Many-body quantum chaos: Classical to quantum and back again

Sumilan Banerjee

Centre for Condensed Matter Theory, Department of Physics,
Indian Institute of Science

ICTS Seminar
June 8, 2021

Support: DST SERB, India ECR grant

Collaborators:

Surajit Bera
(Physics, IISc)

Venkata Lokesh K. Y
(CHEP, IISc)

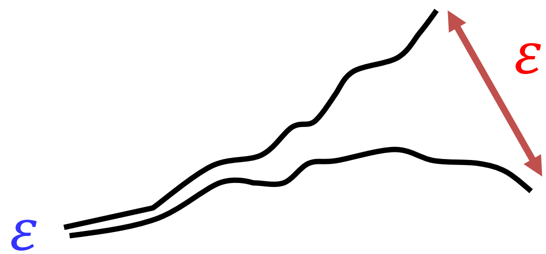
Sibaram Ruidas
(Physics, IISc)

1. S. Bera, V. Lokesh Y & SB, arXiv:2105.13376 (2021).
2. S. Ruidas & SB, arXiv:2007.12708 (2020).

Classical Chaos

Single-particle chaos

Sensitivity to initial condition ⇒



$\varepsilon e^{\lambda_L t} \lesssim \mathcal{O}(1)$

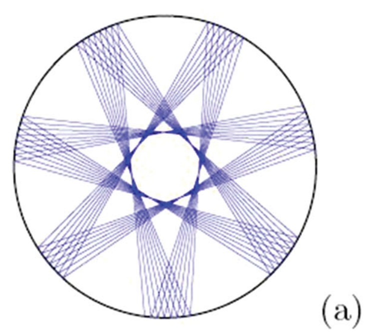
$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda_L t}$$

Lyapunov regime

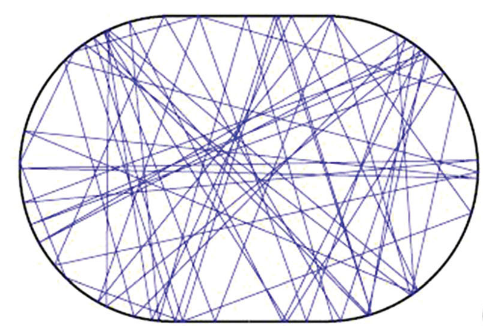
$$\lambda_L^{-1} < t < t^* \sim \lambda_L^{-1} \log\left(\frac{1}{\varepsilon}\right)$$

λ_L, Lyapunov exponent

Non chaotic billiard



Chaotic billiard



Quantum chaos: From classical to quantum

- Classical chaos $\dfrac{\partial x(t)}{\partial x(0)} \sim e^{\lambda_L t}$

  $= \{x(t), p(0)\}$ Poisson bracket

→ Quantum chaos $\Rightarrow [x(t), p(0)]/i\hbar$ Larkin & Ovchinikov (1969)

Out-of-time order commutator $\mathcal{D}(t) = -\langle [x(t), p(0)]^2 \rangle \sim \hbar^2\, e^{2\lambda_L t}$

Generalize to quantum chaotic (interacting) many-body systems

$\mathcal{D}(t) = -\langle [A(t), B(0)]^2 \rangle \qquad [A, B] = 0$

Out-of-time-order correlator (OTOC)

$$F(t) = \langle A(t)B(0)A(t)B(0) \rangle \sim \# - \mathcal{D}(t) \sim \# - \epsilon e^{\lambda_L t}$$

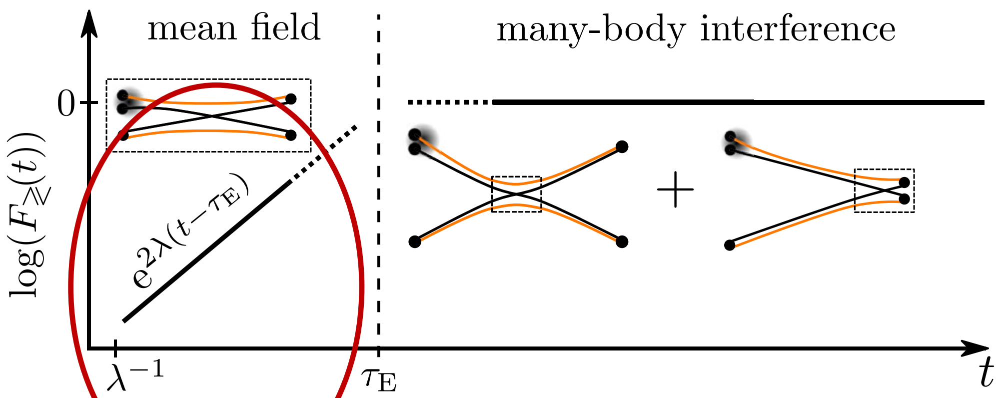
'Scrambling'

*There are other characterizations of quantum chaos at much longer time scales
-- random-matrix theory (RMT) energy-level statistics, ..

Quantum effects and the 'semiclassical limit'

Rammensee et al, PRL (2018)

OTOC



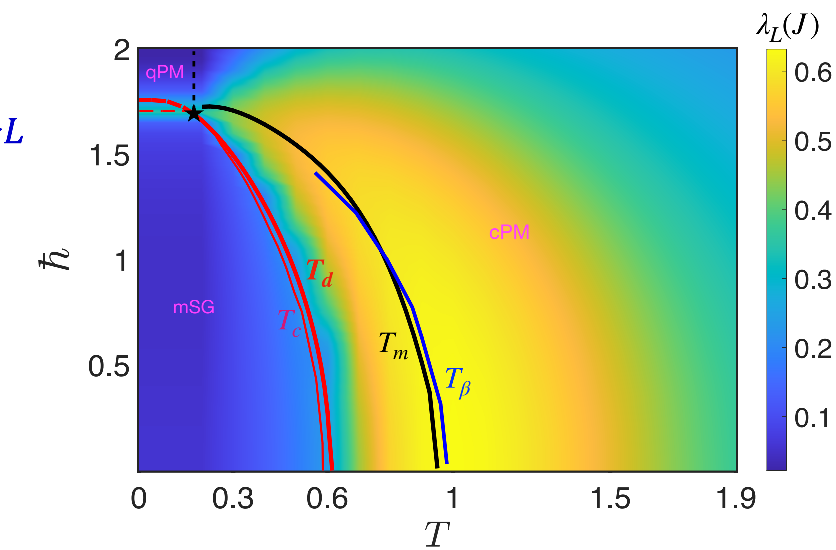
Non-interacting systems
Aleiner & Larkin, PRB (1996)
Tian et al., PRL (2004)
Rozenbaum et al, PRL (2017),
....

Quantum interference effects for $t > \tau_E$ or t^*

$t^* \sim \frac{1}{\lambda_L} \ln\left(\frac{1}{\hbar_{eff}}\right)$, $\hbar_{eff} \sim \hbar$ or $1/N$ in our case

How quantum fluctuations or '$\hbar$' affects λ_L for $t < t^* \sim \frac{1}{\lambda_L} \ln N$?

Quantum spin glass ⇒



Spatio-temporal evolution of many-body chaos

Ballistic spread of chaos

$$\mathcal{D}(\boldsymbol{r},t) = -\langle [A_{\boldsymbol{r}}(t), B_0(0)]^2 \rangle$$

$$\sim e^{\lambda_L\left(t-\frac{r}{v_B}\right)}$$

or $$\sim e^{\lambda_L t\left(1-\left(\frac{r}{v_B t}\right)^{\nu}\right)}$$

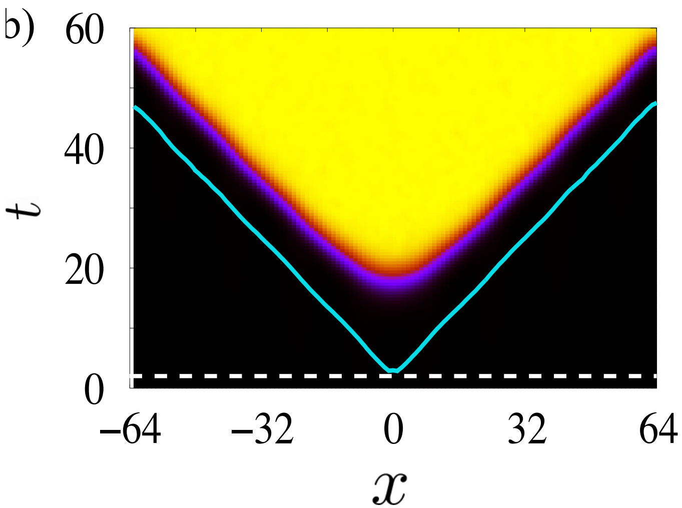
depending on the systems

v_B, Butterfly speed

Chaos can spread ballistically
even in a
diffusive or anomalous diffusive systems

A Das et al., PRL (2014)

S. Ruidas & SB (2020).



Kosterlitz-Thouless phase with
anomalous diffusion in
the classical limit of XXZ model

Why care about chaos?

○ Remarkable upper bound for Lyapunov exponent for any quantum system

$$\lambda_L \leq 2\pi k_B T/\hbar$$

Maldacena, Shenker & Stanford (2016)

Black holes, some non-Fermi liquids saturate the bound

"Fastest scramblers"

Sekino & Susskind (2008)

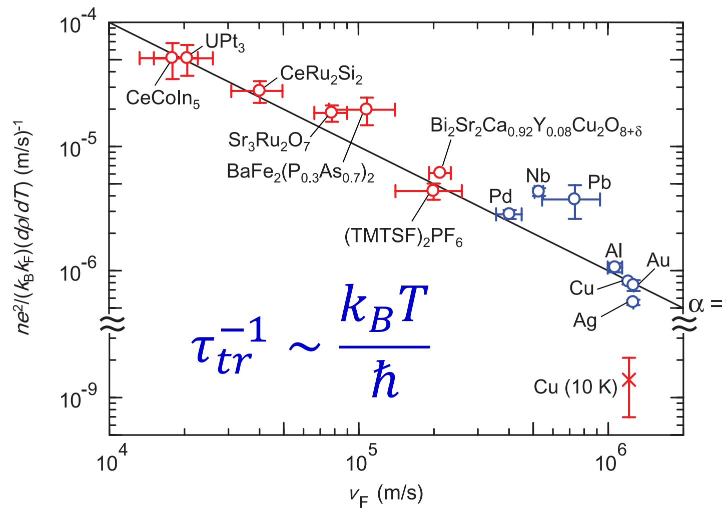
○ In certain strongly correlated systems

Gu, Qi & Stanford (2017)

Diffusion coefficient $D \sim v_B^2/\lambda_L$

Are there bounds on other time scales, like transport scattering rate τ_{tr}^{-1}?

Bruin et al. Science (2013)



Planckian dissipation

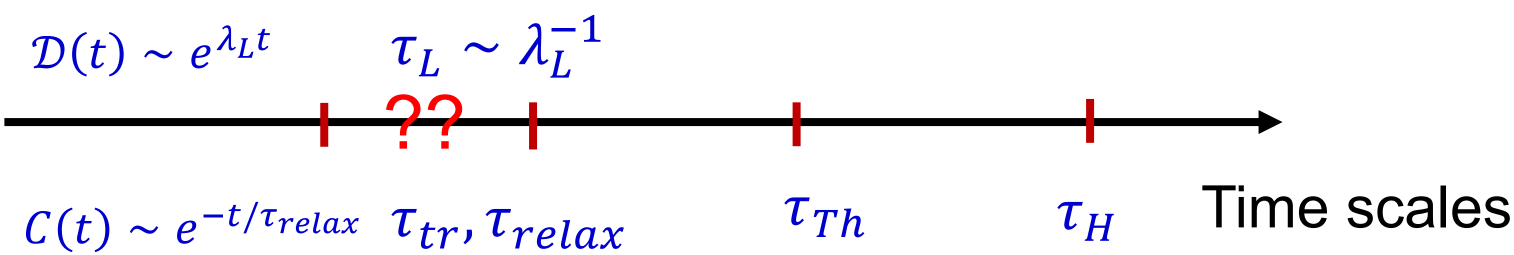
Many questions ....

- How does quantum mechanics lead to fundamental bounds on time scale from going to classical to quantum?

- Connection between chaos and transport?

- Role of chaos in thermalization? Entanglement growth?

- What happens to chaos across phase transitions? Is there critical slowing down? Effect of collective modes on chaos?



Outline for rest of the talk

- Chaos in large-N models: Sachdev-Ye-Kitaev (SYK).
- Quantum to classical crossover in chaos in a quantum spin glass.
- Chaos across thermal phase transitions in two dimensions in the classical XXZ model.
- Conclusions.

Zero-dimensional Model for NFL: Sachdev-Ye-Kitaev (SYK) model

Sachdev & Ye, PRL (1993)
Kitaev, KITP (2015)
Sachdev, PRX (2015)

$$H_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

$$P(J_{ijkl}) \sim e^{-\frac{|J_{ijkl}|^2}{J^2}}$$

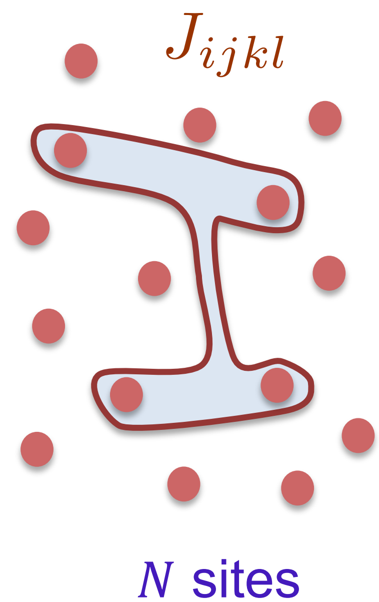
- Solvable in strong coupling for large N

→Non-Fermi liquid ground state.

- 'Maximally chaotic' with Lyapunov exponent,

$$\lambda_L = 2\pi k_B T/\hbar$$

OTOC $\langle c_i^\dagger(t) c_j(0) c_i^\dagger(t) c_j(0) \rangle$

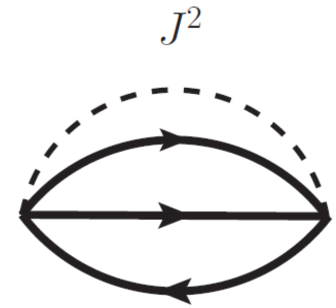


'Upper bound' to quantum chaos as in a black hole

Maldacena, Shenker & Stanford (2016)

Kitaev → Solvable model for holography

Computation of λ_L in large N



- Disordered averaged saddle point for $N \to \infty$

$$G(\tau) = -\overline{\langle \mathcal{T}_\tau c_i(\tau) c_i^\dagger(0) \rangle}$$

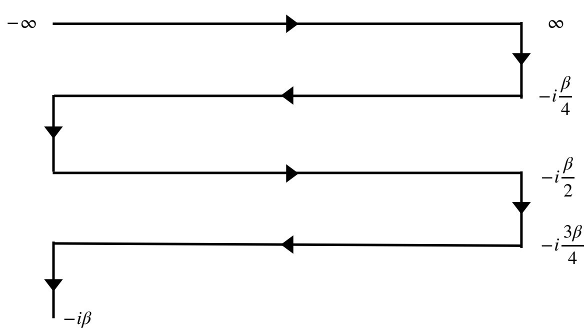
$$G^{-1}(\omega) = \omega + \mu - \Sigma(\omega)$$

Out-of-time-order correlation (OTOC)

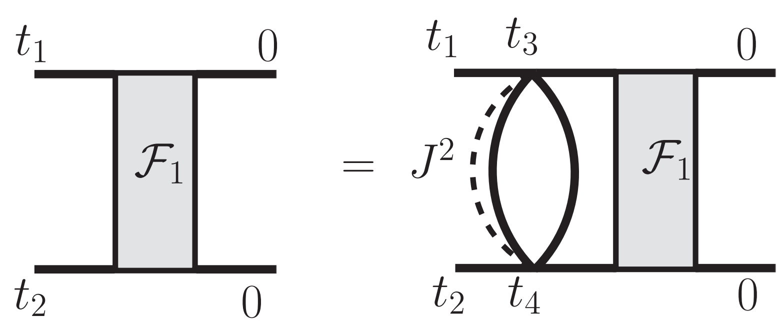
Kitaev, KITP (2015)

$$F(t,t) = \overline{\langle c_i^\dagger(t) c_j^\dagger(0) c_i(t) c_j(0) \rangle} \simeq f_0 - \frac{f_1}{N} e^{\lambda_L t} + \cdots$$

Keldysh contour



Ladder series →

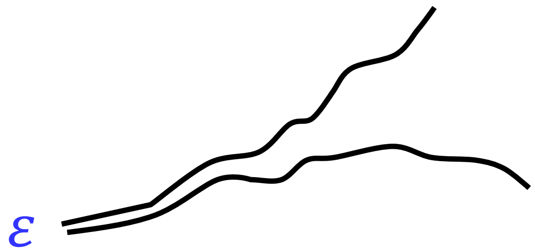


Kernel equation $\mathcal{F}(t_1,t_2) = \int dt_3 dt_4 K(t_1,t_2,t_3,t_4) \mathcal{F}(t_3,t_4)$

Saddle-point Green's function $G(\omega)$

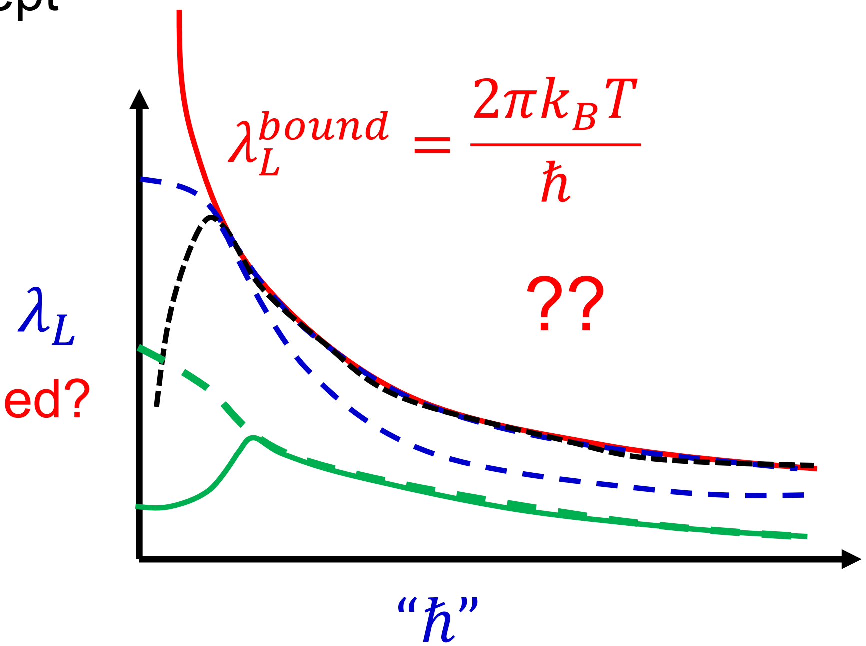
Quantum to classical crossover in chaos in a quantum spin glass

- Originally chaos is a classical concept



What happens to the chaos bound
as the classical limit $\hbar \to 0$ is approached?

Classically there is no bound!



- How the classical limit is approached?
- Is the limit always analytic?
- Could there be non-monotonic evolution as a function of $\hbar$?

Need a model where "$\hbar$" can be varied from quantum to classical limit.

* SYK model won't work! Fermionic model, no classical limit

Solvable model of quantum spin glass

Quantum spherical p-spin glass model   Cugliandolo et al, Phys. Rev. B (2001)

$$H = \sum_i \frac{\pi_i^2}{2M} + \sum_{i_1<\cdots<i_p} J_{i_1 \ldots i_p} S_{i_1} \ldots S_{i_p}$$

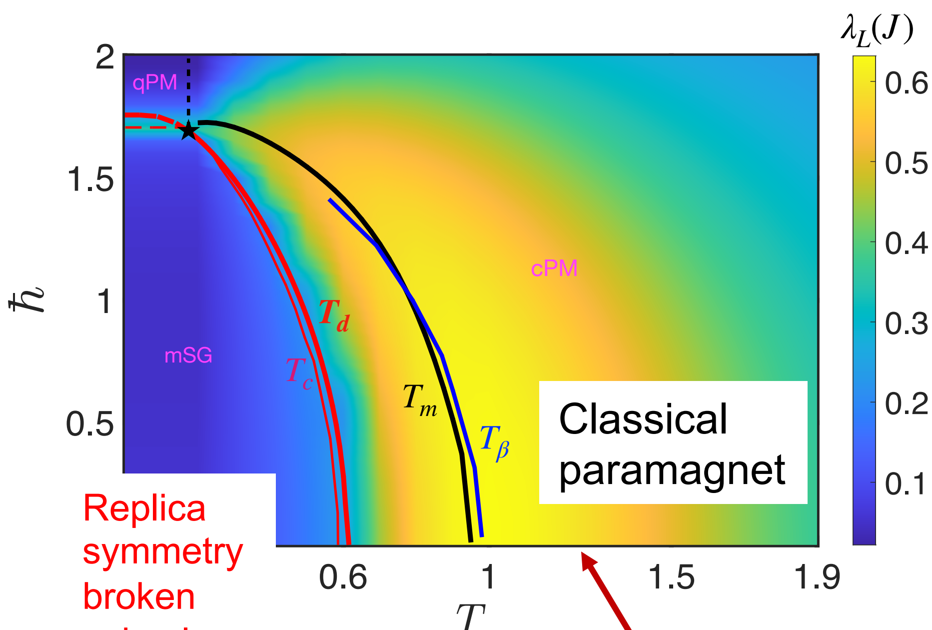
$i = 1, \ldots, N$

Spherical constraint $\sum_i S_i^2 = N$

Quantum dynamics

$$[S_i, \pi_j] = i\hbar\delta_{ij}$$

- $\hbar$ can be continuously varied in this model
- Chaos across quantum and classical spin glass (SG) transition.



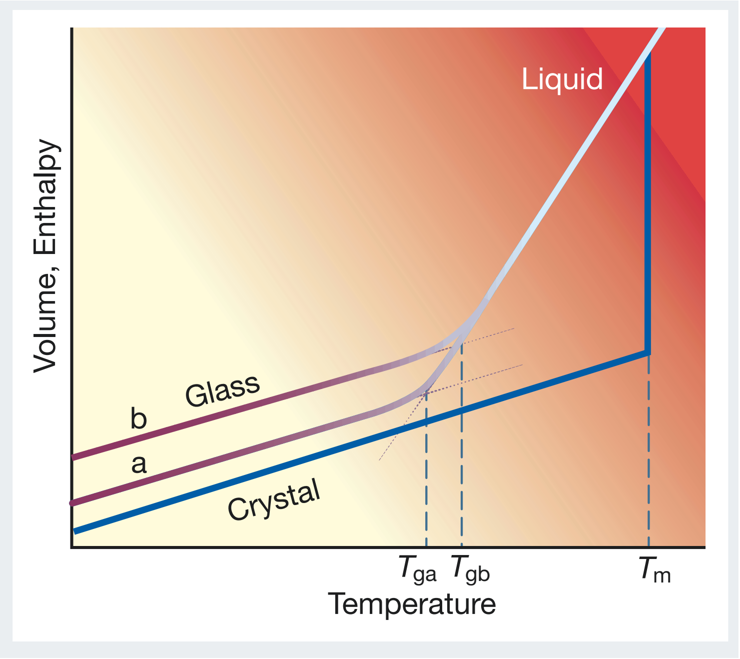
Replica symmetry broken spin glass (SG)

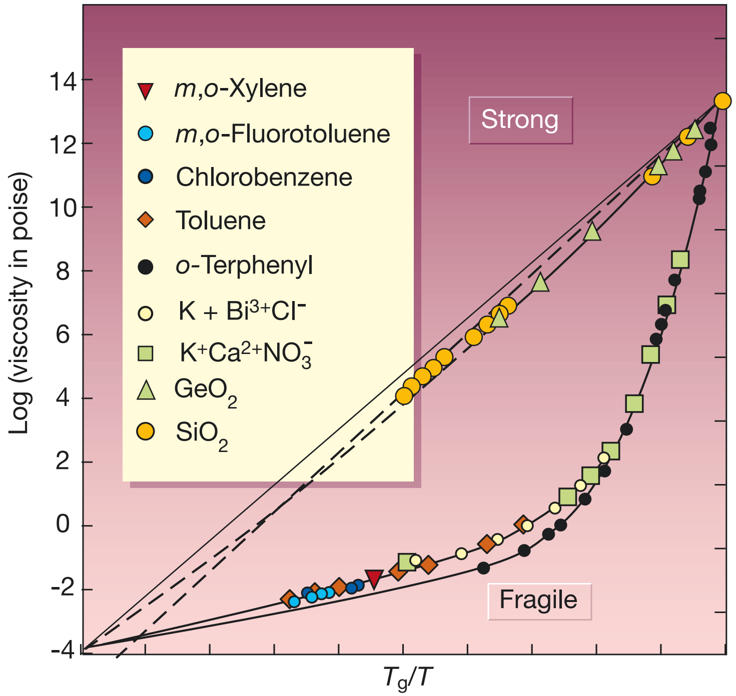
Interplay of quantum fluctuations, glassy dynamics, replica symmetry breaking and chaos?

$\hbar \to 0$ limit, dynamics of classical supercooled liquids

Digression: Glassy dynamics and dynamical transition to non-ergodic phase

Supercooled liquids in structural glasses



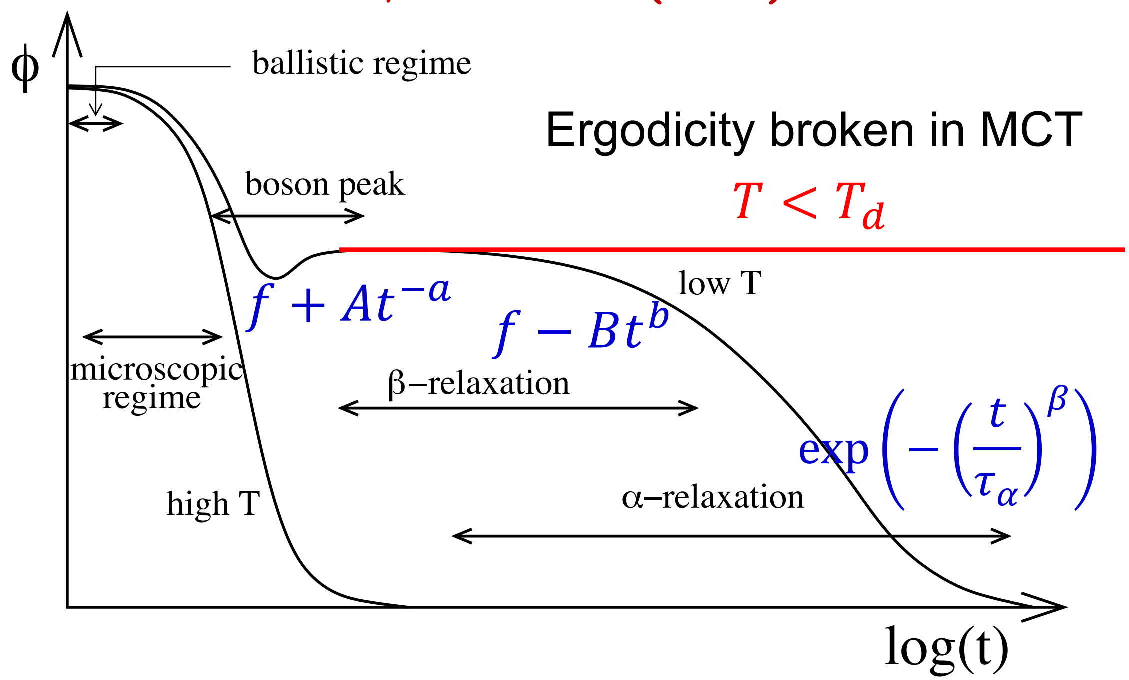


Debenedetti and Stillinger, Nature (1994).

Angell et al., Nuovo Cimento D (1994).

Digression: Glassy dynamics and dynamical transition to non-ergodic phase

Kob, Les Houche (2002).



Two-step relaxation and β-relaxation: Power-law decays

Mode-coupling theory (MCT) for supercooled liquids $\equiv$ p-spin glass dynamics

$$\tau_\alpha \sim (T - T_d)^{-\gamma}$$

Diverging time scale

What happens to λ_L^{-1}?

$p = 3$ model

Spin-glass order parameter

$$Q_{ab}(\tau, \tau') = \frac{1}{N}\sum_i \langle S_{ia}(\tau) S_{ib}(0) \rangle$$

Spherical constraint

$Q_{aa}(0) = 1$

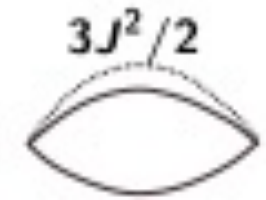
Replica index

$a = 1, \dots, n$ (n→ 0 limit)

Schwinger-Dyson equation

$$(Q^{-1})_{ab}(\omega) = \left(\frac{\omega^2}{\Gamma} + z\right)\delta_{ab} - \Sigma_{ab}(\omega)$$

$$\Sigma_{ab}(\tau) = \frac{3J^2}{2} Q_{ab}(\tau)^2$$

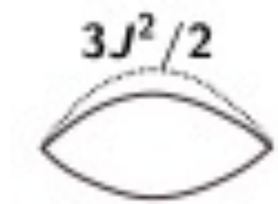


Dimensionless quantum parameter $\Gamma = \hbar^2/MJ$

- Paramagnetic state, replica symmetric and diagonal $Q_{ab}(\omega_k) = Q(\omega_k)\delta_{ab}$

Matsubara frequency $\omega_k = 2k\pi T/\hbar$ $(k_B = 1)$

$$(Q^{-1})_{ab}(\omega) = \left(\frac{\omega^2}{\Gamma} + z\right)\delta_{ab} - \Sigma_{ab}(\omega)$$



- Spin glass state, one-step replica symmetry breaking (1RSB)

Break $n \times n$ matrix into n/m diagonal $m \times m$ blocks

$$Q_{ab}(\tau) = (q_d(\tau) - q_{EA})\delta_{ab} + q_{EA}\epsilon_{ab}$$

Cugliandolo et al, Phys. Rev. B (2001)

$$Q(\omega_k) = \begin{bmatrix} q_d(\omega_k) & \tilde{q}_{EA} & 0 & 0 \\ \tilde{q}_{EA} & q_d(\omega_k) & 0 & 0 \\ 0 & 0 & q_d(\omega_k) & \tilde{q}_{EA} \\ 0 & 0 & \tilde{q}_{EA} & q_d(\omega_k) \end{bmatrix}$$

n replicas in groups, with each having m replicas

Example: $\mathrm{n} = 4, m = 2$

$$\tilde{q}_{EA} = \beta q_{EA}\delta_{\omega_k,0}$$

Eventually take $n \to 0$ limit

Edwards-Anderson order parameter q_{EA}

- Thermodynamic spin glass, extremize Free energy $\Rightarrow$ $\frac{\partial F}{\partial m} = 0 \Rightarrow \; m$
  - Marginal spin glass, set the replicon or transverse eigenvalue of the Gaussian fluctuation matrix around 1RSB saddle point to zero

$\Rightarrow \quad m$ (break point)

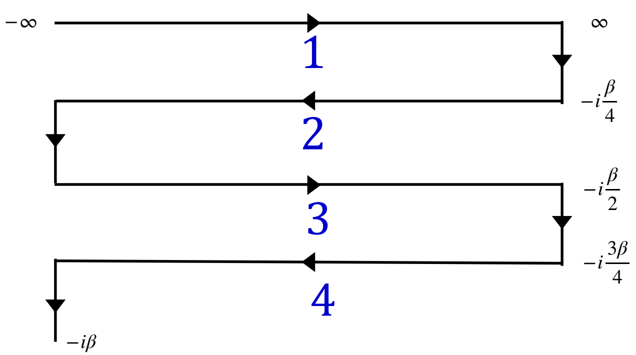
OTOC in the paramagnetic and marginal Spin glass phase

Regularized OTOC $F(t_1,t_2) = \overline{\mathrm{Tr}[yS_i(t_1)yS_j(0)yS_i(t_2)yS_j(0)]}$ $\quad y^4 = \dfrac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}$

Schiwnger-Keldysh contour for generating function

$$\mathcal{Z} = \frac{1}{\mathrm{Tr}[e^{-\beta H}]} Tr[e^{-\frac{\beta H}{4}} U(t_0,t_f) e^{-\frac{\beta H}{4}} U(t_f,t_0) e^{-\frac{\beta H}{4}} U(t_0,t_f) e^{-\frac{\beta H}{4}} U(t_0,t_f)]$$

$t_0 \to -\infty, \qquad t_f \to \infty$



Typically, replicas are not needed in Keldysh for disorder averaging

Works for paramagnetic phase

For dynamical correlations in spin glass phase, need replicas even in Keldysh

$\Rightarrow \mathcal{Z}^{\mathrm{n}}$ Houghton, Jain, Young (1983)
Cugliandolo et al. (2019)

$$F_a(t_1,t_2) = \overline{\left\langle y s_{ia}^{(4)}(t_1) y s_{ja}^{(3)}(0) y s_{ia}^{(2)}(t_2) y s_{ja}^{(1)}(0) \right\rangle} = \# - \left(\frac{1}{N}\right) \mathcal{F}_a(t_1,t_2)$$

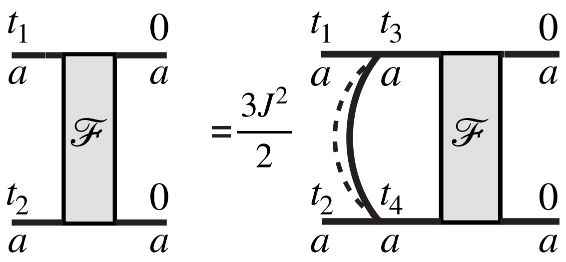
$$\mathcal{F}_a(t,t) \sim e^{\lambda_L t}$$

How does replica symmetry breaking enter in chaos?

Kernel equation $\mathcal{F}_a(t_1,t_2)=\int dt_3 dt_4 K_a(t_1,t_2,t_3,t_4)\mathcal{F}_a(t_3,t_4)$

$$K_a(t_1,t_2,t_3,t_4)=3J^2 Q^R_{aa}(t_1-t_3)Q^R_{aa}(t_2-t_4)Q^W_a(t_3-t_4)$$

Wightmann correlation



$$Q^W_a(\omega)=[2\pi\delta(\omega)q_{EA}+q^W_{reg}(\omega)]$$

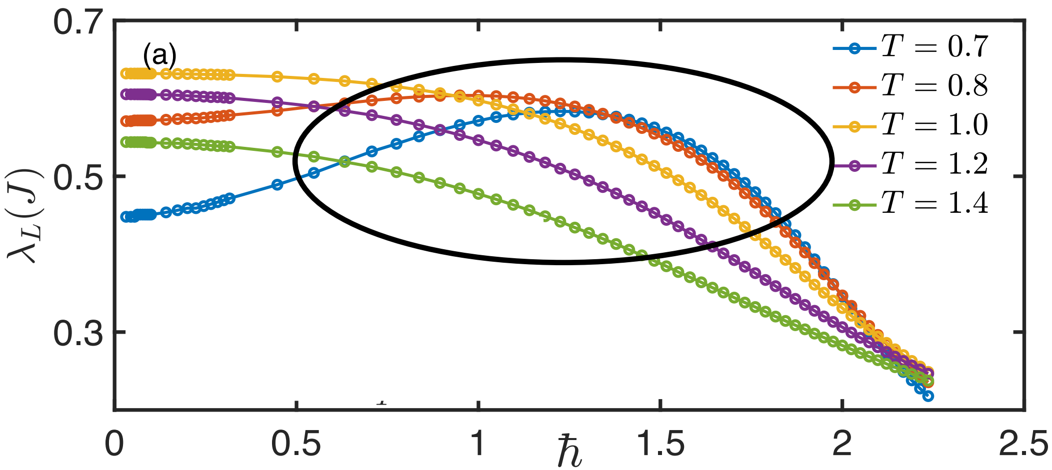
$q_{EA}=0$ for paramagnetic phase, $q_{EA}\neq 0$ for spin glass

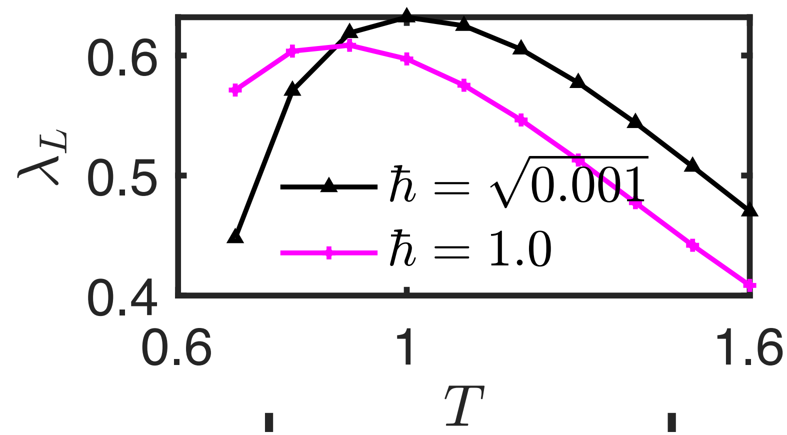
Solve the Kernel equation with growth ansatz

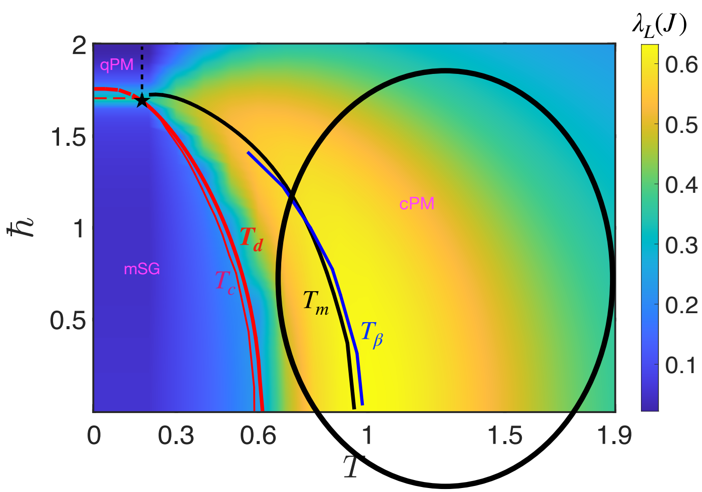
$$\mathcal{F}_a(t_1,t_2)=e^{\frac{\lambda_L(t_1+t_2)}{2}}f(t_1-t_2)$$

Self consistently find λ_L

Lyapunov exponent in the paramagnetic phase

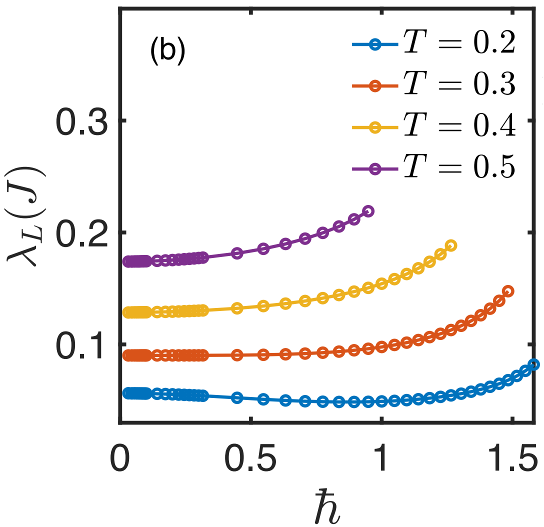




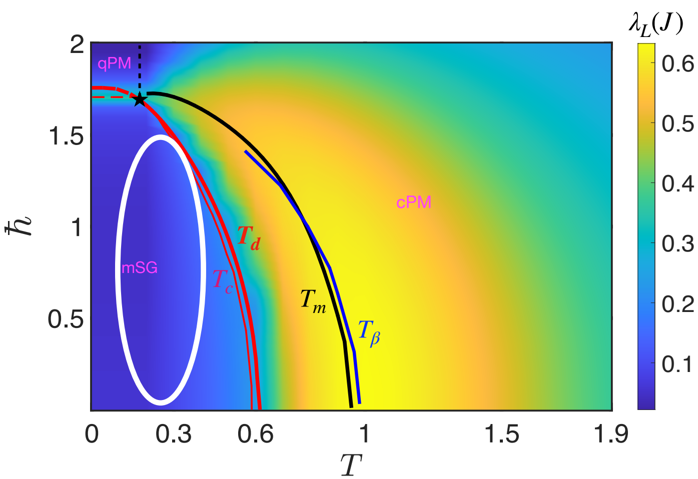


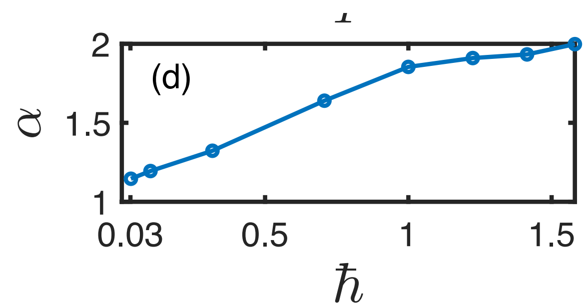
- Quantum fluctuations typically reduces chaos in the paramagnetic phase.
- Non-monotonic approach to classical limit for $T \gtrsim T_d$.
- Broad maximum above the dynamical transition temperature

Lyapunov exponent in the (marginal) spin glass phase



- Quantum fluctuations increases chaos in the spin-glass phase.





- Spin glass phase, Power-law T dependence

$$\lambda_L \sim T^{\alpha}$$

$$1\ (classical) < \alpha < 2 (quantum)$$

Analytical result for low-T quantum limit

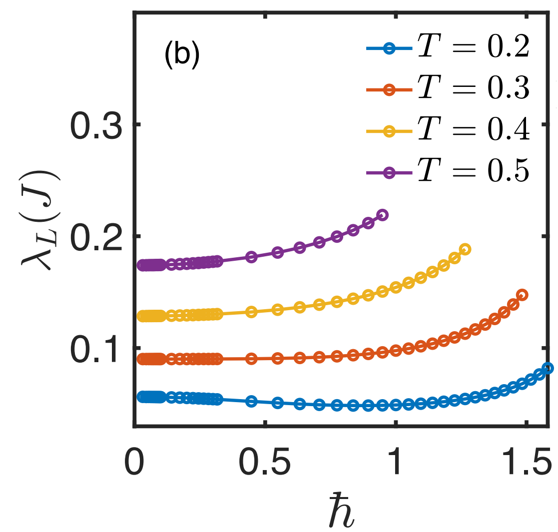
Kernel equation $\mathcal{F}_a(t_1,t_2) = \int dt_3 dt_4 K_a(t_1,t_2,t_3,t_4)\mathcal{F}_a(t_3,t_4)$

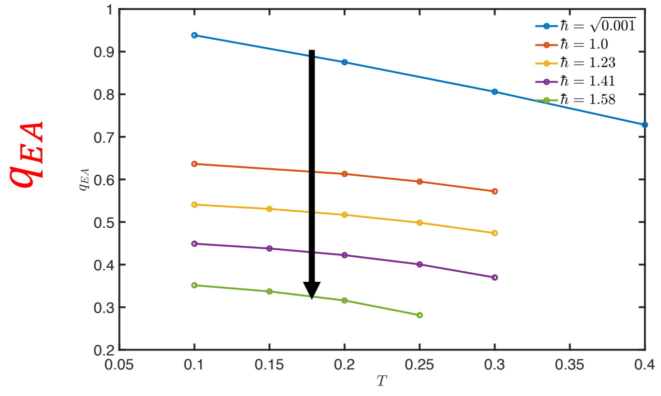
Reduce to a Schrodinger equation $\Rightarrow$

Poschl-Teller potential Similar to a Fermi liquid Kim, Cao, Altman (2020)

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial t^2} - \operatorname{sech}^2 t\right) f(t) = -\frac{1}{3}\left(\frac{\tilde{\lambda}_L}{T^2}+1\right) f(t)$$

$$\Rightarrow \qquad \lambda_L \sim \left(\frac{T^2}{J}\right)\left(\frac{1}{q_{EA}^{3/2}}\right)$$

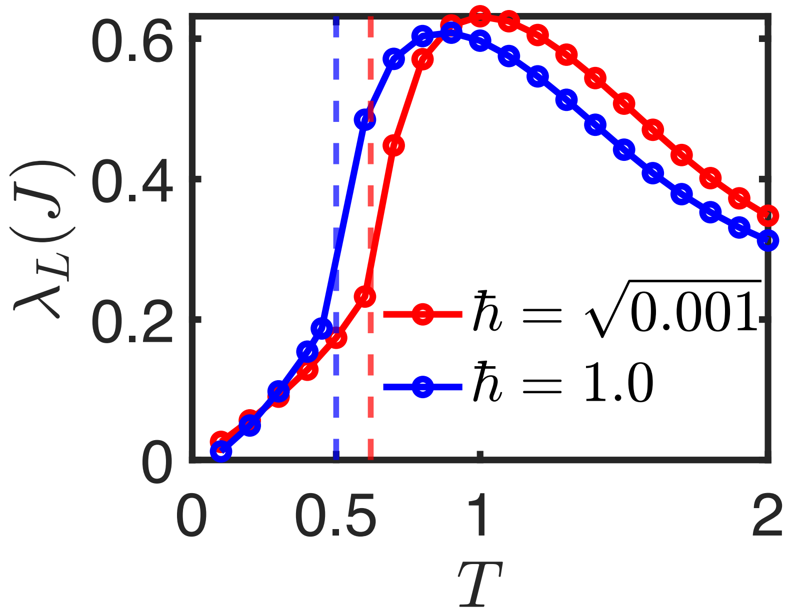




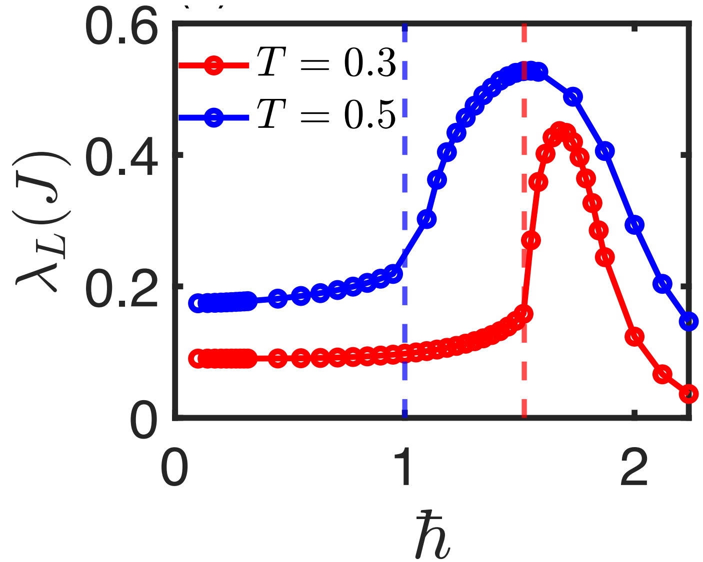
Quantum fluctuations reduces q_{EA} ('stiffness') and thus increases λ_L

Chaos across spin glass transitions

Thermal transition



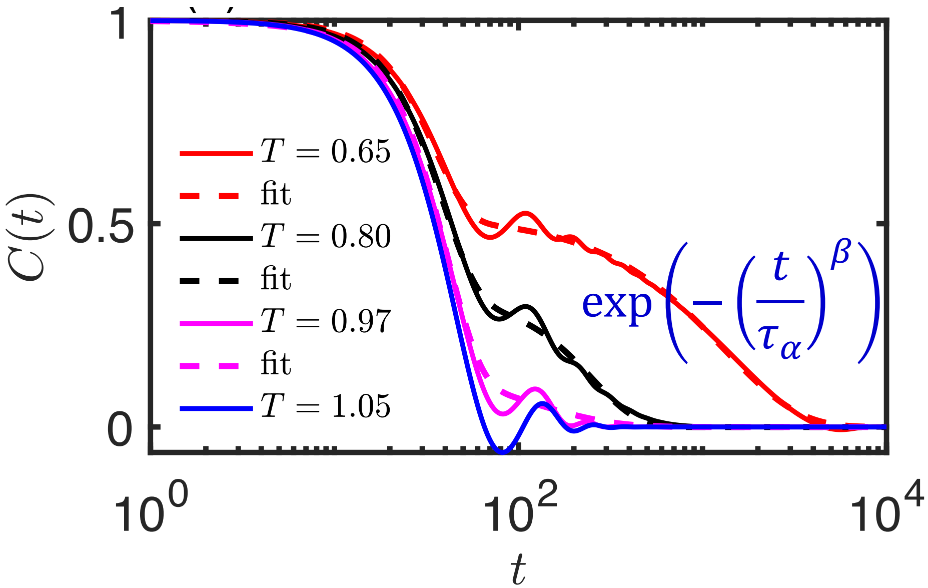
Quantum fluctuation induced transition



- Crossover across transition (slope change?).
- Broad maximum above the transition, but not at the transition.

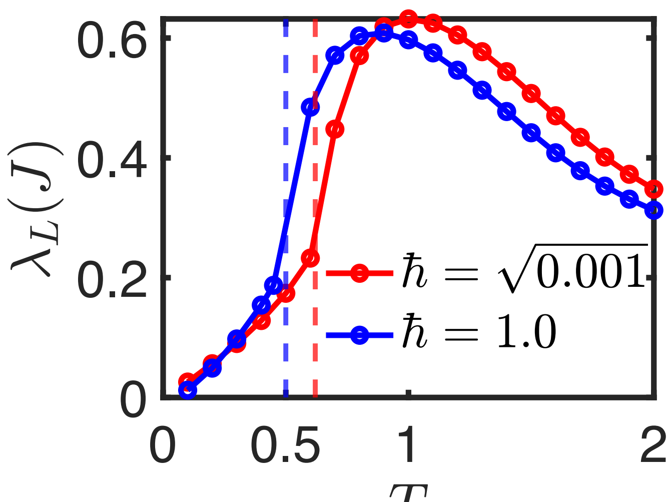
Glassy dynamics and Chaos

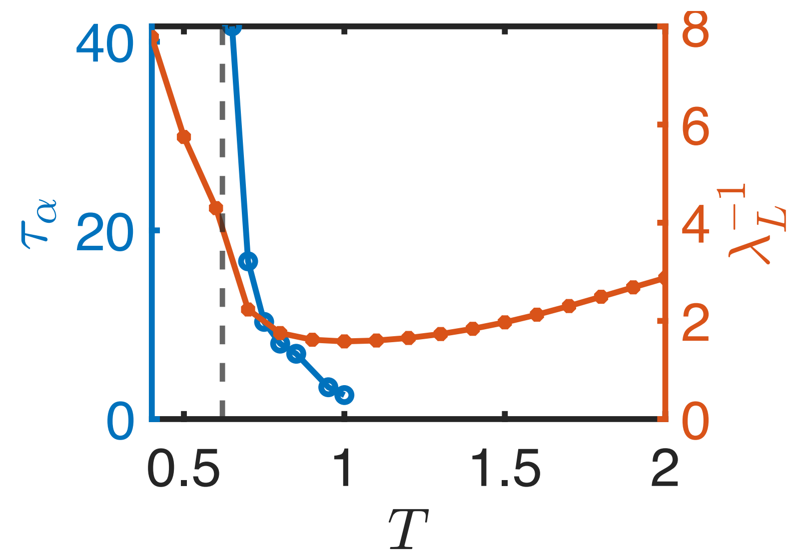
Two-step relaxation



Relaxation time diverges at the dynamical transition

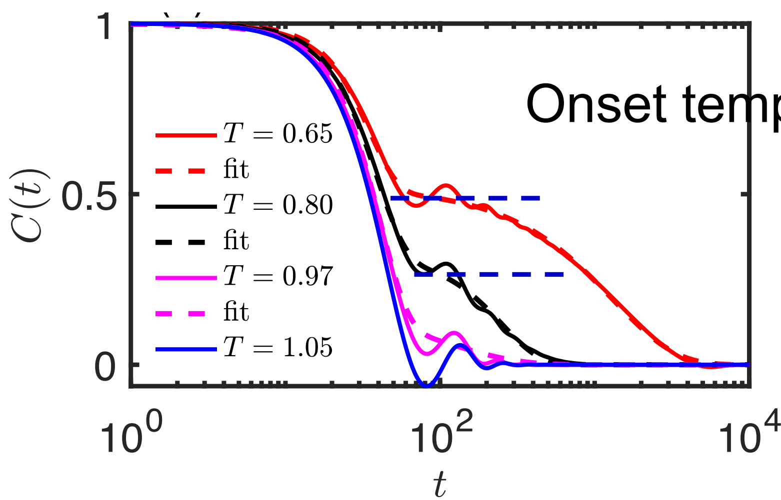
Thermal transition



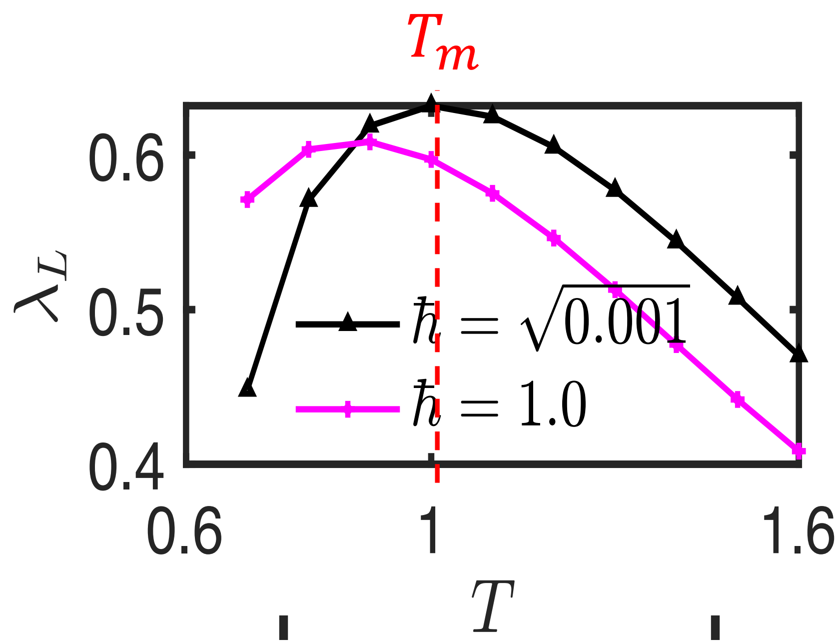


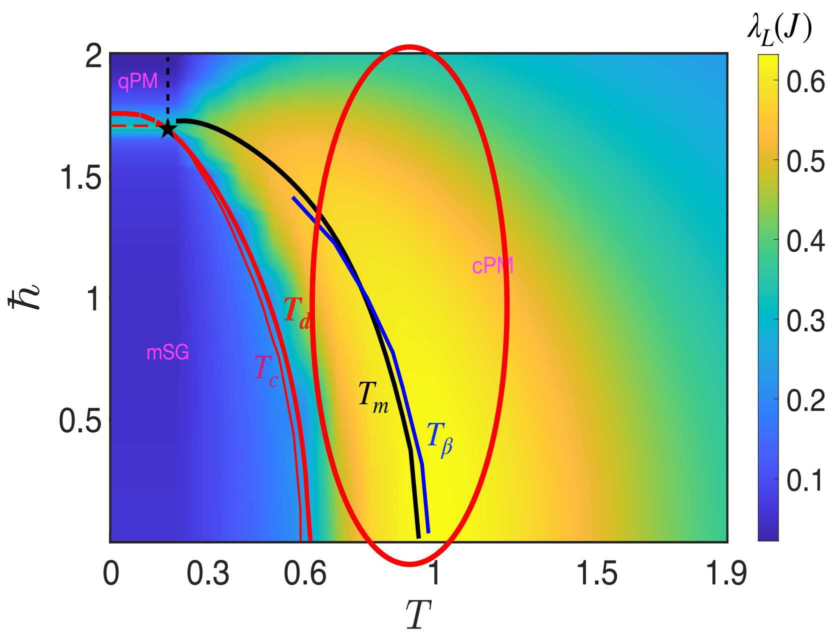
Glassy dynamics and Chaos

Onset of two-step relaxation



Onset temperature T_β





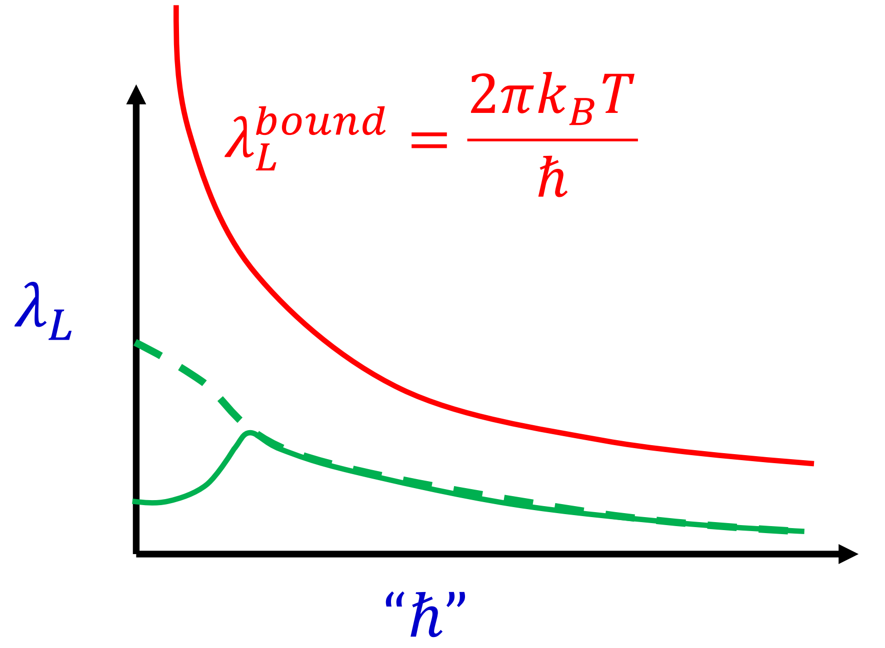
Crossover from strong to weak chaos coincides with the onset of two-step or β relaxation

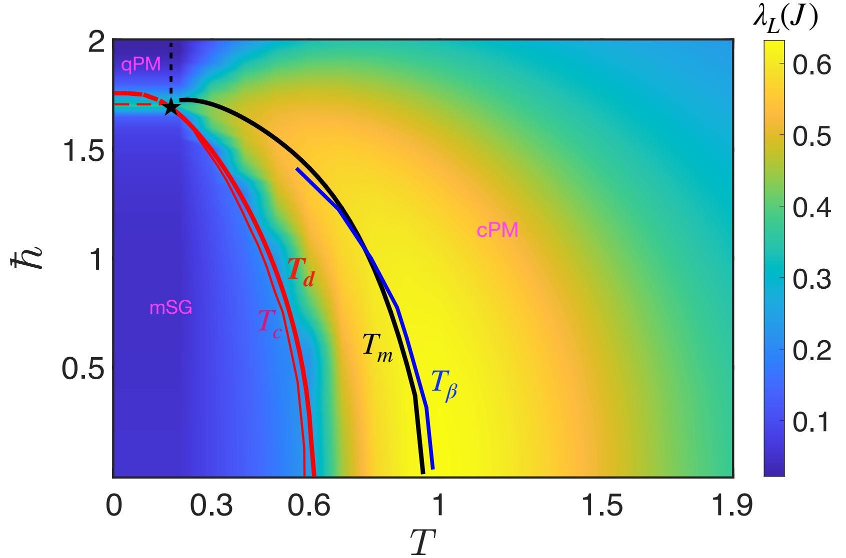
- Phase diagram of chaos in a solvable quantum spin glass model.

- The approach to classical limit in terms of chaos could be non-trivial and non-monotonic as a function of $\hbar$

- Quantum fluctuations can make system more chaotic or less chaotic depending on the phase.

- Non-monotonic temperature dependence of Lyapunov exponent in the supercooled liquid regime in the p-spin glass model.



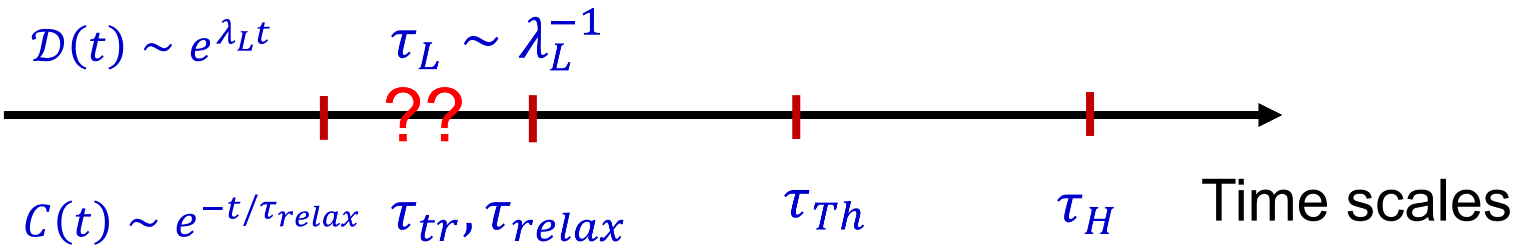


Chaos across thermal phase transitions in two dimensions in the classical XXZ model.

- Spatio-temporal growth of chaos?
- Relation between transport and chaos?

  Diffusion coefficient $D \sim v_B^2/\lambda_L$?
- What happens to chaos across phase transitions? Is there critical slowing down? Effect of collective modes on chaos?

- Relation between different time scales?



Strongly quantum lattice models with local interactions and finite-dimensional local Hilbert space do not show exponential growth.

However, quantum effects should be negligible close to transition with diverging length and time scale (Hohenberg & Halperin)

$\Rightarrow$ Exponential growth near the transition even in quantum models?

Study a well known lattice spin model in the large S ($\rightarrow \infty$) or classical limit.

Model and dynamics

Classical anisotropic Heisenberg Hamiltonian on 2D square lattice —

$$\mathcal{H} = -\frac{J}{2}\sum_{\mathbf{r},\delta}(S^x_{\mathbf{r}}S^x_{\mathbf{r}+\delta} + S^y_{\mathbf{r}}S^y_{\mathbf{r}+\delta} + \Delta S^z_{\mathbf{r}}S^z_{\mathbf{r}+\delta})$$

Two paradigmatic phase transitions —

- Kosterlitz-Thouless (KT) for $\Delta < 1$ (Model E critical dynamics, $z = 1$)
- Ising for $\Delta > 1$ (Model D critical dynamics, $z = 4 - \eta$)

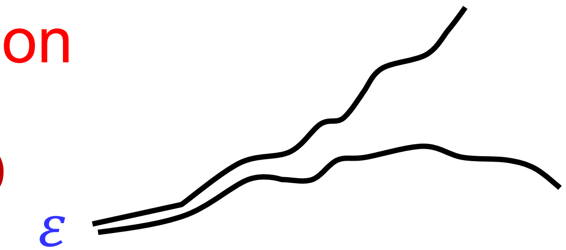
Poisson bracket dynamics:

$$\frac{d\mathbf{S}_{\mathbf{r}}}{dt} = \{\mathbf{S}_{\mathbf{r}}, \mathcal{H}\} = \mathbf{S}_{\mathbf{r}} \times J\sum_{\delta}(S^x_{\mathbf{r}+\delta}\hat{\mathbf{x}} + S^y_{\mathbf{r}+\delta}\hat{\mathbf{y}} + \Delta S^z_{\mathbf{r}+\delta}\hat{\mathbf{z}})$$

Conserved quantities E_{tot} and $S^z_{\text{tot}} = \sum_i S^z_i$ for $\Delta \neq 1$

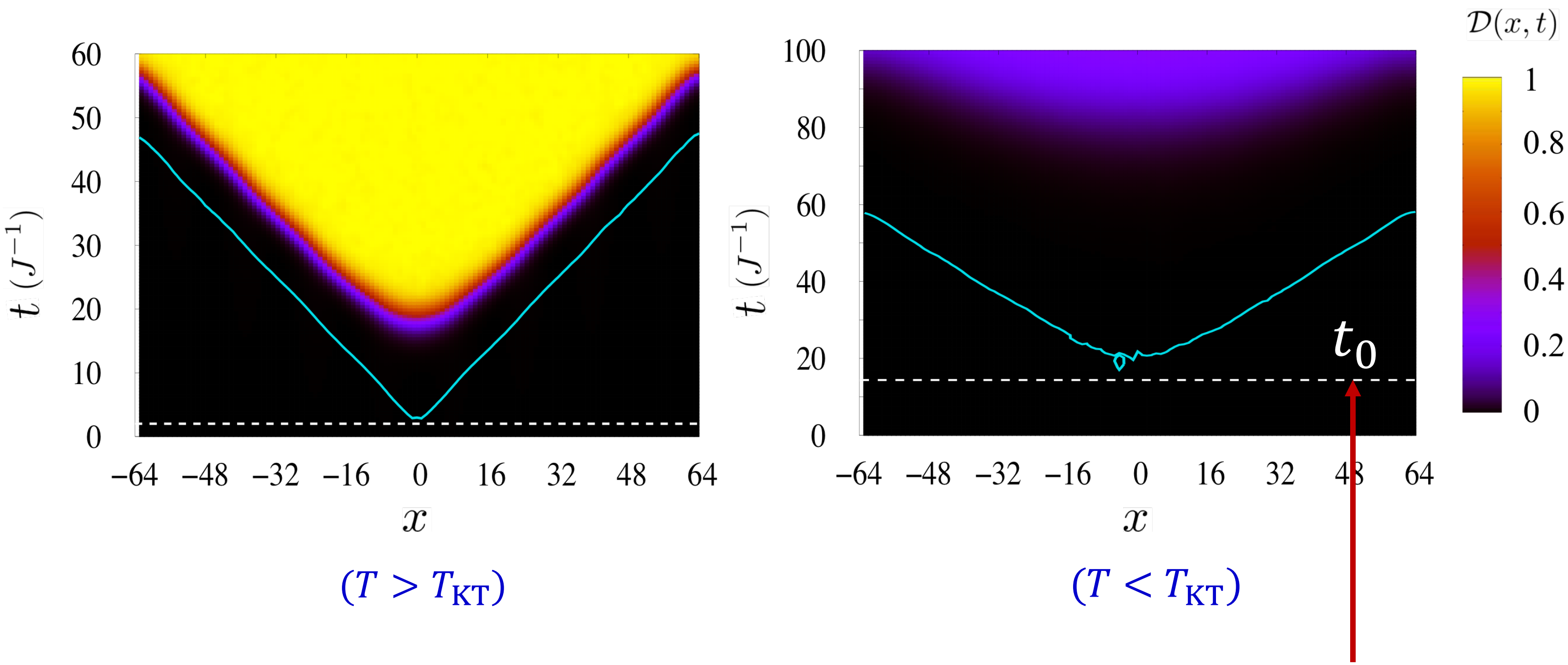
Classical OTOC or “decorrelation” function

The de-correlation function Das et.al, PRL (2018)



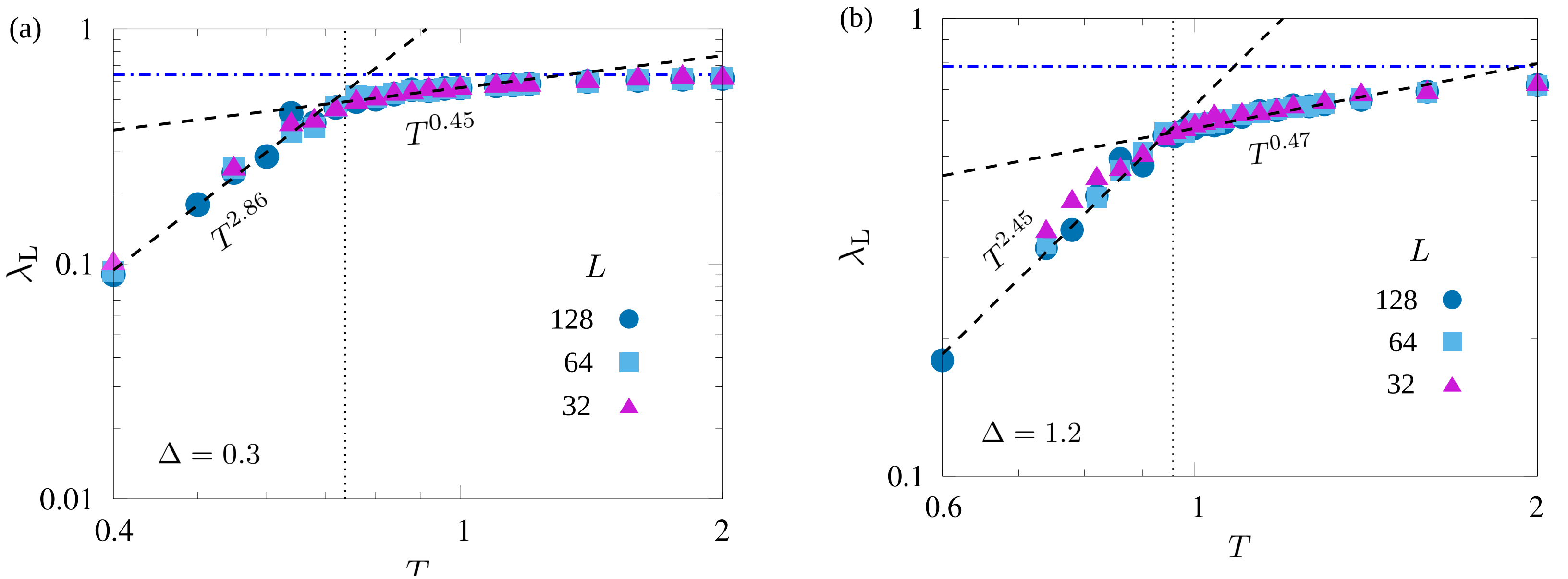
$$\mathcal{D}(\mathbf{r},t) = 1 - \langle \mathbf{S}_{a\mathbf{r}}(t)\cdot \mathbf{S}_{b\mathbf{r}}(t)\rangle$$

$$= 1 - \mathcal{F}(\boldsymbol{r},t)$$



Delay in the onset of light cone

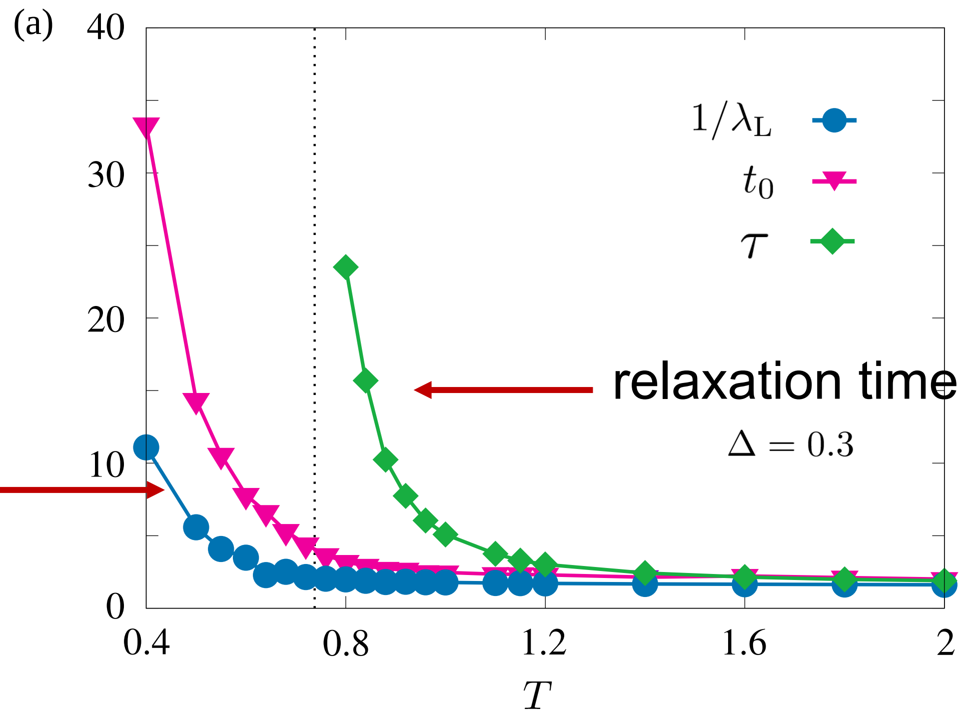
Temperature dependence of Lyapunov exponent across KT and Ising transitions



$(\Delta = 0.3)$ $(\Delta = 1.2)$

- Clear crossover across both the transitions
- Power law dependence with temperature above and below the transitions
- Not much finite size effects at the transition ⇒ Absence of critical slowing down and signature of diverging length and time scale in chaos

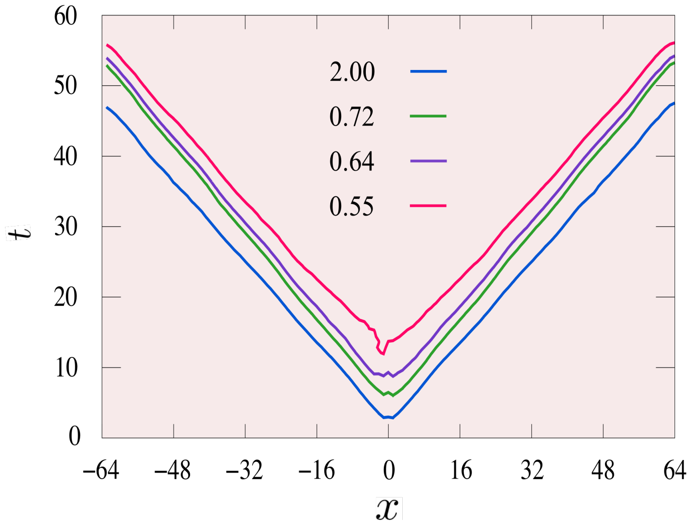
Comparison of time scales



relaxation time

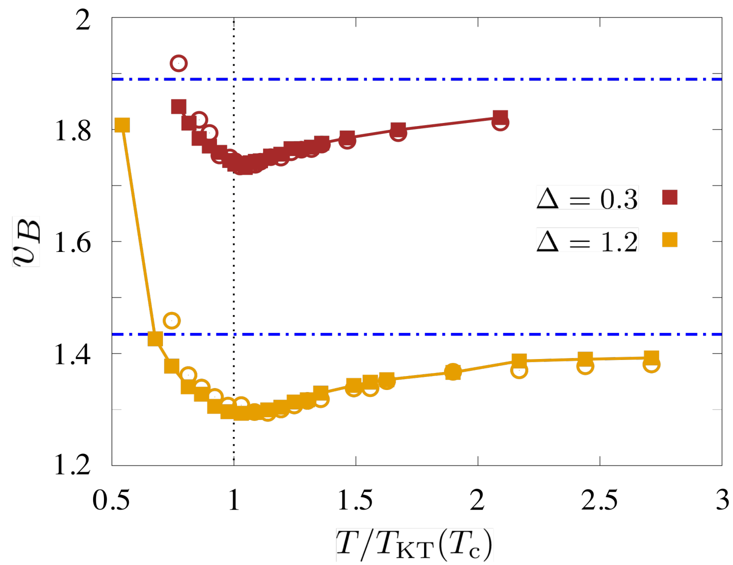
chaos time

Spatial spread of chaos



The chaos front spreads ballistically

Temperature dependence of butterfly speed



- Clear minima at the transitions
- Only relatively sharp signature of the phase transition in terms of chaos

Scaling for classical OTOC

$$\mathcal{F}(r,t) = \Phi\left(\frac{L}{\xi}, \frac{r}{\xi}, \xi^{-z}t\right)$$

$\Lleftarrow$

$$v_B \sim \xi^{1-z} \quad L \to \infty$$
$$\sim \quad L^{1-z} \quad \xi \to \infty$$

No system-size dependence for KT $(z = 1)$

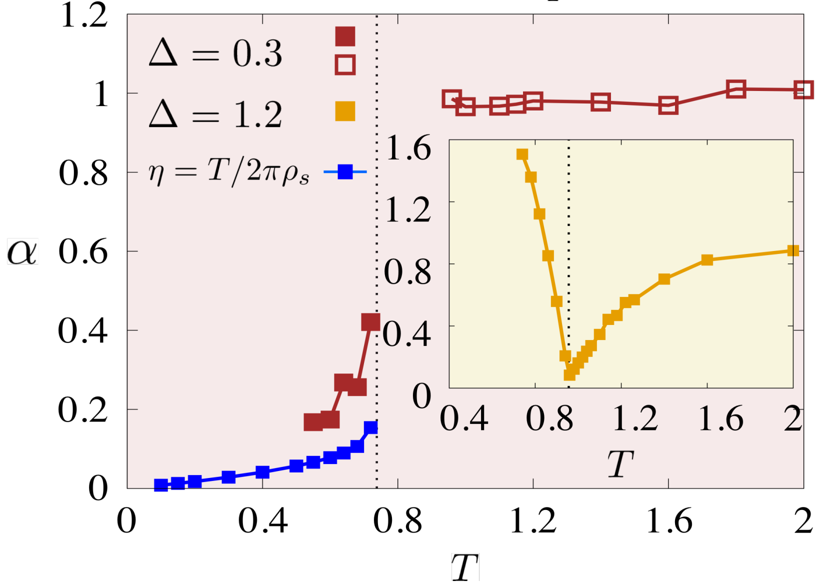
Strong system-size dependence for Ising $(z = 4 - \eta)$

Transport and chaos

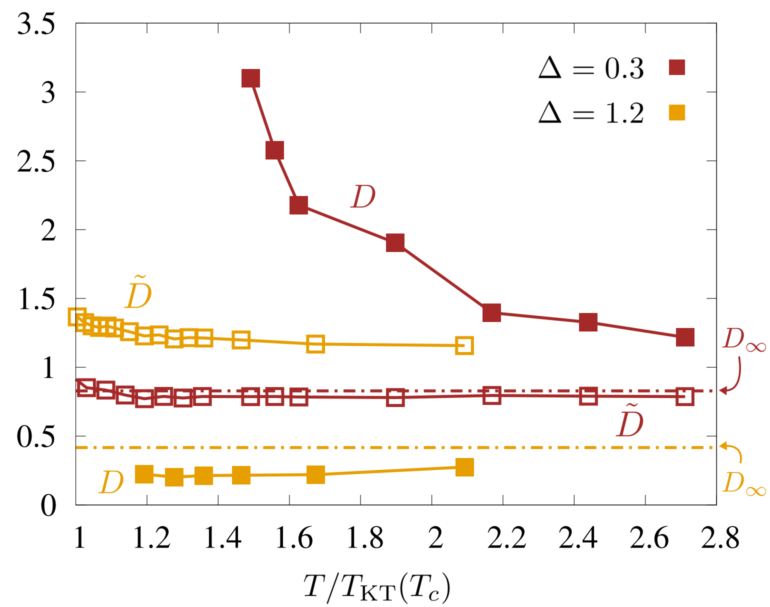
Spin-spin autocorrelation function

$i \in x, y, z \qquad \mathcal{C}_{ii}(t) = \frac{1}{N}\sum_{\mathbf{r}} S^i_{\mathbf{r}}(t) S^i_{\mathbf{r}}(0)$ $\sim e^{-t/\tau}$, exponential decay

or $\sim 1/t^{\alpha}$, powerlaw decay



⇒ Anomalous diffusion at low temperature and across the transitions

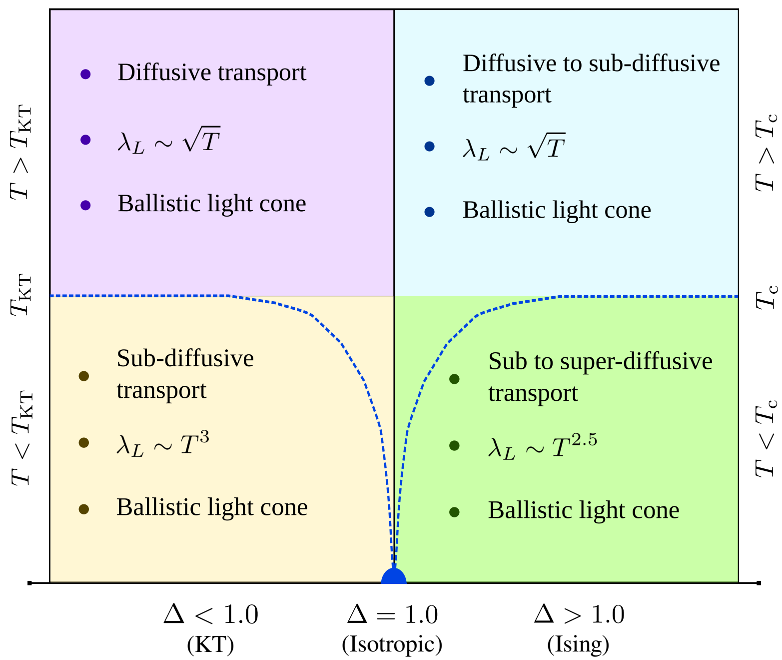


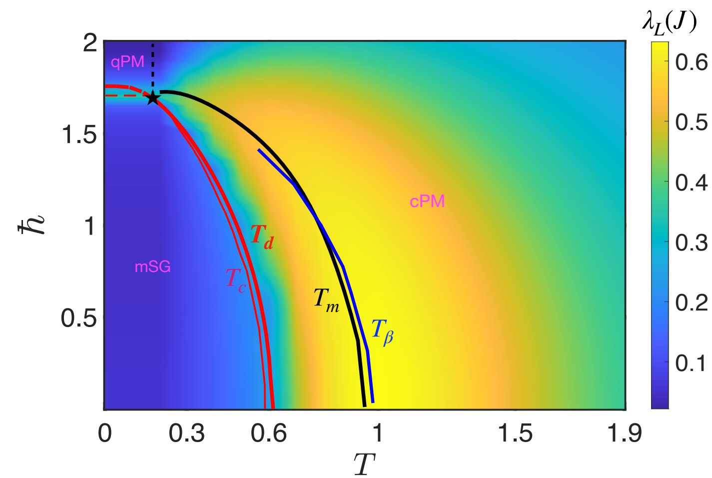
In general the diffusion constant D and $\tilde{D} \sim v_B^2/\lambda_L$ are not related

- Crossover in chaos across KT and Ising transitions.
- Chaos is unaffected by critical slowing down.
- Chaos spreads ballistically in phases with normal and anomalous
- diffusions.
- In general, $D \neq v_B^2/\lambda_L$, connection between chaos and transport

is not always straightforward.

Chaos in quantum spin glass and classical quantum crossover



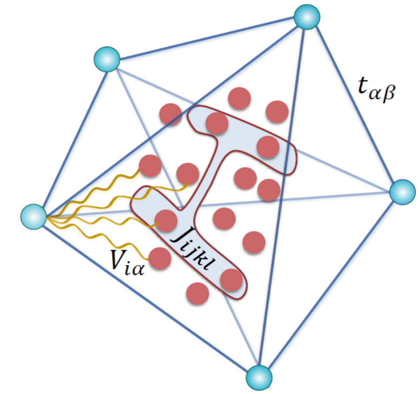


Chaos and transport across KT and Ising transitions

Thank You!

Fast to slow scrambling across dynamical transition from non-Fermi liquid to Fermi liquid

$$H = \frac{1}{(2N)^{3/2}}\sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{M}}\sum_{\alpha\beta} t_{\alpha\beta}\psi_\alpha^\dagger \psi_\beta + \frac{1}{(MN)^{\frac{1}{4}}}\sum_{i\alpha}(V_{i\alpha} c_i^\dagger \psi_\alpha + h.c.)$$



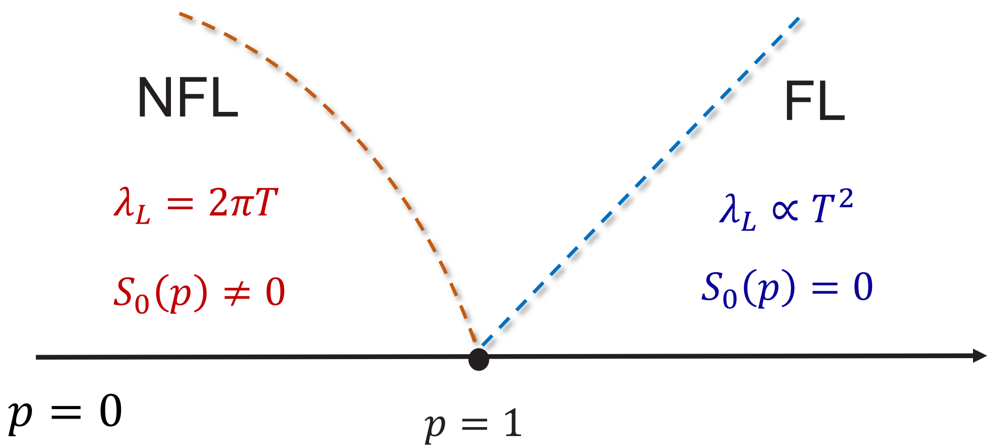
SB & E. Altman, PRB (2017)

Large N limit

$p = M/N$

$N, M \to \infty$

SYK fermions -- N sites
"Lead" fermions -- M sites



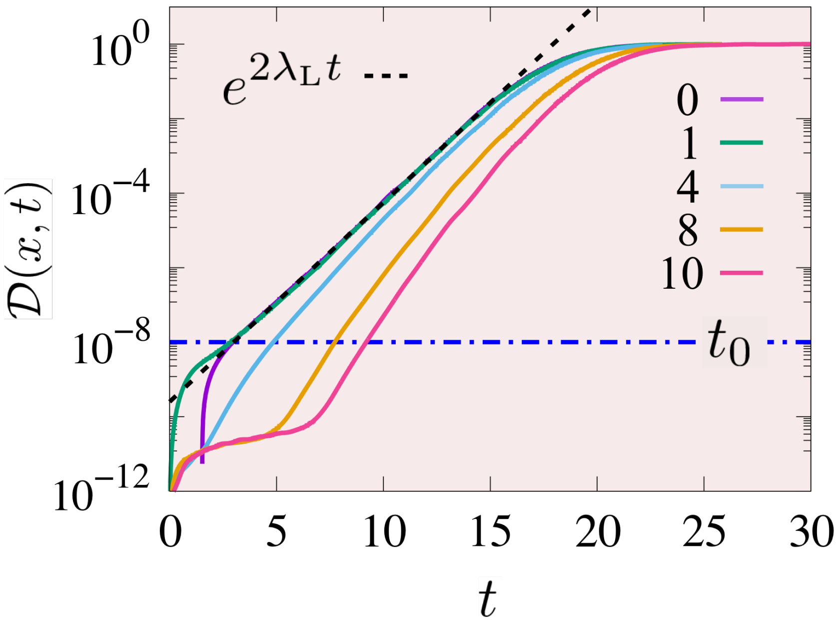
Transition from fast to slow scrambling due to 'quantum fluctuations' as SYK fermions get screened or hybridized. Quantum parameter is no explicit!

Lyapunov exponent

We need to extract λ_{L} and v_B from decorrelation.

$$\mathcal{D}(x,t) = \varepsilon^2 e^{2\lambda_L(x,t)t}$$

Lyapunov exponent
$$\lambda_L \equiv \lambda_{\mathrm{L}}(0,t) = \frac{1}{2t}\ln\left[\mathcal{D}(0,t)/\varepsilon^2\right]$$



Arguments for the existence of chaos bound

The proof for the bound, $\lambda_L \leq 2\pi k_B T/\hbar$, is not a rigorous proof!

- Maldacena-Shenker-Satnford $\Rightarrow$ Analytical properties of regularized OTOC + some physical assumptions

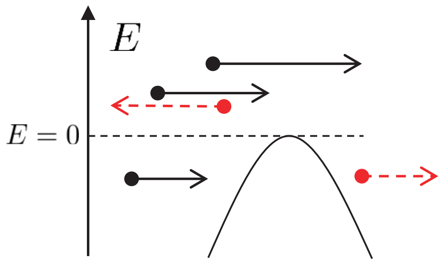
$$F(t) = \frac{1}{Z} Tr[e^{-\frac{\beta H}{4}} A(t) e^{-\frac{\beta H}{4}} B(0) e^{-\frac{\beta H}{4}} A(t) e^{-\frac{\beta H}{4}} B(0)]$$

- Energy-time uncertainty type argument (very crude)

$$\lambda_L^{-1} k_B T \geq \hbar$$

- Murthy and Srednicki, PRL (2019) $\Rightarrow$ Eigenstate thermalization hypothesis + assumptions.

- Morita, SciPost (2021) $\Rightarrow$ Effective model for classical system with Lyapunov exponent $\rightarrow$ inverse Harmonic potential



$$P(E) := \frac{1}{\exp(\beta_L |E|) + 1}$$

Analogous Hawking radiation temperature

$$T_L := \frac{1}{\beta_L} = \frac{\hbar}{2\pi} \lambda_L$$

Thermal equilibrium $\Rightarrow T \geq T_L \Rightarrow \lambda_L \leq 2\pi k_B T/\hbar$