

$$:\psi^\dagger \psi: = \partial_\tau \phi . \tag{3.19}$$

We can then see Eq. (3.18) coming out immediately from the Bose field correlator.

In the next section we investigate the interactions of this Bose field which are subleading in $1/N$.

## 4. The Three-Point Function of Density Fluctuations

For fermions satisfying the Dirac equation, the three-point function of density is zero. This is a consequence of the charge conjugation symmetry of the Dirac Hamiltonian. In other words, it is a consequence of the symmetry of the problem under reflection about the Fermi level. However, we know that this symmetry is broken in the nonrelativistic model and this is caused by the second term in the Hamiltonian. This term, treated as a perturbation, should provide systematic order by order contributions to the three-point function.

The lowest order contribution to the three-point function is obtained as follows. Take the region $t_1 > t_2 > t_3$ :

$$\tilde{G}^{(3)}(1, 2, 3) = \langle 0 | :\psi_+^\dagger(1)\psi_+(1): :\psi_+^\dagger(2)\psi_+(2): :\psi_+^\dagger(3)\psi_+(3): | 0 \rangle + (+ \to -) . \tag{4.1}$$

Now,

$$\langle 0|:\psi_-^\dagger(1)\psi_-(1)::\psi_-^\dagger(2)\psi_-(2)::\psi_-^\dagger(3)\psi_-(3):|0\rangle$$

$$= S_p^{(-)}(1,2)S_p^{(-)}(2,3)S_h^{(-)}(1,3) - S_h^{(-)}(1,2)S_h^{(-)}(2,3)S_p^{(-)}(1,3)\,. \quad (4.2)$$

This is zero if $S_p = S_h = S$, which is true in the lowest order. If we consider the first order corrections $\Delta S_p$ and $\Delta S_h$ to the propagator, then we have

$$S_p^{(-)}(1,2)S_p^{(-)}(2,3)S_h^{(-)}(1,3) - S_h^{(-)}(1,2)S_h^{(-)}(2,3)S_p^{(-)}(1,3)$$

$$\approx \Delta S_p^{(-)}(1,2)S_p^{(-)}(2,3)S_h^{(-)}(1,3) - \Delta S_h^{(-)}(1,2)S_h^{(-)}(2,3)S_p^{(-)}(1,3)$$

$$+ S_p^{(-)}(1,2)\Delta S_p^{(-)}(2,3)S_h^{(-)}(1,3) - S_h^{(-)}(1,2)\Delta S_h^{(-)}(2,3)S_p^{(-)}(1,3)$$

$$+ S_p^{(-)}(1,2)S_p^{(-)}(2,3)\Delta S_h^{(-)}(1,3) - S_h^{(-)}(1,2)S_h^{(-)}(2,3)\Delta S_p^{(-)}(1,3)\,. \quad (4.3)$$

Now $\Delta S_p$ and $\Delta S_h$ can be calculated in various ways. One way is to calculate the first order correction due to the second term of the Hamiltonian through relativistic perturbation theory. The same result is obtained if we use directly the WKB solution with its full energy dependence and extract the leading departure from a plane wave after giving the transformation (which is, essentially, division by a reference state $\rho_0^{1/2}e^{\pm iN\Theta_0}$). The "reference state" used in the transformation is a solution to the Schrödinger equation for the energy which is midway between $\epsilon_N$ and $\epsilon_{N+1}$. We indicate quantities related to this state by the index $N+\frac{1}{2}$.

$$\phi_{N+m+1}(\lambda) \sim \left(\frac{\omega_{N+m+1}}{\pi}\right)^{1/2} \frac{1}{(2(\epsilon_{N+m+1} - V(\lambda)))^{1/4}}$$

$$\cdot \exp\left\{\pm iN \int^{\lambda} d\lambda'(2(\epsilon_{N+m+1} - V(\lambda')))^{1/2}\right\}$$

$$\tilde{\phi}_{N+m+1}(\lambda) = \left(\frac{\omega_{N+1/2}}{\pi}\right)^{1/2} \frac{\phi_{N+m+1}(\lambda)}{\phi_{N+1/2}(\lambda)}$$

$$= \left(\frac{\omega_{N+1/2}}{\pi}\right)^{1/2} \left(\frac{\omega_{N+m+1}}{\omega_{N+1/2}}\right)^{1/2} \left(\frac{\epsilon_{N+1/2} - V(\lambda)}{\epsilon_{N+m+1} - V(\lambda)}\right)^{1/4}$$

$$\cdot \exp\left[\pm iN \int^{\lambda} d\lambda'\{(2(\epsilon_{N+m+1} - V(\lambda')))^{1/2}\right.$$

$$\left. - (2(\epsilon_{N+1/2} - V(\lambda')))^{1/2}\}\right]\,. \quad (4.4)$$

We consider $\omega$ as a function $\epsilon$ (when we use $\epsilon$, $\omega$ or $\partial\omega/\partial\epsilon$ in a formula, we can replace it by its value at Fermi level).

$$\tilde{\phi}_{N+m+1}(\lambda) = \left(\frac{\omega}{2\pi}\right)^{1/2}\left[1 + \frac{(m+\frac{1}{2})}{2N}\frac{\partial\omega}{\partial\epsilon} - \frac{(m+\frac{1}{2})\omega}{2N}\frac{1}{2(\epsilon - V)} \pm i\,\frac{(m+\frac{1}{2})^2\omega}{2N}\frac{\partial\omega}{\partial\epsilon}\tau \right.$$

$$\left. \mp i\,\frac{((m+\frac{1}{2})\omega)^2}{2N}\int^{\tau}\frac{d\tau'}{2(\epsilon - V)}\right] e^{\pm i(m+1/2)\omega\tau}\,. \tag{4.5}$$

Hence

$$\delta\tilde{\phi}_{N+m+1}(\lambda) = \left(\frac{\omega}{2\pi}\right)^{1/2}\frac{(m+\frac{1}{2})\omega}{2N}\left[\omega^{-1}\frac{\partial\omega}{\partial\epsilon}\left\{1 \pm i\left(m+\frac{1}{2}\right)\omega\tau\right\}\right.$$

$$\left. - \left\{\frac{1}{2(\epsilon - V)} \pm i\left(m + \frac{1}{2}\right)\omega\int^{\tau}\frac{d\tau'}{2(\epsilon - V)}\right\}\right] e^{\pm i(m+1/2)\omega\tau}\,. \tag{4.6}$$

$$\Delta S_p^{(-)}(1,2) = \delta\left\{\sum_{m=0}^{\infty}\langle\tau_1|N+m+1\rangle\langle N+m+1|\tau_2\rangle\, e^{-i(\epsilon_{N+m}-\epsilon_N)(t_1-t_2)}\right\}$$

$$= \frac{\omega}{2N\pi}\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)\omega e^{-i(m+1/2)\omega t_{\bar{12}}}\left[\omega^{-1}\frac{\partial\omega}{\partial\epsilon}\left\{1 - \frac{i}{2}\left(m+\frac{1}{2}\right)\omega t_{\bar{12}}\right\}\right.$$

$$\left. - \frac{1}{2}\left\{\frac{1}{2(\epsilon - V(1))} + \frac{1}{2(\epsilon - V(2))} + i\left(m+\frac{1}{2}\right)\omega\int_{\tau_2}^{\tau_1}\frac{d\tau}{2(\epsilon - V)}\right\}\right], \tag{4.7}$$

where we have used

$$\epsilon_{N+m+1} - \epsilon_{N+1/2} = \left(m+\frac{1}{2}\right)\omega\left(\frac{\epsilon_{N+1/2} + \epsilon_{N+m+1}}{2}\right)$$

$$= \left(m+\frac{1}{2}\right)\omega + \frac{(m+\frac{1}{2})^2\omega}{2N}\frac{\partial\omega}{\partial\epsilon}\,. \tag{4.8}$$

Similarly

$$\Delta S_h^{(-)}(1,2) = -\frac{\omega}{2N\pi}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\omega e^{-i(n+1/2)\omega t_{\bar{12}}}\left[\omega^{-1}\frac{\partial\omega}{\partial\epsilon}\left\{1 - \frac{i}{2}\left(n+\frac{1}{2}\right)\omega t_{\bar{12}}\right\}\right.$$

$$\left. - \frac{1}{2}\left\{\frac{1}{2(\epsilon - V(1))} + \frac{1}{2(\epsilon - V(2))} + i\left(n+\frac{1}{2}\right)\omega\int_{\tau_2}^{\tau_1}\frac{d\tau}{2(\epsilon - V)}\right\}\right]. \tag{4.9}$$

Therefore we get

$$\begin{aligned}
&\Delta S_p^{(-)}(1,2)S_p^{(-)}(2,3)S_h^{(-)}(1,3)\\
&= \frac{\omega^3}{8N\pi^3}\sum_{\ell,m,n=0}^{\infty}\exp\left\{-i\left(\ell+\frac{1}{2}\right)\omega t_{12}^- - i\left(m+\frac{1}{2}\right)\omega t_{23}^- - i\left(n+\frac{1}{2}\right)\omega t_{13}^-\right\}\\
&\cdot\left(\ell+\frac{1}{2}\right)\omega\left[\omega^{-1}\frac{\partial\omega}{\partial\epsilon}\left\{1-\frac{i(\ell+\frac{1}{2})\omega}{2}t_{12}^-\right\}\right.\\
&\left.-\frac{1}{2}\left\{\frac{1}{2(\epsilon-V(1))}+\frac{1}{2(\epsilon-V(2))}+i\left(\ell+\frac{1}{2}\right)\omega\int_{\tau_2}^{\tau_1}\frac{d\tau}{2(\epsilon-V)}\right\}\right]\\
&= \frac{\omega^3}{8N\pi^3}\sum_{\ell,m,n=0}^{\infty}\exp\{-i(\ell+n+1)\omega t_1^- + i(\ell-m)\omega t_2^- + i(m+n+1)\omega t_3^-\}\\
&\cdot\left(\ell+\frac{1}{2}\right)\omega\left[\omega^{-1}\frac{\partial\omega}{\partial\epsilon}\left\{1-\frac{i(\ell+\frac{1}{2})\omega}{2}(t_1^- - t_2^-)\right\}\right.\\
&-\frac{1}{2}\left\{\frac{1}{2(\epsilon-V(1))}+\frac{1}{2(\epsilon-V(2))}\right.\\
&\left.\left.+i\left(\ell+\frac{1}{2}\right)\omega\left(\int^{\tau_1}\frac{d\tau}{2(\epsilon-V)}-\int^{\tau_2}\frac{d\tau}{2(\epsilon-V)}\right)\right\}\right].
\end{aligned} \tag{4.10}$$

Combining all the six terms in (4.3) we get

$$\begin{aligned}
&\tilde{G}^{(3)}(1,2,3)\\
&= \frac{\omega^3}{8N\pi^3}\sum_{\ell,m,n=0}^{\infty}\exp\{-i(\ell+n+1)\omega t_1^- - i(\ell-m)\omega t_2^- + i(m+n+1)\omega t_3^-\}\\
&\cdot\left[(\ell-n)\omega\left(\omega^{-1}\frac{\partial\omega}{\partial\epsilon}-\frac{1}{2(\epsilon-V(1))}\right)\right.\\
&+(\ell+m+1)\omega\left(\omega^{-1}\frac{\partial\omega}{\partial\epsilon}-\frac{1}{2(\epsilon-V(2))}\right)\\
&+(m-n)\omega\left(\omega^{-1}\frac{\partial\omega}{\partial\epsilon}-\frac{1}{2(\epsilon-V(3))}\right)\\
&+i\left\{\left(n+\frac{1}{2}\right)^2-\left(\ell+\frac{1}{2}\right)^2\right\}\omega^2\left(\omega^{-1}\frac{\partial\omega}{\partial\epsilon}t_1^- + \int^{\tau_1}\frac{d\tau}{2(\epsilon-V)}\right)
\end{aligned}$$

$$+ i\left\{\left(\ell + \frac{1}{2}\right)^2 - \left(m + \frac{1}{2}\right)^2\right\} \omega^2 \left(\omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_2^- + \int^{\tau_2} \frac{d\tau}{2(\epsilon - V)}\right)$$

$$+ i\left\{\left(m + \frac{1}{2}\right)^2 - \left(n + \frac{1}{2}\right)^2\right\} \omega^2 \left(\omega^{-1} \frac{\partial \omega}{\partial \epsilon} t_3^- + \int^{\tau_3} \frac{d\tau}{2(\epsilon - V)}\right)\Bigg] . \tag{4.11}$$

Doing some partial summations (see App. A) we obtain the correlator for the region $t_1 > t_2 > t_3$.

$$\tilde{G}^{(3)}(1, 2, 3) = \sum_{j_1, j_2 = 1}^{\infty} \frac{\omega^4}{8\pi^3} e^{-ij_1 \omega t_1^- - i(j_2 - j_1)t_2^- + ij_2 t_3^-}$$

$$\cdot \Bigg[ \omega^{-1} \frac{\partial \omega}{\partial \epsilon} \{ j_2(j_1 - j_2)\Theta(j_1 - j_2)(1 - ij_1 \omega t_1^-)$$

$$+ j_2 j_1 (1 - i(j_2 - j_1)\omega t_2^-) + j_1(j_2 - j_1)\Theta(j_2 - j_1)(1 + ij_2 \omega t_3^-)\}$$

$$- j_2(j_1 - j_2)\Theta(j_1 - j_2)\left(\frac{1}{2(\epsilon - V(1))} + ij_1 \omega \int^{\tau_1} \frac{d\tau}{2(\epsilon - V)}\right)$$

$$- j_2 j_1 \left(\frac{1}{2(\epsilon - V(2))} + i(j_2 - j_1)\omega \int^{\tau_2} \frac{d\tau}{2(\epsilon - V)}\right)$$

$$- j_1(j_2 - j_1)\Theta(j_2 - j_1)\left(\frac{1}{2(\epsilon - V(3))} - ij_2 \omega \int^{\tau_3} \frac{d\tau}{2(\epsilon - V)}\right)\Bigg]$$

$$+ \left(t^- \to t^+ \quad \text{and} \quad \int^{\tau} \frac{d\tau'}{2(\epsilon - V)} \to - \int^{\tau} \frac{d\tau'}{2(\epsilon - V)}\right), \tag{4.12}$$

$\Theta(x)$ being the Heaviside function.

In the next section we explore the implications of this quantity for the effective action.

## 5. The Structure of the Effective Action

We want to keep only terms up to the order of $1/N$ in the equation of motion. It can be easily seen from $N$ counting that if we normalize the two-point connected Green function to be order 1, then the order of the $n$ point connected function is $N^{2-n}$. Hence we need to consider only the two- and the three-point function. The leading contribution to the three-point function is of the order $1/N$. The $1/N$ contribution to the two-point function cancels off. This is because the two-point function of the density is $S_h(1, 2)S_p(1, 2)$, which is $S(1, 2)^2$ in the lowest order. The next order is $\delta(S_h(1, 2)S_p(1, 2))$. Since in the lowest order $\Delta S_h = -\Delta S_p$, the first correction to the two-point function is zero. The correction is therefore $\sim O(1/N^2)$.

Hence, from what we have done till now, we can reconstruct in the lowest order quadratic and cubic pieces of the effective action. The quadratic piece is going to be that of a free boson field which is $2\pi \int dt d\tau \partial_+ \phi \partial_- \phi$. We need to choose a three-vertex which gives the correct three-point function. This three-point function has two pieces. One is proportional to $\omega^{-1}(\partial\omega/\partial\epsilon)$, the other is not. The first term is the dominant one for fixed $\lambda_i$, if we calculate $\langle \prod_i \rho(\lambda_i, t_i) \rangle$. However, if we change over to a scaled variable like $\tau - a$ then, since

$$\omega \sim |\ln \Delta\epsilon|^{-1}, \qquad \Delta\epsilon = V(\lambda_0) - \epsilon,$$
$$\frac{1}{N\omega}\frac{\partial\omega}{\partial\epsilon} \sim \frac{1}{|\ln \Delta\epsilon|}\frac{1}{N\Delta\epsilon} \to 0, \tag{5.1}$$

if $N\Delta\epsilon$ is held fixed when $N \to \infty$. On the other hand, quantities like

$$\frac{1}{N}\frac{1}{2(\epsilon - V)} \sim \frac{1}{N\Delta\epsilon}\frac{1}{\sinh^2(\tau - a)} \tag{5.2}$$

remain finite. Hence we pay less attention to the piece proportional to $\partial\omega/\partial\epsilon$. The other piece is the sum of two chiral contributions. This indicates that the vertex is made of $\partial_+\phi$ and $\partial_-\phi$. $\partial_\pm = \partial/\partial t^\pm$. In fact one can show (App. B) that the required interaction piece of the effective action is of the form

$$\Gamma_{\rm int} = \frac{-2\pi^2}{3N}\int dt d\tau \rho_f^2(\tau)\{(\partial_+\phi)^3 - (\partial_-\phi)^3\}. \tag{5.3}$$

It is remarkable that a very similar action can be obtained if one tries to bosonize the fermion theory naively by using the Mandelstam formulas,[18]

$$\psi_\pm^\dagger(\tau_1)\psi_\pm(\tau_2) = \mp\frac{i}{2\pi(\tau_1 - \tau_2)}\exp\left\{-\pi i\int_{\tau_1}^{\tau_2} d\tau(\dot\phi \pm \phi') + O(\tau_1 - \tau_2)^2\right\}. \tag{5.4}$$

(Note that our normalization of $\phi$ is different from Mandelstam's.) Now, one can separately differentiate in $\tau_1$ and $\tau_2$ and then take the limit $\tau_2 \to \tau_2$ and use the result in Eq. (2.26) to obtain the bosonic expression for the perturbation.

We know that the Mandelstam formulas depend crucially on the short distance properties of the Green function, which can be modified if the perturbation is singular. This is precisely the case here. Yet this procedure gives the same leading order effective action, except for $1/N \int dt d\tau \rho_f^2 \partial_\tau^3 \phi$ term (which, if genuinely present, should shift the background $\phi$ from zero to a value $\sim O(1/N)$ and in that process give $O(1/N^2)$ correction to the two-point function which no longer remains translation-invariant). It is possible that there is a generalization of the Mandelstam formulas in our case, where terms more singular than $1/(\tau_1 - \tau_2)$ appear, but they are always multiplied by higher powers of $1/N$ (or $g_{\rm str}$).

The equation of motion in the lowest order looks like

$$\partial_+\partial_-\phi = \frac{\pi}{2N}[\partial_+\{\rho_f^2(\partial_+\phi)^2\} - \partial_-\{\rho_f^2(\partial_-\phi)^2\}] , \tag{5.5}$$

since

$$\rho_f^2(\tau) \sim \frac{1}{4\mu \sinh^2(\tau - a)} , \tag{5.6}$$

for large $\tau - a$, i.e. for points far away from the turning point,

$$\rho_f^2(\tau) \sim \frac{e^{-2(\tau-a)}}{\mu} . \tag{5.7}$$

Then

$$\partial_+\partial_-\phi = \frac{\pi}{N\mu} e^{-2(\tau-a)}[-(\partial_+\phi)^2 + (\partial_-\phi)^2 + \partial_+\phi\partial_+^2\phi - \partial_-\phi\partial_-^2\phi] . \tag{5.8}$$

This is very similar to the tachyon equation. Note, however, that the interaction terms consist solely of derivatives of $\phi$ and not $\phi$ itself. Also, it can be written entirely in terms of the currents $j_\pm = \partial_\pm\phi$ + higher order terms.

One interesting feature of this bosonic theory is the existence of infinitely many conserved quantities. This is a consequence of the theory's being equivalent to a noninteracting fermion theory which is trivially integrable. The conserved quantities in the fermion theory would be $\int d\tau\chi^\dagger\hat{H}^n\chi$ for $n = 1, 2, 3, \ldots$ . It will be interesting to find the corresponding bosonic expressions for $n \geq 2$, which are nontrivial constants of motion of the time evolution described by Eq. (5.8).

## 6. Comparison With Continuum Results

There are two major issues to be discussed in this context. One involves the energy region in which the matrix model under consideration simulates the "Polyakov string." The other is regarding the choice of the "right" gravitational dressing. We discuss them in this order.

The naive discretization[19,8] of the Polyakov action through the large $N$ method would involve using an exponential propagator $e^{-\alpha' p^2}$. For the low momentum region $p \ll 1/\sqrt{\alpha'}$, $e^{\alpha' p^2} \sim \alpha'(1/\alpha' + p^2 + O(\alpha' p^4))$. Hence in that region we have the same results as those for propagator $1/(\alpha' p^2 + 1)$.

Thus $(\alpha')^{1/2}$ sets a target-space length scale so that only for distances much larger than that may the matrix theory resemble the Polyakov string. Note that this scale remains finite and does not go to zero. Therefore the Green functions show some correspondence to the continuum theory[12] only in the low momentum region.

The other point is that the KPZ formula[12] for the gravitational dressing gives two roots. Usually, one branch of the solution to this quadratic equation is chosen on the basis of semiclassical results. However, there is no compelling reason in favor of this specific resolution of the ambiguity.

On the other hand, matrix models should provide an unambiguous dressing in the associated theory of two-dimensional gravity. To see what it is exactly, we examine the two- and the three-point function.

Let us look at a correlator of the form $\langle \mathrm{tr}\, M^{n_1}(t_1)\ \mathrm{tr}\, M^{n_2}(t_2)\rangle$ for finite $n$.[20] This corresponds to a random surface sum with two points fixed in the one-dimensional target space, one at $t = t_1$ and the other at $t = t_2$. This depends on $t_1 - t_2$ only. We define

$$G(P) = \int_{-\infty}^{\infty} dt\, e^{iPt}\langle \mathrm{tr}\, M^{n_1}(t)\ \mathrm{tr}\, M^{n_2}(0)\rangle\ .$$

In terms of the random surface theory $G(P)$ can be thought of as the correlator of the dressed tachyon vertex operators with energy $P$ and $-P$.

$$\langle \mathrm{tr}\, M^{n_1}(t)\ \mathrm{tr}\, M^{n_2}(0)\rangle = \int d\lambda_1 d\lambda_2 \lambda_1^{n_1}\lambda_2^{n_2}\langle \mathrm{tr}\, \delta(M(t) - \lambda_1)\ \mathrm{tr}\, \delta(M(0) - \lambda_2)\rangle$$

$$= \int d\tau_1 d\tau_2 \lambda^{n_1}(\tau_1)\lambda^{n_2}(\tau_2)\tilde{G}^{(2)}(t, \tau_1; 0, \tau_2)\ .$$

Hence

$$G(P) = \int d\tau_1 d\tau_2 \lambda^{n_1}(\tau_1)\lambda^{n_2}(\tau_2)\int dt\, e^{iPt}\tilde{G}^{(2)}(t, \tau_1; 0, \tau_2)\ ,$$

$$\int dt\, e^{iPt}\tilde{G}^{(2)}(t, \tau_1; 0, \tau_2) = \partial_{\tau_1}\partial_{\tau_2}\sum_{j=-\infty}^{\infty}\frac{2\omega i}{\pi^2}\frac{e^{ij\omega\tau_{12}}}{P^2 - (j\omega)^2 + i\epsilon}\ .$$

This sum can be done by using the Poisson summation formula.[21] Finally, we have

$$G(P) = \frac{-iP}{\pi^2}\sum_{m=-\infty}^{\infty}\int d\tau_1 d\tau_2 \lambda^{n_1}(\tau_1)\lambda^{n_2}(\tau_2)\exp\left(iP\left|(\tau_1 - \tau_2) - \frac{2\pi m}{\omega}\right|\right),$$

$$\lambda(\tau) \sim \lambda_0 + \sqrt{2\mu}\cosh(\tau - a)$$

$$\sim \lambda_0 + \sqrt{\mu/2}e^{(\tau - a)} \quad \text{(for large } \tau - a)\ .$$

If we consider the sum that expresses $G(P)$, the term $m = 0$ does not depend upon $\omega$ strongly. This is the term that corresponds to the continuum tachyon propagator. The next term is the first indication of the discreteness of the spectrum. It is proportional to $e^{2\pi Pi/\omega}$. In Euclidean energy it becomes $e^{-2\pi|P|/\omega}$. This corresponds to the naive KPZ/DDK result. In the continuum analysis of a conformal field theory coupled to gravity, for each primary field in the conformal field theory, there are two allowed solutions for the gravitational dimension of the operator obtained by the gravitational dressing primary field. This is because the gravitational dimensions are determined by a quadratic equation which involves the central charge of the matter theory and the scaling dimension of the primary field. For the tachyon operator, $e^{iPt(z,\bar{z})}$, in the one-dimensional case these are $|P|$ and $-|P|$ for Euclidean time. If one calculates the correlation function of the dressed versions of the primary fields $e^{iPt}$ and $e^{-iPt}$ for the $d = 1$ case, then the answer is the cosmological constant raised to a power which is the sum of the two gravitational dimensions. Given the ambiguity of gravitational dressing, there are three possibilities. From the matrix model we seem to be getting two of these possibilities. The leading contribution is $\omega$-independent, indicating that the roots chosen must have been from *different branches* of the *solution* (i.e. $|P|$ and $-|-P|$, say). The subdominant contribution corresponds to the choice of similar roots (i.e. $|P|$ and $|-P|$).

This seems to be the case for the three-point function also. There we have three energies $P$, $Q$, $-(P+Q)$ appearing. The momentum dependence of the leading singular contribution, in Euclidean energy, is not $e^{-\pi(|P|+|Q|+|P+Q|)/\omega}$ as might be naively thought, but $e^{-2\pi|P|/\omega}$ or $e^{-2\pi|Q|/\omega}$ or $e^{-2\pi|P+Q|/\omega}$.[21] These will come if we choose a nonstandard sign in front of any one of the roots. Therefore we are tempted to conjecture that the matrix model provides both kinds of dressing and hence gives rise to more than one gravitationally dressed operator. However, there does not seem to be any simple scheme of local operator mixing which accounts for these results.

## 7. Possible Ambiguities of the $d = 1$ String

We have already indicated that the choice of the kinetic term of the matrix model restricts the momentum region in which universal behavior is observed. It is possible that even after the choice of the ordinary kinetic term and the choice of a particular kind of criticality, we still have ambiguities left in the theory. Until now we have not paid much attention to the way the formally bottomless potential, which appears in the double-scaling theory, is being regulated. One can try to argue that the precise boundary condition, chosen for the regularization, affects the theory at most through one cutoff parameter. This indeed seems to be the case for the large negative energy asymptotics of the single-particle density of states (see Brézin, Kazakov, Zamolodchikov[9]). However, one can raise two questions here:

(a) Are the scaled exponential corrections to this asymptotic expansion also universal?

(b) Even if the density of states is universal, is there any other nonuniversal property which is relevant to the one-dimensional string?

One laboratory for studying these questions is a set of $d = 1$ unitary matrix models. In the singlet sector of the unitary matrix model, this reduces to a problem of $N$ noninteracting fermions on a circle, moving under a given potential. The model has the same large $N$ critical behavior as the Hermitian models.[14,10] The compactness of the model provides a natural regularization since continuous potentials always have a minimum there. Usually the wave functions of the fermions on the circle are either periodic or antiperiodic, depending upon whether $N$ is odd or even. We shall extend this to the quasi-periodicity. This can be achieved by adding to the unitary matrix Lagrangian a total derivative term of the form

$$L_{\text{phase}} = \frac{\theta}{2\pi} \frac{d}{dt} \operatorname{tr} \ln U \,,$$

where $U$ is the unitary matrix and $\theta$ is the phase difference that gives rise to quasi-periodicity.

What are the consequences of a nonzero $\theta$ term? The obvious ones are existence of an average current and breaking of time reversal symmetry. What role these (or some similar effect) have in the associated string theory is an interesting question for future investigations.

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## Appendix A

Putting $l + n + 1 = j_1$ and $m + n + 1 = j_2$ in Eq. (4.11), we have

$$\begin{aligned}
\tilde{G}^{(3)}(1, 2, 3) = \frac{\omega^3}{8N\pi^3} \sum_{j_1, j_2 = 1}^{\infty} & \exp\{-ij_1\omega t_1^- + i(j_1 - j_2)\omega t_2^- + ij_2\omega t_3^-\} \\
\cdot \sum_{n=0}^{\min(j_1, j_2)-1} & \Bigg[ (j_1 - 1 - 2n)\omega \left( \omega^{-1} \frac{\partial\omega}{\partial\epsilon} - \frac{1}{2(\epsilon - V(1))} \right) \\
& + (j_1 + j_2 - 1 - 2n)\omega \left( \omega^{-1} \frac{\partial\omega}{\partial\epsilon} - \frac{1}{2(\epsilon - V(2))} \right) \\
& + (j_2 - 1 - n)\omega \left( \omega^{-1} \frac{\partial\omega}{\partial\epsilon} - \frac{1}{2(\epsilon - V(3))} \right) \\
& - ij_2(j_2 - 1 - n)\omega^2 \left( \omega^{-1} \frac{\partial\omega}{\partial\epsilon} t_1^- + \int^{\tau_1} \frac{d\tau}{2(\epsilon - V)} \right) \\
& + i(j_1 - j_2)(j_1 + j_2 - 1 - 2n)\omega^2 \left( \omega^{-1} \frac{\partial\omega}{\partial\epsilon} t_2^- + \int^{\tau_2} \frac{d\tau}{2(\epsilon - V)} \right) \\
& + ij_2(j_2 - 1 - 2n)\omega^2 \left( \omega^{-1} \frac{\partial\omega}{\partial\epsilon} t_3^- + \int^{\tau_3} \frac{d\tau}{2(\epsilon - V)} \right) \Bigg] \,. \qquad \text{(A.1)}
\end{aligned}$$

Now we use the relation

$$\sum_{n=0}^{\min(j_1,j_2)-1} (a-1-2n) = \min(j_1,j_2)(a-\min(j_1,j_2)) \tag{A.2}$$

to get the identities

$$\begin{aligned}
&\sum_{n=0}^{\min(j_1,j_2)-1} (j_1+j_2-1-2n) = j_1 j_2\,,\\
&\sum_{n=0}^{\min(j_1,j_2)-1} (j_2-1-2n) = \Theta(j_2-j_1)(j_2-j_1)j_1\,,\\
&\sum_{n=0}^{\min(j_1,j_2)-1} (j_1-1-2n) = \Theta(j_1-j_2)(j_1-j_2)j_2\,,
\end{aligned} \tag{A.3}$$

Using these, one gets Eq. (5.1) trivially.

## Appendix B

To check that the vertex is really $(2\pi^2/3N)\rho_f^2(\tau)\{(\partial_-\phi)^3-(\partial_+\phi)^3\}$, we calculate the leading order connected three-point correlator of density using this vertex.

$$\begin{aligned}
&\langle\tilde\rho(1)\tilde\rho(2)\tilde\rho(3)\rangle\\
&\quad=\langle\partial_{\tau_1}\phi(1)\partial_{\tau_2}\phi(2)\partial_{\tau_3}\phi(3)\rangle\\
&\quad\approx\frac{2\pi^2 i}{3N}\int dt d\tau\rho_f^2(\tau)\left\langle\{(\partial_-\phi)^3-(\partial_+\phi)^3\}\prod_i\partial_{\tau_i}\phi(i)\right\rangle_{\text{free}}\\
&\quad=4\pi^2 i\int dt d\tau\rho_f^2(\tau)\left\{\prod_i\partial_{\tau_i}\partial_{t^-}K(\tau_i,t_i;\tau,t)-\prod_i\partial_{\tau_i}\partial_{i^+}K(\tau_i,t_i;\tau,t)\right\},
\end{aligned} \tag{B.1}$$

where

$$K(\tau_i,t_i;\tau,t)=\langle\phi(1)\phi(2)\rangle_{\text{free}}\,, \tag{B.2}$$

and the subscript "free" refers to correlators calculated for the free Bose theory.

Since we are going to make the leading order estimate, we can use the continuum form of $K(1;2)$ and ignore the discreteness of allowed energy and momenta. Then

$$\partial_{\tau_i}\partial_{t^\mp}K(\tau_i,t_i;\tau,t)=\pm\frac{1}{4\pi^2}\frac{1}{(t^\mp-t_i^\mp-i\epsilon s(t-t_i))^2}\,, \tag{B.3}$$

where $s(x)=x/|x|$.

From this point onward we concentrate on one chirality (say, right chirality) only. The other chirality also works out the same way. The relative sign between these two pieces originates because of the ± on the right-hand side of (B.3).

The contribution of the right chirality is

$$\tilde{G}_-(1, 2, 3) = \frac{i}{(2\pi)^4} \int d\tau \rho_f^2(\tau) \int dt \prod_i \frac{1}{(t^- - t_i^- - i\epsilon s(t - t_i))^2} . \tag{B.4}$$

Consider the case $t_1 > t_2 > t_3$. The time integral in (B.4) has four regions, in each of which $s(t - t_i)$'s have definite signs. Using the identity

$$\begin{aligned}\int^x &\frac{dy}{\{(y-a)(y-b)(y-c)\}^2}\\ &= \left[-\frac{1}{(x-a)(a-b)^2(a-c)^2} + \frac{2b+2c-4a}{(a-b)^3(a-c)^3}\ln(x-a)\right]\\ &\quad + (a, b, c \to b, c, a) + (a, b, c \to c, a, b) ,\end{aligned} \tag{B.5}$$

along with

$$\frac{1}{u - i\epsilon} - \frac{1}{u + i\epsilon} = 2\pi i \delta(u) , \tag{B.6}$$

and its consequence

$$\ln \frac{u + i\epsilon}{u - i\epsilon} = -2\pi\Theta(u) + \text{const} , \tag{B.7}$$

where the constant depends upon convention, we get

$$\tilde{G}_-(1, 2, 3) = -\frac{1}{8\pi^3}\left[\rho_f^2(\tau_1) - \int d\tau \rho_f^2(\tau)\partial_{t_1^-}\right]\frac{1}{(t_{12}^- t_{13}^-)^2} + \text{other, similar terms} . \tag{B.8}$$

This is what one gets when one takes the dominant part of (4.12) and converts the discrete sums into integrals.

## References

1. S. R. Das, S. Naik and S. R. Wadia, *Mod. Phys. Lett.* **A4** (1989) 1033.
2. A. Dhar, T. Jayaraman, K. S. Narain and S. R. Wadia, *Mod. Phys. Lett.* **A5** (1990) 863.
3. S. R. Das, A. Dhar and S. R. Wadia, *Mod. Phys. Lett.* **A5** (199) 799.
4. A. B. Zamolodchikov, *JETP Lett.* **43** (1986) 731.

5. S. R. Das, G. Mandal and S. R. Wadia, *Mod. Phys. Lett.* **A4** (1989) 745.
6. J. Polchinski, Texas preprint UTTG-02-89 (1989).
7. T. Banks and J. Lykken, Univ. of California, Santa Cruz preprint (May, 1989).
8. V. A. Kazakov and A. A. Migdal, *Nucl. Phys.* **B320** (1989) 654.
9. E. Brézin, V. A. Kazakov and Al. B. Zamolodchikov, Ecole Normale preprint LPS-ENS 99-19R; G. Parisi, *Phys. Lett.* **B238** (1990) 209; D. Gross and N. Miljković, *Phys. Lett.* **B238** (1990) 217; P. Ginsparg and J. Zinn-Justin, Harvard preprint, HUTP-90/A004 (1990).
10. S. R. Das, A. Dhar, A. M. Sengupta and S. R. Wadia, *Mod. Phys. Lett.* **A5** (1990) 891.
11. E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, *Commun. Math. Phys.* **59** (1978) 35.
12. A. M. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 899; V. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819.
13. F. David, *Mod. Phys. Lett.* **A3** (1988) 1651; J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509.
14. S. R. Wadia, *Phys. Lett.* **B93** (1980) 403.
15. S. R. Wadia, talk given at Cargese Summer School, on "Random Surfaces, Quantum Gravity and Strings," May 28–June 2, 1990. In the same meeting, D. Gross also discussed the Dirac representation.
16. J. Polchinski, Texas preprint UTTG-15-90.
17. S. R. Das and A. Jevicki, Brown preprint (1990).
18. S. Mandelstam, *Phys. Rev.* **D11** (1975) 3026.
19. F. David, *Nucl. Phys.* **B257** (1985) 45; V. Kazakov, *Phys. Lett.* **150B** (1985) 282; J. Ambjørn, B. Durhuus and J. Fröhlich, *Nucl. Phys.* **B257** (1985) 433.
20. I. Kostov, *Phys. Lett.* **B215** (1988) 499.
21. S. Ben-Menahim, SLAC preprint, SLAC-PUB-5652 (1990).