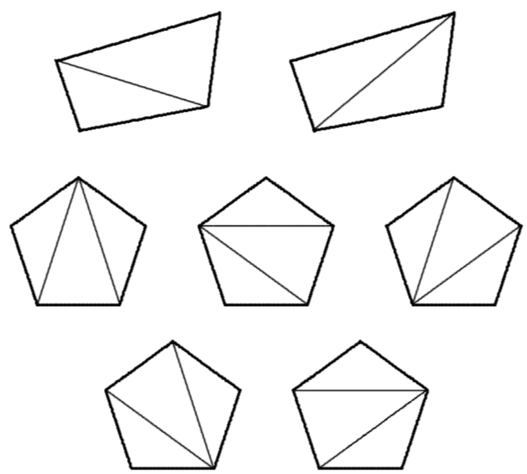
Maths Circle: Explorations

1. (a) Find the number of ways that one can cut a polygon[1] with $n+2$ sides into triangles by cutting along non-intersecting diagonals, for $n = 1, 2, 3, ..., 8$. The figure below shows the cases $n = 2, 3$.

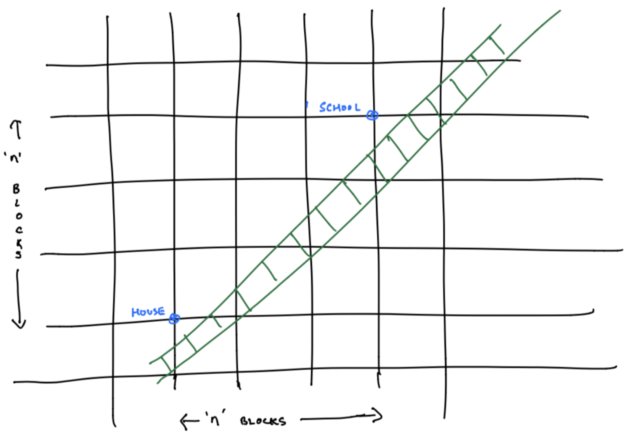


For large values of n, can you come up with a method to calculate the number of ways to cut a polygon with $n+2$ sides as described above?

(b) In Nayan's town Hunasemarahalli, the cycling tracks are arranged in an evenly spaced grid, with the tracks running from south to north and west to east. Her school is exactly n blocks north and n blocks east of her home. There is railway track that passes immediately east of her home, travels in a straight line northeast,

[1] Our polygons are required to be convex. This means that for any two points in our polygon, the line segment connecting them is contained in the polygon.

and passes immediately east of her school (see the figure below). A *good cycling route* for Nayan is one that never involves travelling south or west, and never crosses the railway track. How many good cycling routes are there from Nayan's home to her school?



MAP OF HUNASEMARAMALLI

(c) Find a rule that pairs each way of cutting a convex $(n+2)$-sided polygon into triangles with a good cycling route, so that

- Each cutting is paired with exactly one good cycling route
- Each good cycling route is paired with some way of cutting the convex $(n+2)$-sided polygon into triangles.

(d) What did you learn about parts $(a), (b)$ from part (c)? Can you use part (c) to come up with a formula (in terms of n) for the number you were asked to calculate in part (a)?

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2. Suman does not like irrational numbers and fractions. She likes to define numbers only using integers. When I asked her what $\sqrt{2}$ is, she said it is a solution to the equation $x^2 - 2 = 0$. Notice that the equation

has only integer coefficients. The above equation is said to be of degree 2, since the highest power of x that appears in the equation is 2.

(a) I was curious about her approach. So I asked her how she defines the number $\sqrt{2} + \sqrt{3}$. She gave me an equation with integer coefficients for which my number was a solution. And it turned out the degree was as small as possible. What was the equation? Can you write down all the solutions to this equation? Do you think there is only one such equation?

(b) Can you find an equation with integer coefficients that has the number $\sqrt{2} + \sqrt{3} + \sqrt{5}$ as a solution? What is the degree of the equation and what are all its solutions? Does there exist an equation of smaller degree that also has $\sqrt{2} + \sqrt{3} + \sqrt{5}$ as a solution?

(c) Repeat part (b) for the number $\sqrt{2} + \sqrt{3} + \sqrt{6}$. Before finding an explicit equation, try guessing the degree of the lowest degree equation that will do the job. Verify your guess.

(d) (*Hard*) Do you think there is a polynomial equation with integer coefficients that has $\sqrt[3]{2} + \sqrt[5]{2}$ as a solution? Support your answer with a convincing argument. If you think that such an equation does exist, what can you say about it? For example, what is the smallest possible degree of such an equation, and what are all its solutions? Can you construct the equation explicitly?

3. Recall that Suman likes to think of non-integral numbers in terms of the equations they solve (see Exploration 2 above). One evening, while explaining her ideas to Arnab, she writes down the equation $x^2 + 1 = 0$. She introduces the symbol α to describe the hypothetical number that solves this equation. Then she observes that $\beta = (-1)(\alpha)$ must be another solution, since $(-1)^2 = 1$. She also notes that she could just as well have started with β as her first solution, and written $\alpha = (-1)(\beta)$: *she has no way of distinguishing one solution from the other.*

   Suman thinks of α and β as being *reflections* of each other, in the same spirit that one thinks of one triangle as being a reflection of another in a line. Just as in the geometric case, if you do one of Suman's reflections twice you end up where you started.

Arnab is deeply troubled by these ideas. "How do you know these *chimeral numbers* actually exist," he asks.

To this, she replied "I don't know what it means for them to 'actually exist', or even that it is meaningful to ask that question. The ancient Greeks were troubled by the question of the existence of $\sqrt{2}$. I prefer to leave such questions to the philosophers and take a more utilitarian point of view. What matters to me is that I can reason with my chimeral numbers just as we do with 'ordinary' numbers, and arrive at interesting conclusions without ever contradicting myself. For instance, I can add and multiply chimeral numbers to form chimeral numbers like $7 + 10\alpha^{47} + 2\alpha^{52}$, and the rules for addition and multiplication behave just as one would hope. I can even do new things like 'reflecting' a number by replacing all the α's by β's and vice versa."

"But what use is all of this make-believe?" retorted Arnab. To show Arnab the power of her ideas, Suman challenged him with this problem, giving him the hint below:

(a) A whole number n is called a *gem* if it is (i) the area of a square whose side length is a whole number, or (ii) the area of a square, whose side is the hypotenuse of a right angled triangle whose other two sides have lengths that are whole numbers. Show that if n and m are gems, then so is nm.

HINT: Define the *size* of a chimeral number to be the product of the number with its reflection. Then the size of a product of chimeral numbers is the product of their sizes.

Do you follow Suman's line of reasoning and can you reconstruct the details of her solution to the above problem? What property of Suman's reflections are key to her solution and implicit in her discussion?

(b) Suppose there are whole numbers a, b, c, d such that $n = a^2+b^2-ab$ and $m = c^2 + d^2 - cd$. Find whole numbers e, f, such that $nm = e^2 + f^2 - ef$.

HINT: Let ω be a solution to the equation $x^2 + x + 1 = 0$. Show that ω^2 is another solution to the same equation. Now argue as suggested in the hint to part (a).

(c) Suppose $n = x^3 + y^3 + z^3 - 3xyz$ and $m = r^3 + s^3 + t^3 - 3rst$

for some whole numbers x, y, z, r, s, t. Are there whole numbers e, f, g such that $nm = e^3 + f^3 + g^3 - 3efg$?

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4. A very thin circular metal ring is heated unevenly by a collection of candles that are placed at various points along the ring. A few milliseconds after the heat source is removed, some parts of the ring are hot while others are cold. The only thing that we know is that at any given moment the temperature varies continuously: an ant walking along the ring would not experience an abrupt change in the temperature at any point. Armed with just this knowledge, Hana claims that there exist two points on the ring that are diametrically opposite to each other, and are also at exactly the same temperature. Is she correct? Justify your response with a convincing argument.

5. When Homi heard of Hana's statement about the heated ring (see Exploration 4 above), he made the following claim:

   Assume that the surface of the earth is a perfect sphere, and that the atmospheric concentration of NOx (nitrogen oxides) and SOx (sulphur oxides) never jump abruptly as one move's across the surface of the earth (a mathematician expresses this by saying that concentration of NOx and SOx are assumed to be *continuous* functions). Then there exist a pair of diametrically opposite points P and Q on the earth's surface, such that the concentration of NOx at P equals the concentration of NOx at Q *and* the concentration of SOx at P equals the concentration of SOx at Q.

   Do you believe Homi's claim? If not, how would you set about disproving him? If yes, can you furnish a convincing argument that he is correct?

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6. Josh and Ishan are playing a game. The goal of the game is to find a transformation of the Euclidean plane onto itself that leaves distances unchanged, and carries a given triangle ABC onto another given triangle $A'B'C'$. Examples of transformations that leave distances unchanged include:

   - Rotation counterclockwise about a point O through an angle θ.
   - Reflection in a line l. This is a transformation that does not move the points on l and moves every other point. For example, if l is the x-axis, then reflection in l sends the point (x, y) to $(x, -y)$.
   - Translation by a vector $v = (\alpha, \beta)$. This sends the point (x, y) to the point $(x + \alpha, y + \beta)$.

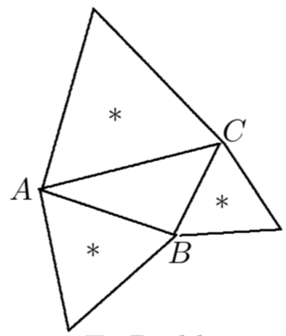
   Josh claims that given triangles ABC and $A'B'C'$ with the same side lengths, he can find a finite sequence of lines such that successively performing reflections in those lines takes the first triangle to the second. In order to challenge Josh, Ishan starts with a triangle ABC and constructs

   - A triangle $A'B'C'$ by rotating the triangle ABC counterclockwise through an angle θ about a point O.
   - A triangle DEF by translating ABC by a non-zero vector v.

   (a) Can Josh succeed in transforming ABC into $A'B'C'$ and DEF using only a series of reflections in lines? Justify your answer.

   (b) Josh constructs a new triangle UVW by rotating counterclockwise the triangle $A'B'C'$ constructed by Ishan through an angle ϕ about a point O'. Ishan claims that he can transform the triangle ABC into the triangle UVW using only a single rotation about a point O''. Is Ishan's claim true? If yes, what is the point O'', and what is the angle through which he must rotate ABC?

7. (a) Let ABC be an arbitrary triangle. Draw equilateral triangles outward on the edges of ABC (see the figure below). Show that the centres of these new triangles form an equilateral triangle themselves.



(b) Kiran, who was preparing for the mathematical olympiads, showed his junior Sana the problem in part (a), which had stumped him. Sana hasn't yet taken a geometry class at school, and so she doesn't know any of the complicated theorems from Euclidean geometry. However, she was watching Josh and Ishan closely as they played their game of transformations of the plane (see Exploration 6 above). She had a very good grasp of how successive reflections and rotations could be used to transform one geometric figure into another congruent one. In particular, she knew that

- if l and l' are parallel lines separated by a distance a, then reflection in l followed by reflection in l' results in translation through a distance $2a$.
- counterclockwise rotation through an angle θ about a point O, followed by counterclockwise rotation through an angle ϕ about a point O' is equivalent to
  - a translation if $\theta + \phi = 360°$
  - rotation through an angle $\theta + \phi$ about a point O'' otherwise.

Furthermore, she knew how to find the point O''.

Using only her rudimentary knowledge of transformations of the plane, Sana was able to solve Problem 4 (a). How did she do it?

8. Recall that a symmetry of a mathematical object is a transformation that leaves all the relevant features of the object unchanged. For example, in Euclidean geometry, a symmetry of a geometrical figure is a transformation of the figure that leaves it unchanged. In the previous explorations, we saw that studying the symmetries present in a given problem can teach us a great deal about the problem at hand. This suggests that it would be useful to study symmetries themselves as mathematical objects in their own right.

   Let ABC be an equilateral triangle with centre O, and let L_P denote the straight line passing through a given point P and the point O. Let r_θ denote the operation of counterclockwise rotation through an angle θ about the point O, and let f_P denote reflection in the line L_P. Then, clearly, $r_0, r_{120}, r_{240}, f_A, f_B, f_C$ define symmetries of the triangle ABC (here, we are measuring the angles in degrees). Note that the symmetry r_0 is the trivial symmetry – it does not move any points!

   (a) Are there any more symmetries of the triangle ABC? Can you justify your answer?

   (b) If f, g are symmetries, then show that the result of first "doing" g and then doing f is also a symmetry. We denote this symmetry by $f \circ g$, and say that it is obtained by *composing* f with g. For example, $f_C \circ r_{120} = f_A$ (verify this!). We can keep track of all the ways in which symmetries can be composed by writing a "multiplication table":

$\circ$	r_0	r_{120}	r_{240}	f_A	f_B	f_C
r_0	r_0	r_{120}	r_{240}	f_A	f_B	f_C
r_{120}	r_{120}					f_B
r_{240}	r_{240}					
f_A	f_A	f_B	f_C			
f_B	f_B					
f_C	f_C	f_A				

   The symbols on top of the horizontal line label the columns of the table, and the symbols on the left of the vertical line label the rows of the table. The element in the row labelled by a symmetry f and a column labelled by a symmetry g is $f \circ g$.

(c) Complete the table in part (b). Do you see any patterns in this multiplication table? Show that

  (i) For any three symmetries f, g and h, we have

$$f \circ (g \circ h) = (f \circ g) \circ h$$

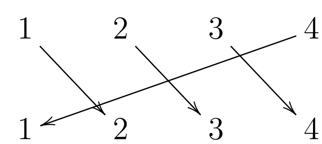
  In words: it does not matter whether we first compose f with g and then compose the result with h or first compose g with h and compose f with the result.

  (ii) There is a symmetry ϕ with the property that $\phi \circ f = f \circ \phi$ for any other symmetry f.

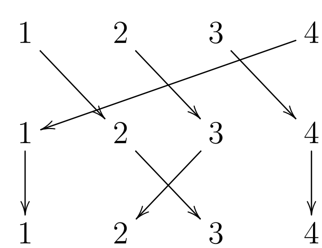
  (iii) For any symmetry f, there is a symmetry g such that $f \circ g = \phi$ and $g \circ f = \phi$.

(d) Convince yourself that the symmetries of any object satisfy conditions (i), (ii) and (iii) above.

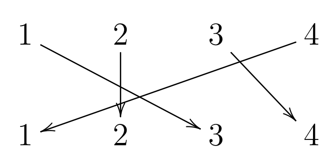
(e) A symmetry of the set $S_n = \{1, 2, \ldots, n\}$ is a permutation of $1, 2, \ldots, n$,



We can compose permutations by performing one after the other: in other words, $(f \circ g)(s)$ is defined to be $f(g(s))$.



The composite $f \circ g$ is obtained by following along the arrows from top to bottom. In the example above, the composite is:



Write down the multiplication table for the symmetries of $\{1, 2, 3\}$.

(f) What is the relationship between the multiplication in part (b) and that in part (e)? Can you explain this relationship?

(g) Write down the multiplication table for the symmetries of

- a square
- a regular[2] pentagon
- a regular hexagon

(h) How many symmetries does a regular polygon with n sides have? How many symmetries does the set $\{1, 2, \ldots, n\}$ have? What is the relationship between the symmetries of a regular polygon with n-sides and the symmmetries of the set $\{1, 2, \ldots, n\}$?

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9. Madhav thinks that there is nothing wrong with setting a particular number to 0. For instance, he notices that setting $5 = 0$ will still let him do arithmetic. Then he will get lots of equalities: $30 = 0, 4 = -1, 232 = -3$ and so on. Let us call these numbers "Madhav numbers".

   (a) First demonstrate that there are only 5 different numbers in his number system. Can you add, multiply and divide by any number in this number system? Let us call this number system Ma_5.

   (b) Generalize the above construction for a general number, i.e. $n = 0$. Can you add, multiply and divide by any number in this number system Ma_n?

   (c) After Suman (from Exploration I and II) talks to Madhav, Suman wants to know what is $\sqrt{2}$ in Madhav's number system. Suman tells Madhav her definition and Madhav says that for some numbers n, there is no number that equals $\sqrt{2}$ in the number system Ma_n. Find all the n for which this is true.

[2] a polygon is *regular* if all its sides have equal length

(d) In order to understand the equation $x\sqrt{2}+y\sqrt{6}+z\sqrt{6}=0$, Suman ponders if there is a number system Ma_n where $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ exists as numbers. Can you find such a number n?

(e) Listening to Suman's idea, Madhav wonders if there is a number system Ma_n where $\sqrt{2}, \sqrt{3}$ exists as numbers but $\sqrt{6}$ is not a number. Again, would you help him find such a number n?

10. Firdaus likes to do arithmetic using the Madhav number systems Ma_p (see Exploration 9 above) where p is any prime number, and $p \neq 2$. He claims that 2^{p-1} is always equal to 1 in the number system Ma_p, no matter what p is. Is he correct? Can you give a convincing argument for or against his claim?