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QUANTIZATION OF THE LIOUVILLE MODE AND STRING THEORY

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We discuss the space-time interpretation of bosonic string theories, which involve d scalar fields coupled to gravity in two dimensions, with a proper quantization of the world-sheet metric. We show that for $d > 25$, the theory cannot describe string modes consistently coupled to each other. For $d = 25$ this is possible; however, in this case the Liouville mode acts as an extra timelike variable and one really has a string moving in 26-dimensional space-time with a Lorentzian signature. By analyzing such a string theory in background fields, we show that the $d = 25$ theory possesses the full 26-dimensional general covariance.

Recently, there has been important progress in understanding the structure of two-dimensional field theories coupled to gravity. This is of obvious importance in string theory which involves bosonic and fermionic fields coupled to the worldsheet metric. Most of contemporary discussions of string theory regard the worldsheet metric as a *classical* variable. Consistency then requires that, with some *given* metric, quantities like the propagator, amplitudes, etc. are independent of the metric. In the conformal gauge, the only degree of freedom of the metric is the Liouville mode, and this requirement is equivalent to the requirement that the two-dimensional field theory is conformally invariant. For the free bosonic string, e.g., this happens when the space-time-dimensionality d is 26. If the metric is treated as a proper quantum variable, requiring conformal invariance thus does not make any sense and d is not *a priori* restricted to 26.

The problem of quantizing the Liouville mode has been studied for quite some time.[1,2] In particular, Gervais and Neveu (GN) have studied exact quantization using a particular method of quantization. Recently, Polyakov and Knizknik, Polyakov and Zamolodchikov (KPZ)[3] have solved the Liouville problem exactly by using a light cone gauge for the worldsheet metric and utilizing algebraic techniques. The most important consequence of the work of KPZ is the calculation of anomalous dimensions of generic statistical models coupled to gravity in two dimensions.

In this letter, we shall address some basic issues about the string-theoretic aspects of the integration over the Liouville mode. We shall argue that for $d > 25$, this mode does not give rise to a consistent theory of string modes interacting in space-time. We shall also demonstrate in a precise manner how the standard 26-dimensional bosonic string emerges as a string propagating in a space with Lorentzian signature where the Liouville mode itself acts as a timelike coordinate. We shall present the treatment in the conformal gauge.[4,5] Our discussion is in the context of the bosonic string; the supersymmetric string may be dealt with along similar lines.

The free string propagator is given, in the Polyakov formulation, by the functional integral

$$\int \mathscr{D}_g g_{ab}(\xi)\mathscr{D}_g X^\mu(\xi)\exp\left[-\frac{1}{2\alpha'}\int d^2\xi\sqrt{g}g^{ab}\partial_a x^\mu\partial_b x^\nu\eta_{\mu\nu}\right] \tag{1}$$

where $X^\mu(\xi)(\mu = 1, 2, \ldots d)$ denotes the spacetime location of a point on the worldsheet (these are simply d scalar fields in two dimensions). ξ^a are coordinates on the world sheet and $g_{ab}(\xi)$ is its metric. Let us choose a gauge $g_{ab}(\xi) = e^{\phi(\xi)}g^0_{ab}$, where g^0_{ab} is a fiducial metric. (In this paper, we shall only consider worldsheets of spherical topology. Surfaces of higher genus may also be treated by well known methods, but do not add anything important to the essential physics discussed in this work). At the critical point, i.e., where a continuum theory may be defined, the worldsheet cosmological constant vanishes and (1) becomes:

$$\int \mathscr{D}_g\phi\,\mathscr{D}_g X\mathscr{D}_g b\mathscr{D}_g c\,\exp[(d-26)S_L - S_M - S_{gh}]\,, \tag{2}$$

where

$$S_L = -\frac{1}{48\pi}\int d^2\xi\sqrt{g_0}(g_0^{ab}\partial_a\phi\partial_b\phi + R_0\phi) \tag{3}$$

$$S_M = \int d^2\xi\sqrt{g_0}(g_0^{ab}\partial_a X^\mu\partial_b X^\nu\xi_{\mu\nu}) \tag{4}$$

$$S_{gh} = \int d^2\xi\sqrt{g_0}(b^{ab}\nabla_b c_a)\,. \tag{5}$$

Here b, c are the ghost fields and R_0 is the scalar curvature of the metric g_0^{ab}. However, just from its definition, (2) must be invariant under the transformations $g^0_{ab} \to e^{\gamma(\xi)}g^0_{ab}$. We can choose

$$g^0_{ab} = e^{\sigma(\xi)}\delta_{ab} \tag{6}$$

without loss of generality; (2) must be now independent of $\sigma(\xi)$. If quantum fluctuations of $\phi(\xi)$ are ignored, this can happen only for $d = 26$. This, however, is not correct since $\phi(\xi)$ is a quantum field at par with X^μ.

The difficulty in dealing with the full quantum theory of $\phi(\xi)$ is the fact that the various measures $\mathscr{D}_g\phi$, $\mathscr{D}_g X$, etc. are all ϕ-dependent. However, as explained in Ref. 3, one can transform to new variables which have a flat measure in the space of $\phi(\xi)$, whence (2) becomes

$$\int \mathscr{D}\phi\mathscr{D}X\mathscr{D}b\mathscr{D}c \exp[-\lambda S_L - S_M - S_{gh}]\,. \tag{7}$$

(7) is simply a field theory with the fields $X^\mu(\xi)$,$\phi(\xi)$, $b(\xi)$, $c(\xi)$ living on a curved two-dimensional space with the metric $g^0_{ab} = e^\sigma\delta_{ab}$. Then, independence of (7) on $\sigma(\xi)$ means that this theory is conformally invariant. For the free string, this means that the total central charge must vanish. The central charge of ϕ field is $1 + \lambda$ (λ is the contribution from the classical piece $R_0\phi$ in the action), that of the X^μ is d, while that of the ghosts is -26. Thus λ is determined to be

$$\lambda = 25 - d\,.$$

We wish to emphasize that the requirement that the two-dimensional theory given by (7) is conformally invariant is not an additional requirement in the formalism, as is for conventional formulations which ignore the fluctuations of ϕ. If one knew how to deal with (2), this would be automatic. Equation (7) is an ansatz for the evaluation of (2), consistent with the basic condition that it should be independent of the conformal mode of g_0; that is why one has to impose the condition of conformal invariance to determine λ. In physical terms, in a two-dimensional field theory coupled to gravity, a change of the conformal mode of the metric means a change of the cutoff. Since the metric is integrated over, the theory is automatically scale-invariant. In usual string theory, conformally invariant points correspond to on-shell theory while departure from conformal invariance constitute an off-shell generalization. *The above considerations seem to imply that when the Liouville mode is integrated there is no notion of such an off-shell continuation.*

It is convenient to rescale ϕ to write the action in (6) as

$$\int d^2\xi\sqrt{g_0}\left[\frac{1}{2\alpha'}g_0^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2\alpha'}g_0^{ab}\partial_a X^\mu\partial_b X_\mu + \sqrt{\frac{\lambda}{\alpha'}}\,\phi R_0 + (\text{ghosts})\right]. \tag{8}$$

Introducing the notation $X^i = (X^\mu, \phi)$ (i now runs from 1 to $d + 1$), this becomes

$$\int d^2\xi\sqrt{g_0}\left(\frac{1}{2\alpha'}g_0^{ab}\partial_a X^i\partial_b X^j\eta_{ij} + R_0\mathscr{D}(x^i)\right),$$

where

$$\mathscr{D}(x^i) = \phi\sqrt{\frac{\lambda}{\alpha'}}$$

which looks like a $(d+1)$-dimensional string moving in a *given* dilation background $\mathscr{D}(x^i) = \phi\sqrt{\frac{\lambda}{\alpha'}}$. This given background, of course, breaks the $(d+1)$-dimensional Lorentz symmetry to a d-dimensional symmetry.

When $d = 25$, i.e., $\lambda = 0$, ϕ can be truly interpreted as a $(d+1)$-dimensional coordinate, and the theory is a 26-dimensional bosonic string. However, as explained in Refs. 2 and 3, various exponents of the conformal field theory described by (8) become complex for $1 < d < 25$. Thus $d = 25$ must be approached from above. This means that λ is purely imaginary. To make sense of the theory, ϕ must be analytically continued to $i\phi$, i.e., the contour of integration over ϕ must be chosen along the imaginary axis. As a consequence, the kinetic term of ϕ flips sign. This implies that at $d = 25$, ϕ becomes a *time-like* coordinate of a 26-dimensional Minkowski space theory.

We would like to emphasize that the emergence of the Liouville mode as the time-like coordinate is a description in the conformal gauge. KPZ originally studied the problem in the light cone gauge. There the theory of the degree of freedom of the worldsheet metric h_{++} becomes a SL(2, R) current algebra. For $d < 25$, the central charge of this current algebra becomes complex. The role of the Liouville mode is played by a combination of a central charge -2 ghost system and the h_{++} field which has a central charge $28 - d$. It is intriguing that the description of "time" at $d = 25$ is rather complicated in this gauge, while it is extremely simple in the conformal gauge.

To understand the physical meaning of such a string theory for general values of λ, let us first consider the scalar spectrum described by operators $T(x^i) = T(x^\mu, \phi)$. The spectrum is obtained, as usual, by requiring

$$\frac{\delta}{\delta\sigma}\int d^2\xi\sqrt{g_0}\,\langle T(x^\mu, \phi)\rangle = 0\,, \tag{9}$$

the expectation value being taken in the theory defined by (8). Using standard methods, this yields

$$\alpha'[\partial^i\partial_i - 4\pi(\partial_i\mathscr{D})\partial^i]T(x^\mu, \phi) = 2T(x^\mu, \phi)\,. \tag{10}$$

Substituting for the explicit form of $\mathscr{D}$ and continuing $\phi \rightarrow i\phi$, we obtain

$$\alpha' \left[\partial^\mu \partial_\mu - \partial_\phi^2 + 4\pi i \sqrt{\frac{\lambda}{\alpha'}}\, \partial_\phi \right] T(x^\mu, \phi) = 2T(x^\mu, \phi)\,. \tag{11}$$

Once again, for $\lambda = 0$, this becomes the standard mass-shell condition for tachyon in $(25 + 1)$ dimensions. For $\lambda \neq 0$ there is no $d + 1$ dimensional interpretation. Viewed as a theory in d dimensions, (11) describes an infinite number of particles with a continuous spectrum of masses. This is transparent in terms of fourier modes of T

$$T(x^\mu, \phi) = \int dk dq \exp(ik^\mu x_\mu + iq\phi) T_q(k)$$

in which case the mode T_q has a mass

$$m_q^2 = q^2 - 4\pi q \sqrt{\frac{\lambda}{\alpha'}} - \frac{2}{\alpha'}\,. \tag{12}$$

This general feature of having an infinite number of particles with a continuously varying mass persists for each spin. For $d > 25$, $\lambda < 0$ and m_q^2 is complex. We shall examine the consequences of this later.

To gain further insight into the space-time features, we now consider such strings propagating in a background target space metric $G_{\mu\nu}(x)$. The propagator is given, to start with, by

$$Z = \int \mathscr{D}_g g_{ab} \mathscr{D}_g X^\mu \sqrt{G} \exp\left[-\frac{1}{2\alpha'} \int d^2\xi \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(x) \right].$$

As discussed earlier, the measures depend on the full metric g_{ab}, i.e., on the Liouville mode in the conformal gauge. Transforming to variables in terms of which the measure in the ϕ-space is flat, we expect, just as before, a general two-dimensional field theory of the fields X^μ and ϕ. However, due to the presence of non-trivial couplings $G_{\mu\nu}(x)$, renormalization of this field theory will now produce all possible relevant and marginal operators (in the renormalization group sense). Thus, to perform a complete analysis, one must start with all possible operators of naive dimension two. (As is well known, this is sufficient only in a perturbation theory in α'. Non-perturbatively, many more operators are needed. In this paper, we shall be concerned with only the perturbation expansion of the sigma model.) We therefore consider

$$Z = \int \mathcal{D}\phi \mathcal{D}X \mathcal{D}b \mathcal{D}c \exp$$

$$\times \left\{ -\frac{1}{2\alpha'} \int d^2\xi \sqrt{g_0} (g_0^{ab} \partial_a X^i \partial_a X^j G_{ij}(x^k) + R_0 \tilde{D}(x^k) \right\}, \tag{13}$$

where

$$G_{ij}(x^k) = G_{ij}(X^\mu, \phi)$$

is the generalised metric and $\tilde{D}(x^k)$ is given by

$$\tilde{D}(x^k) = \sqrt{\frac{\lambda}{\alpha'}}\, \phi + D(X^\mu, \phi) \tag{14}$$

and $D(X^\mu, \phi)$ is the corresponding generalisation of the dilaton field. The crucial difference between (13) and a standard $(d+1)$-dimensional string in a background is embodied in (14). The "dilaton" $\tilde{D}(X^k)$ always contains the classical piece $\sqrt{\lambda/\alpha'}\ \phi$ which cannot be put to zero. Furthermore, because of this piece, $\tilde{D}(X^k)$ is not a genuine scalar in $(d+1)$ dimensions. For our discussion, we can put $D(X^\mu, \phi) = 0$ without disturbing the essential point.

The theory described by (13) must be conformally invariant, i.e., all its beta functions must vanish. These may be calculated in the loop expansion by standard techniques using normal cordinates; the fact that $\tilde{D}(X^k)$ is not a scalar has to be remembered, but does not present any complication.

The two pertinent beta function are β_{ij} and $\beta_{\tilde{D}}$ associated with the operators $\partial X^i \partial X^j$ and R_0 respectively. The fixed point conditions are

$$\beta_{ij} = -2\sqrt{\lambda\alpha'}\, \Gamma^\phi_{ij} + \alpha' R_{ij} = 0 \tag{15}$$

$$\beta_{\tilde{D}} = \frac{d - 25 + \lambda G^{\phi\phi}}{\alpha'} + \sqrt{\frac{\lambda}{\alpha'}}\, G^{ij}\Gamma^\phi_{ij} - \frac{1}{4}R = 0\,, \tag{16}$$

where R_{ij} is the Ricci tensor constructed out of the full metric G_{ij} and Γ^k_{ij} are the Christoffel symbols. Equations (15) and (16) are the classical equations of motion of the string mode $G_{ij}(X^k)$.

The central charge condition (16) determines λ. From (15), and covariant consistency of the Einstein tensor, it may be verified that

$$\partial_j\left(\frac{1}{2}R\right) = \nabla^i R_{ij} \equiv G^{ik}(\partial_k R_{ij} - \Gamma^m_{ki} R_{mj} - \Gamma^m_{kj} R_{mi})$$

$$= 2\partial_j\left(\sqrt{\frac{\lambda}{\alpha'}}\, G^{ik}\Gamma^\phi_{ik} + \frac{\lambda}{\alpha'} G^{\phi\phi}\right), \tag{17}$$

so that

$$\partial_j\left(\frac{\lambda}{\alpha'} G^{\phi\phi} + \sqrt{\frac{\lambda}{\alpha'}}\, G^{ik}\Gamma^\phi_{ik} - \frac{1}{4}R\right) = 0\,. \tag{18}$$

This means $\beta_{\tilde{D}}$ is a constant (which is a crucial consistency check since $\beta_{\tilde{D}}$ is the central charge).

As expected, the equations of motion (15) and (16) do not possess general coordinate invariance in $(d+1)$-dimensions if $\lambda \neq 0$. However, for $\lambda = 0$, the non-covariant terms involving Γ^ϕ_{ij} and $G^{\phi\phi}$ drops out. Equation (16) shows that one can consistently take $d = 25$, while Eq. (15) shows that at $\lambda = 0$, ϕ can be genuinely interpreted as the $(d+1)$ st. timelike coordinate; the resulting theory being fully generally covariant. This is the standard 26-dimensional bosonic string. The fashion in which this theory has been obtained, however, shows that one of the dimensions is necessarily timelike. We find this necessary emergence of a Lorentzian signature an extremely intriguing feature of the properly quantized string theory.

For $\lambda \neq 0$, the theory is invariant under ϕ-independent transformations of the d coordinates X^μ

$$\phi \rightarrow \phi' = \phi$$

$$X^\mu \rightarrow X^{\mu'} = X^{\mu'}(X^\nu)$$

since under such transformation Γ^ϕ_{ij} is a second rank tensor and $G^{\phi\phi}$ is a scalar. In the d-dimensional language, the metric G_{ij} breaks up into the d-dimensional metric $G_{\mu\nu}$, a vector $A_\mu = G_{\mu\phi}$ and a scalar $B = G_{\phi\phi}$. This is, of course, the Kaluza-Klein mechanism. However, since the full $(d+1)$-dimensional general coordinate invariance is not a symmetry, A_μ is not a gauge field. (In standard Kaluza-Klein theory, gauge transformations are general coordinate transformations of the extra coordinate ϕ, which is not a symmetry here.)

Another crucial difference here is that the extra dimension is not compact, so that there is a continuous, rather than a discrete, mass spectrum. We already saw this for the scalar mode $T(X, \phi)$, this may be seen at spin-2 level by the linearized form of (15).

We now turn to the question of whether the $\lambda \neq 0$ theory admits nontrivial classical solutions. Consider e.g. the equation of motion (15). The two terms are different powers in α'. Thus in the α' expansion, one has to set both the terms individually to zero, i.e.,

$$\Gamma^{\phi}_{ij} = 0, R_{ij} = 0 .$$

In principle, it is possible to find a solution where the two terms cancel each other, provided there is another small parameter and one makes a double expansion in α' and that parameter. It is not clear what that parameter is. The strength of the background $h_{ij} = G_{ij} - \delta_{ij}$ is not one since both terms in (15) start off as linear in h_{ij}. By the same token, for (16) to hold, $G^{\phi\phi}$ must be individually constant.

A two-dimensional field theory coupled to gravity constitutes a string theory if it describes a consistent model of string modes interacting with each other. We now examine whether the $\lambda \neq 0$ model is such a consistent theory in space-time. As we demonstrated above, in this case ϕ cannot be regarded as a space-time coordinate. Consider e.g. the scalar mode $T(X, \phi)$. In terms of the fourier components $T_q(k)$, the mass shell condition (10) is

$$\alpha'\left(-k^2 + q^2 - 4\pi q\sqrt{\frac{\lambda}{\alpha'}}\right) = 0 \tag{19}$$

for $d > 25$, $\lambda < 0$; Denoting $\sqrt{\dfrac{\lambda}{\alpha'}} = iQ$ (Q real), (20) becomes

$$k^2 = q^2 - 4\pi iQq + 2 . \tag{20}$$

This means that the $(\text{mass})^2$ is complex; the space-time momentum k cannot be a hermitian operator even with an indefinite metric. This immediately shows that the states created by $T_q(k)$ cannot be identified with sensible particles in space-time. The culprit is, of course, the anomaly in the Liouville sector. The $q = 0$ sector seems to be free of this particular problem. Is it then possible to construct a theory in which the only excitations are ϕ-independent, i.e., have $q = 0$. To answer this question, one must ask whether the scattering of $q = 0$ particles produces particles with $q \neq 0$. The linear part of the beta function. Eq. (20), cannot decide this issue: since interactions are contained in the quadratic and higher order terms in the beta function. One can, however, provide an answer to this question without performing a detailed beta function calculation. The crucial point is that due to the presence of the terms $Q\phi R\sqrt{g_0}$ in the Lagrangian for ϕ, translations of ϕ are not symmetries. In a slightly different language, there is a background charge Q. This means the quantum number q of the mode $T_q(k)$ is not conserved; rather, in a process of interaction of two modes, the total value of q changes by Q. Thus two $q = 0$ particles produce a particle with $q = Q$. A self-contained interacting theory must include modes with all values of q. This, as argued above, does not give rise to any sensible theory. A similar argument applies to other mass levels.

This problem does not, however, arise if we consider ϕ as a purely internal coordinate for $Q = 0$ (i.e., which does not become a time in the limit $d \to 25$) and one uses operators like $e^{ikx}e^{q\phi}$ for real q to create states. The dispersion relation now becomes

$$K^2 = q^2 + 4\pi Qq + 2.$$

The (mass)^2 is now real, but due to the presence of the linear term in q, the spectrum is unbounded from below. This does not lead to a sensible string theory.

We thus conclude that the Polyakov bosonic string theory does not lead to any sensible model of interacting modes in space-time for $d > 25$. For $d = 25$, the anomaly in the Liouville sector disappears — the Liouville mode ϕ becomes a time-like coordinate and one has a theory in 26 dimensions with a Lorentzian signature. It is not clear what happens for $d < 25$. As shown in Refs. 2 and 3, various exponents become complex if the theory is quantized in the manner described above. The crucial point in this quantization is that the continuum theory is constructed around a critical point where the world sheet cosmological constant vanishes. It is conceivable that for $d < 25$, there exists a different critical point at which the continuum theory makes sense. Indeed, the work of GN seems to indicate that the cosmological constant may be nonzero at such a critical point, though the relationship between the work of GN and the present approach is quite unclear. One way to address this question is to perform a double expansion in the strength of the background and λ. This is because λ is analogous to the ε parameter in usual renormalisation group theory. It is also possible that the description of the theory in terms of the Liouville action and a flat measure itself does not work in the regime $1 < d < 25$. In the KPZ formulation, the continuum description in terms of the current algebra breaks down. These issues are under investigation at this moment. Our discussion can be extended to the supersymmetric string[6] and is being currently pursued.

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References

1. J. L. Gervais and A. Neveu, *Nucl. Phys.* **B199** (1982) 59; *ibid.* **B209** (1982) 125; *ibid.* **B224** (1983) 329; *ibid.* **B238** (1984) 125, 396; *ibid.* **B264** (1986) 557 and *Phys. Lett.* **151B** (1985) 59.

2. T. Curtright and C. Thorn, *Phys. Rev. Lett.* **48** (1982) 1309; E. D'Hoker and R. Jackiw, *Phys. Rev.* **D26** (1982) 3517; T. Yoneya, *Phys. Lett.* **148B** (1984) 111; F. David and E. Guitter, *Europhys. Lett.* **3** (1987) 1169.
3. A. M. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 899; *V. G. Knizhnik, A. M. Polyakov and A. B. Zamoldchikov, Mod. Phys. Lett.* **A3** (1988) 819.
4. F. David, Saclay Preprint SPh/88-132 (1988).
5. J. Distler and H. Kawai, Cornell Preprint CLNS-88/853 (1988).
6. A. M. Polyakov and A. B. Zamoldchikov, *Mod. Phys. Lett.* **A3** (1988) 1213; some aspects of this work also appear in N. D. Hari Dass and R. Sumitra, Matscience Preprint (1988), to appear in *Int. J. Mod. Phys.* **A**.