An exactly solvable model for interacting electrons in a magnetic field

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+
Unpublished results

Acknowledgments
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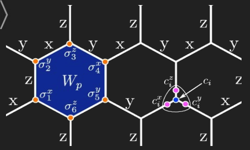
Motivation

$$\sum \sigma_z^i \sigma_z^{i+1} + f\sigma_x^i$$

$$\sum_{\langle ij \rangle} J_{\mu_{ij}} \sigma^i_{\mu_{ij}} \sigma^j_{\mu_{ij}}$$



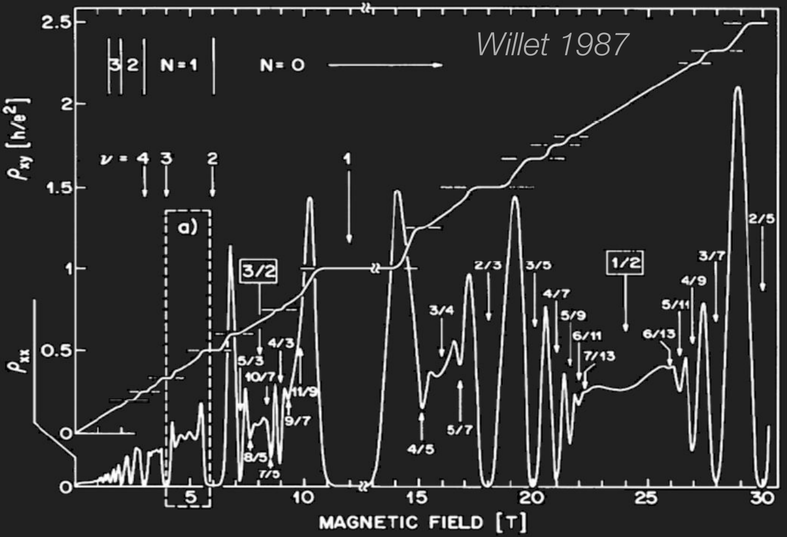
$$\mathcal{M}(p,q)$$

$$-\mu \sum n_i - \sum c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + h.c.$$

$$\hat{H} = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2$$

$$\hat{H} = \frac{J}{4} \sum_{i=1}^{N} (\vec{S}_{j-1} + \vec{S}_j + \vec{S}_{j+1})^2$$

$$-\sum_{\text{plaq}} \left(|\parallel\rangle\langle =| + \text{H.c.} \right) + \sum_{\text{plaq}} \left(|\parallel\rangle\langle \parallel| + |=\rangle\langle =| \right)$$



$$H = \sum_{i<j} \mathcal{P}^{ij}_{L=1}$$ Laughlin + QHs

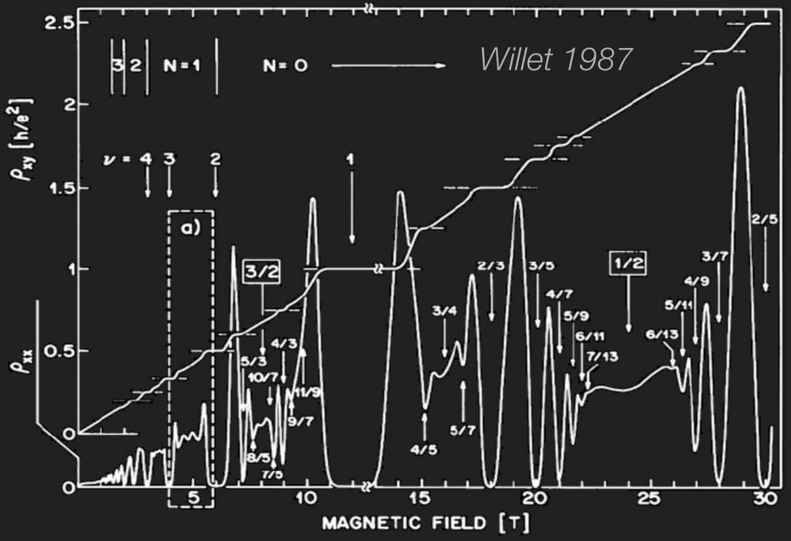
$$H = \sum_{i<j<k} \mathcal{P}^{ijk}_{L=3}$$ Moore Read + QHs

$$H = \nabla^2 \delta(r_1 - r_2)$$ Jain CF state @ 2/5

Pfeutty 1970, Kitaev 2001 Kitaev 2005, Belavin et al 1984, Affleck et al 1987, Majumdar Ghosh 1969, Kivelson, Rokhsar, Sethna 1987

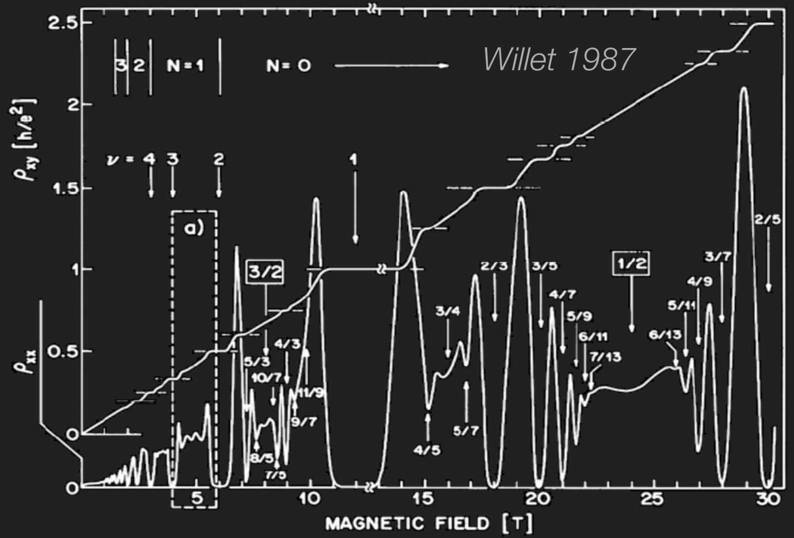
Laughlin 1983, Moore Read 1991, Simon Rezayi Cooper 2007, Trugman Kivelson 1985

CF wavefunctions, CF Landau levels, low energy excitations



$$H = \frac{1}{r}$$

CF wavefunctions, CF Landau levels, low energy excitations



$$H = \frac{1}{r}$$

$$\nu = \frac{n}{2pn+1}$$

$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n$$

Attaches *2p* vortices to each electron

IQH state of CF at an effective *B*

Jain PRL 1989

CF wavefunctions, CF Landau levels, low energy excitations

Ground state at ***n/(2pn+1)*** can be interpreted as an IQH state of the CFs at an effective magnetic field.

Neutral modes, QPs, QHs can be interpreted as corresponding excitations of the IQHE of CFs.

Low energy spectra of the FQH states here becomes identical (at the level of quantum numbers) to the IQH states at an effective magnetic field.

These give wavefunctions with a simple structure that are almost identical to the eigenstates of the Coulomb Hamiltonian in the lowest Landau level.

$$\nu = \frac{n}{2pn+1}$$

$$\underbrace{\prod_{i<j=1}^{N} (z_i - z_j)^{2p}}_{\text{Attaches } 2p \text{ vortices to each electron}} \times \underbrace{\Psi_n}_{\text{IQH state of CF at an effective } B}$$

Jain PRL 1989
Jain Kivelson Trivedi PRL 1990

CF wavefunctions, CF Landau levels, low energy excitations

Laughlin state corresponds to the case where *n=1*. One Landau level of the IQHE of CFs is fully filled.

There is a simple Hamiltonian - *V1* pseudopotential Hamiltonian - of which this state is the incompressible exact ground state. The Coulomb interaction is in the same phase as this Hamiltonian. *(Haldane 1983)*

In spite of the similar and simple structure, the general state at *n/(2pn+1)* is not known to be the eigenstate of any local Hamiltonian.

"... in applying the cut and glue method to composite fermion states we have implicitly assumed that each CF wave function is the exact ground state of some reasonably physical Hamiltonian, often known as a parent Hamiltonian. It is still not known if such Hamiltonians exist... "

- *Henderson, Sreejith, Simon, in preparation;*

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$$\underbrace{\prod_{i<j=1}^{N} (z_i - z_j)^{2p}}_{\text{Attaches } 2p \text{ vortices to each electron}} \times \underbrace{\Psi_n}_{\text{IQH state of CF at an effective } B}$$

Attaches *2p* vortices to each electron

IQH state of CF at an effective *B*

Sreejith, Fremlin, Jeon, Jain 2018; Bandyopadhyay et al 2020
Wilczek, Greiter 2021;
Li Haldane 2008, Dubail, Read, Rezayi 2011, Qi, Katsura, Ludwig 2012

Jain PRL 1989
Jain Kivelson Trivedi PRL 1990

Motivation

Models of strongly interacting systems that are exactly solvable are special

$$H = \sum_{i<j} \mathcal{P}^{ij}_{L=1}$$ Laughlin + QHs

$$H = \sum_{i<j<k} \mathcal{P}^{ijk}_{L=3}$$ Moore Read + QHs

$$H = \sum_{i<j} \nabla^2 \delta(r_i - r_j)$$ Jain CF state @ ⅖

- Only the zero energy GS and zero energy QH states can be written down
- A different model for every FQH state. Each model works for only one FQH state
- Conventional approach → cyclotron gap > interactions ⇒ Project into one/few LLs.
- Such exact Hamiltonians are not known for generic Jain sequence states

Laughlin 1983, Haldane 1983, Moore Read 1991, Greiter Wen Wilczek 1991, Simon Rezayi Cooper 2007, Trugman Kivelson 1985, Rezayi, MacDonald 1991, Jain, Kivelson, Trivedi 1990, Bandhopadhyay et al 2018, 2020, Sreejith, Fremling, Jeon, Jain 2018

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Features of the model that we present
(PRL 126, 136601 (2021))

- Model is defined in the unconventional limit of very strong interactions (relative to the cyclotron energy)
- Strong interaction defines a highly constrained Hilbert space
- All low energy states - GS, QH, QP, neutral modes can be written down exactly
- A single model *H* works for multiple FQH states (for all spin polarized Jain states at filling n/(2pn+1))
- Frustration free

Laughlin 1983, Haldane 1983, Moore Read 1991, Greiter Wen Wilczek 1991, Simon Rezayi Cooper 2007, Trugman Kivelson 1985, Rezayi, MacDonald 1991, Jain, Kivelson, Trivedi 1990, Bandhopadhyay et al 2018, 2020, Sreejith, Fremling, Jeon, Jain 2018

CF wavefunctions

Our construction is motivated by the structure of the CF wavefunction.

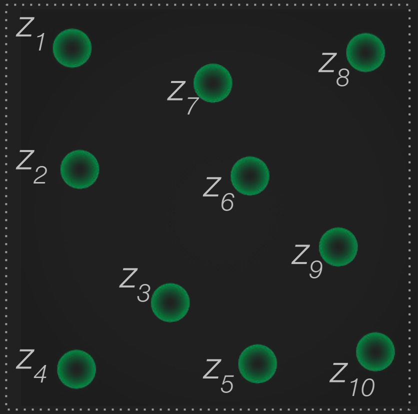
CF wf describing an incompressible state at $\nu = \dfrac{n}{2pn+1}$

$$\underbrace{\prod_{i<j=1}^{N} (z_i - z_j)^{2p}}_{\text{Attaches } 2p \text{ vortices to each electron}} \times \underbrace{\Psi_n}_{\text{IQH state of CF at an effective } B}$$

Jain 1989

Correlations in two particle sectors

$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n$$



Pairs of electrons in this correlated state occur in various possible relative angular momentum channels

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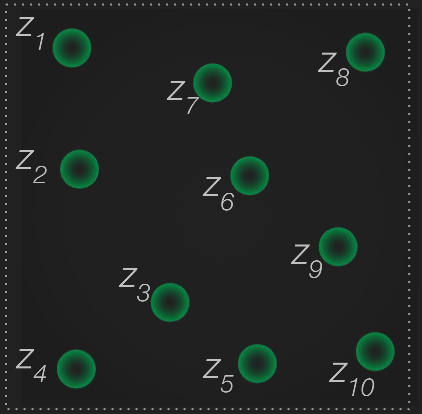

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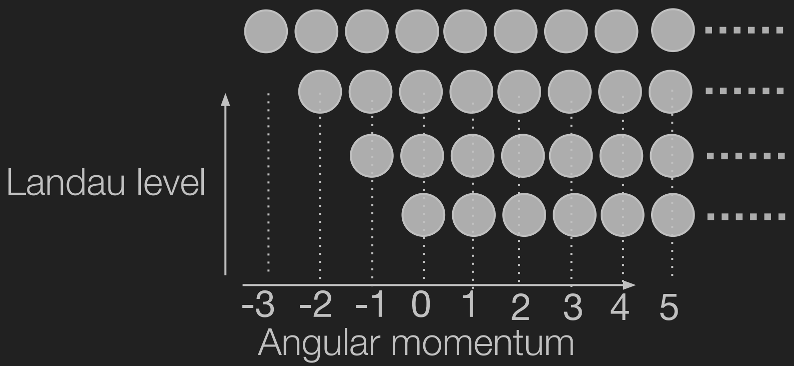

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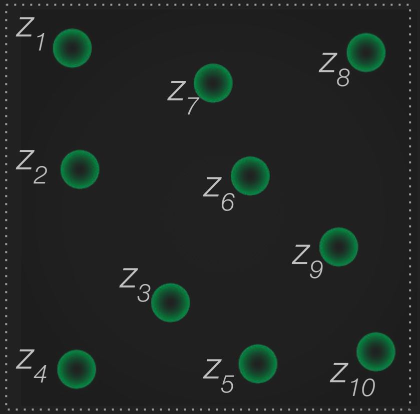

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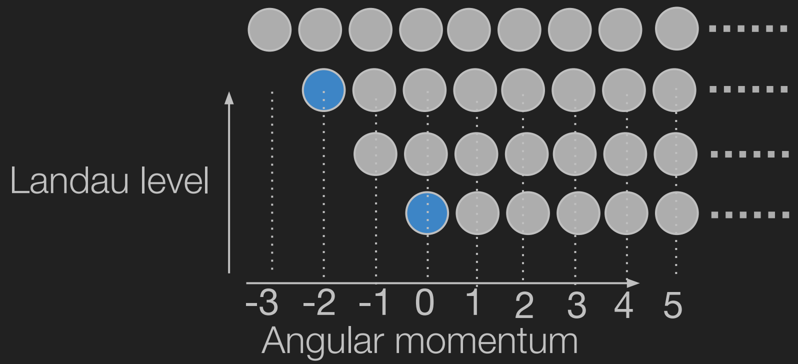

In the IQH state Ψ_γ , minimum relative angular momentum is ***-n-m***

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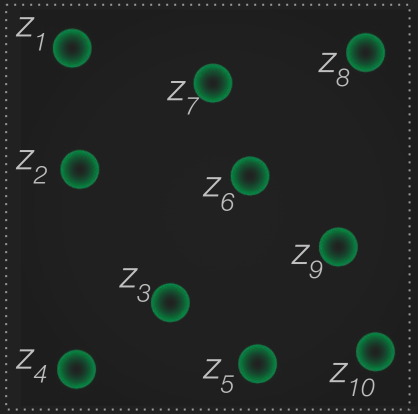

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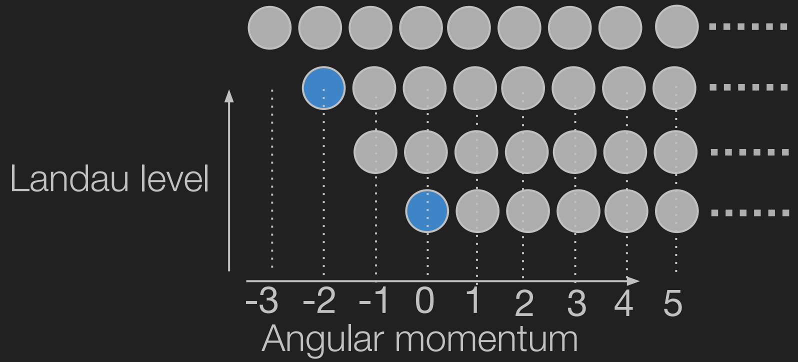
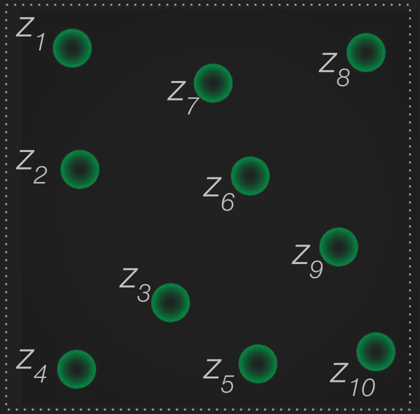


In the IQH state Ψ_ν , minimum relative angular momentum is ***-n-m***

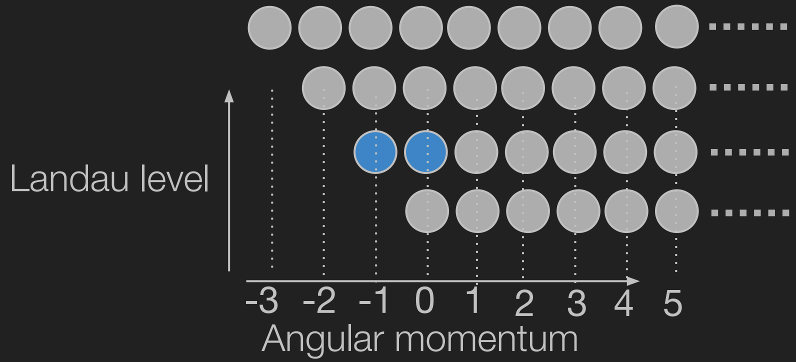
Flux attachment increments this to ***-n-m+2p***

Correlations in two particle sectors

$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n$$



Pairs of electrons in this correlated state occur in various possible relative angular momentum channels

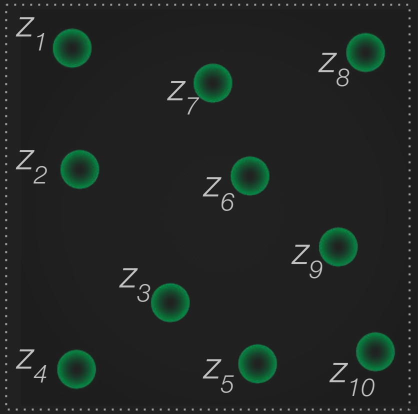


In the IQH state Ψ_γ , minimum relative angular momentum is ***-n-n+1***

Flux attachment increments this to ***-n-n+1+2p***

Correlations in two particle sectors

$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n$$



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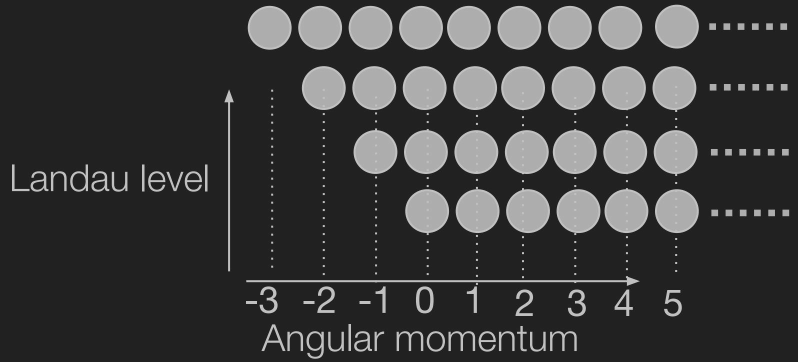
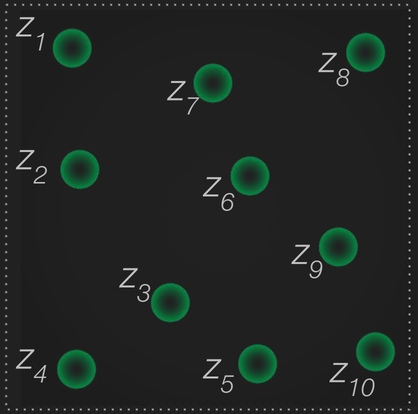


Summary: For two particles in LLs ***n*** and ***m***, relative momenta are lower bound by

$$-n - m + \delta_{nm} + 2p$$

Correlations in two particle sectors

$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n$$



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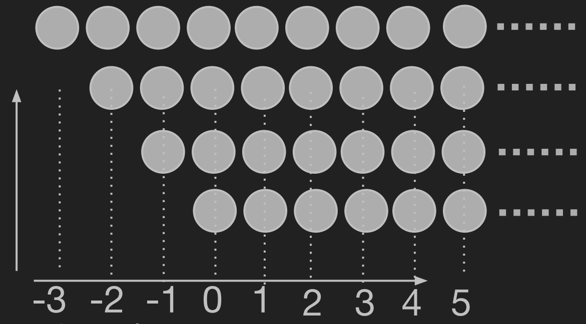


Summary: For two particles in LLs ***n*** and ***m***, relative momenta are lower bound by

$$-n - m + \delta_{nm} + 2p$$

Can this be used to construct a parent Hamiltonian by projecting out the forbidden sectors ?

No. Multiplication by Jastrow factor does not preserve the LL indices.

Jastrow factor of guiding center coordinates

Consider a slightly different wavefunction

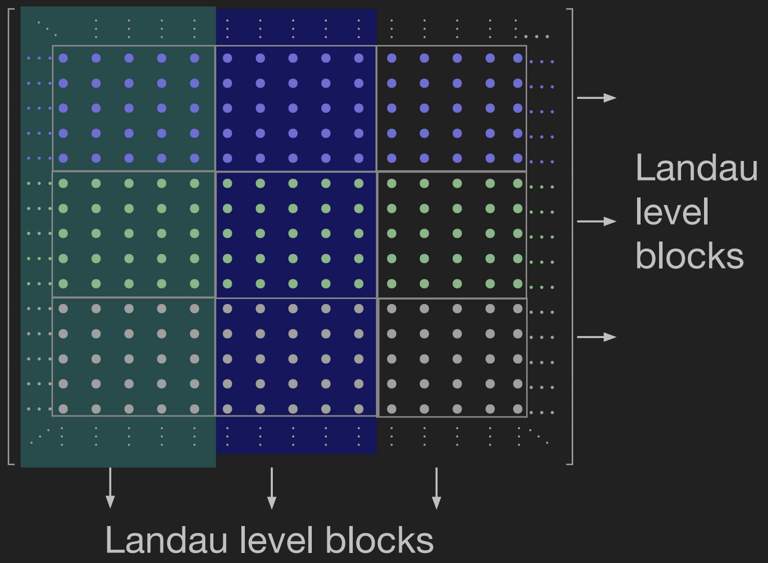
$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n \xrightarrow{\text{replace with}} \prod_{i<j=1}^{N} (\hat{Z}_i - \hat{Z}_j)^{2p} \times \Psi_n$$

Slater determinant
Fermions occupying arbitrary single particle KE eigenstates

-Raises relative angular momenta
-But preserves LL labels

Guiding center coordinates

Position operator scatters between LLs



Guiding center coordinates

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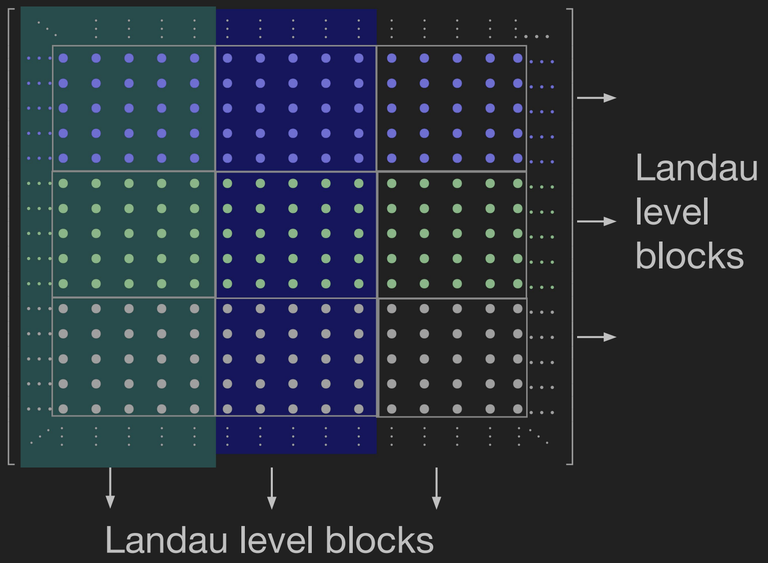
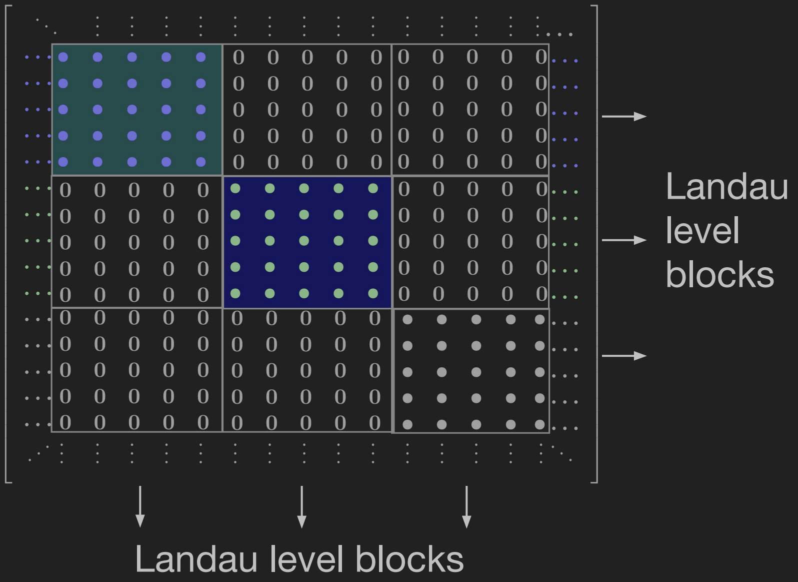


Guiding center coordinates do not scatter between LLs



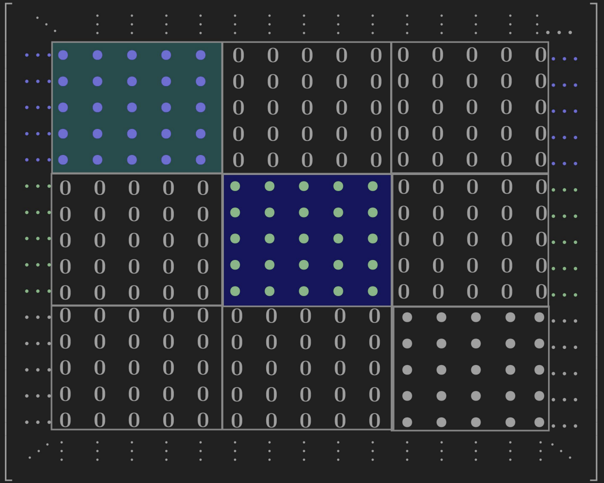
Guiding center coordinates

$$\hat{Z} = \hat{\pi} - (\hat{z} \times \vec{r})$$
$$\sim \hat{z} - \imath(\hat{\pi}_x - \imath\hat{\pi}_y)$$

[T , Z] = 0 ⇒ Preserves Landau level

[L , Z] = Z ⇒ Increments momentum

Guiding center coordinates do not scatter between LLs



$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n \xrightarrow{\text{replace with}} \prod_{i<j=1}^{N} (\hat{Z}_i - \hat{Z}_j)^{2p} \times \Psi_n$$

For two CFs in LLs *n* and *m*, relative momentum in the state is lower bound by

$$-n - m + \delta_{nm} + 2p$$

Jastrow factor of guiding center coordinates

$$\prod_{i<j=1}^{N} (z_i - z_j)^{2p} \times \Psi_n \xrightarrow{\text{replace with}} \prod_{i<j=1}^{N} (\hat{Z}_i - \hat{Z}_j)^{2p} \times \Psi_n \longrightarrow$$

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satisfies this property

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Slater determinant
Fermions occupying arbitrary single particle KE eigenstates

-Raises relative angular momenta
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satisfies this property

For two particles in LLs ***n*** and ***m***, relative momenta in this state is lower bound by

$$-n - m + \delta_{nm} + 2p$$

The wavefunction is different from the Jain CF states → Jastrow factor is replaced with an operator.

Pseudopotential Hamiltonian

We can construct an interaction for which previously mentioned states are exact zero energy states

$$\hat{V} = \sum_{n \leq n'=0}^{\infty} \sum_{M=-n-n'+\delta_{n,n'}}^{-n-n'+\delta_{n,n'}+2p-1} \mathcal{P}_{nn'}^{M}.$$

Sum over Landau levels of particle pairs

Sum over forbidden relative momentum channels

Projector onto relative momentum channels

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Sum over Landau levels of particle pairs

Sum over forbidden relative momentum channels

Projector onto relative momentum channels

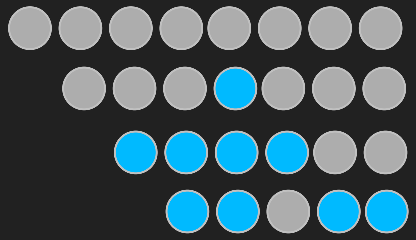
- Imposes an energy cost for pairs in the forbidden relative momentum sectors
- Includes intra LL and inter LL interactions
- Number of particles in each LL is conserved

Interaction Hamiltonian & its zero interaction energy space

$$\hat{V} = \sum_{n\le n'=0}^{\infty} \sum_{M=-n-n'+\delta_{n,n'}}^{-n-n'+\delta_{n,n'}+2p-1} \mathcal{P}^{M}_{nn'}.$$

$$\hat{V}\Phi = 0 \quad \text{for all } \Phi \text{ of the form } \prod_{i<j=1}^{N} (\hat{Z}_i - \hat{Z}_j)^{2p} \times \Psi_n$$

$$\prod_{i<j=1}^{N} (\hat{Z}_i - \hat{Z}_j)^{2p} \times$$



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These states are low energy states of a Hamiltonian $H = \hat{T} + \lambda \hat{V}$ in the limit $\lambda \to \infty$

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$$H\Phi = (T + \lambda V)\Phi = T\Phi = \prod (\hat{Z}_i - \hat{Z}_j)^{2p}\, T\Psi_n = E_n\Phi$$

Energy of the state is same as the KE of the slater determinant Ψ_n

Interaction Hamiltonian & its zero interaction energy space

$$\hat{V} = \sum_{n \le n' = 0}^{\infty} \sum_{M = -n-n'+\delta_{n,n'}}^{-n-n'+\delta_{n,n'}+2p-1} \mathcal{P}_{nn'}^{M} \cdot$$

We conjecture that these type of states exhaust the null space of ***V*** and are linearly independent.

Conjecture tested and found to be exact in every one of ~200 systems studied.

$$\hat{V}\Phi = 0 \quad \text{for all } \Phi \text{ of the form } \prod_{i<j=1}^{N} (\hat{Z}_i - \hat{Z}_j)^{2p} \times \Psi_n$$

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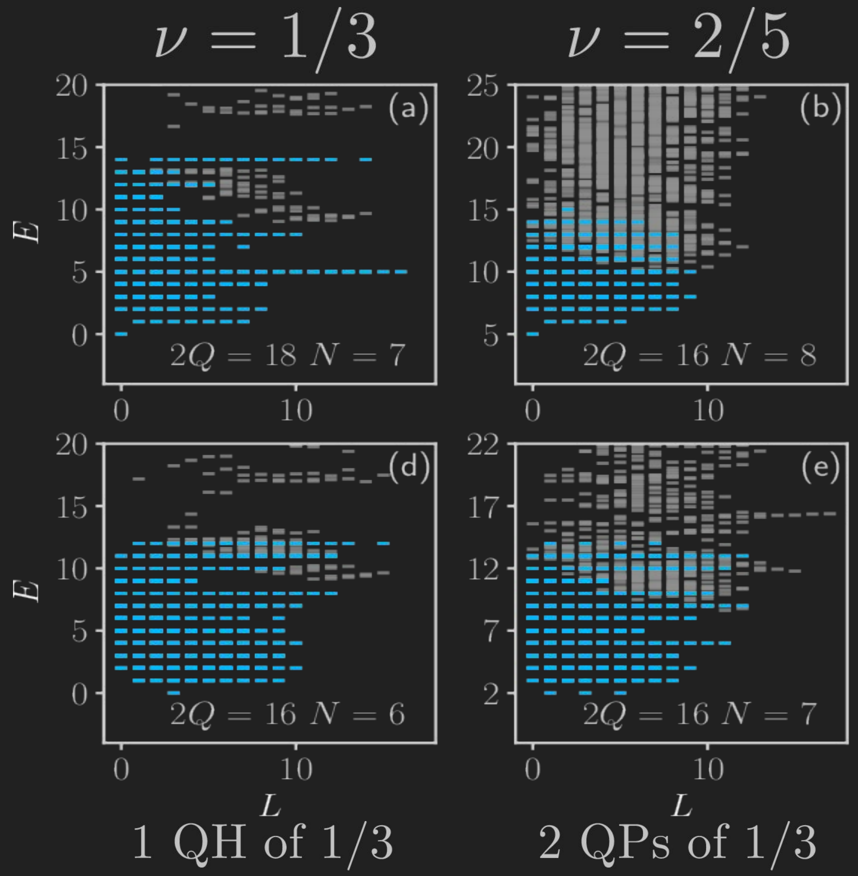
Spectrum of $H=T+V$



Blue color dashes = Null space of interaction *V*

→ Includes QP, QH, neutral modes etc

→ wavefunctions for all these states can be explicitly written down

→ Low energy quantum numbers same as

- those of non-interacting electrons in a reduced field **B***, and
- by implication same as those of Coulomb problem in LLL (Jain 1989)

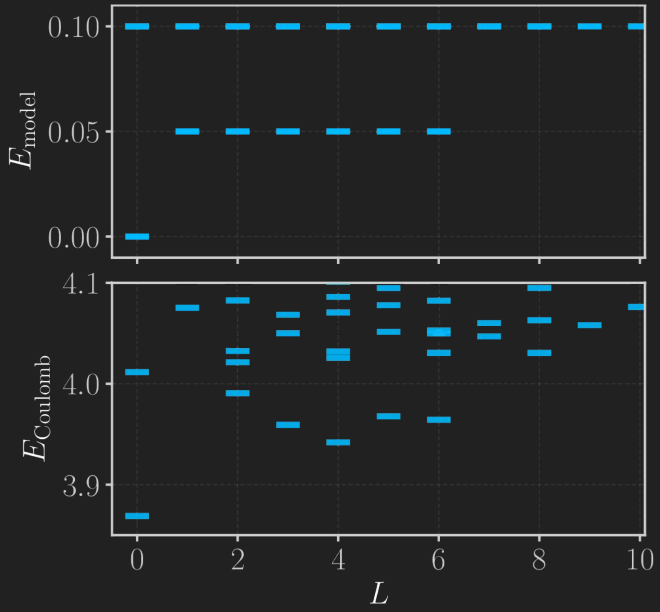
Grey color dashes = Finite interaction energy states

These will pushed to infinite energies if interaction strength is sent to infinity

Spectrum of ***H=T+V*** compared to LLL Coulomb

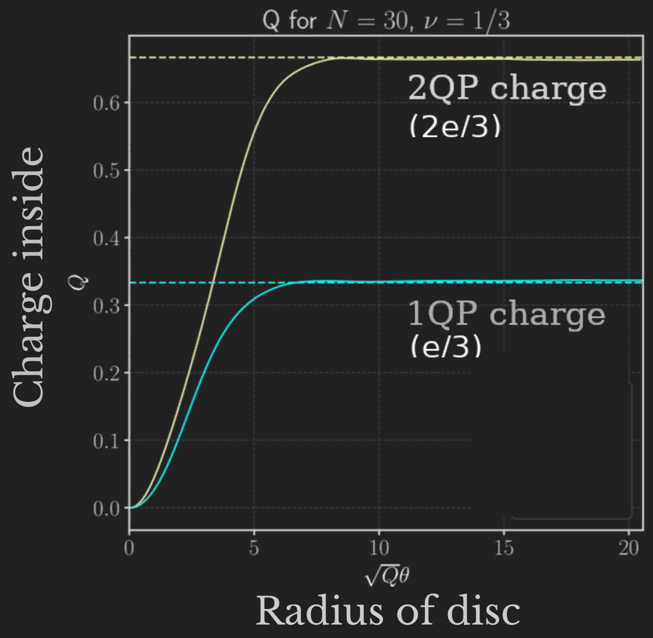
Model

LLL
Coulomb

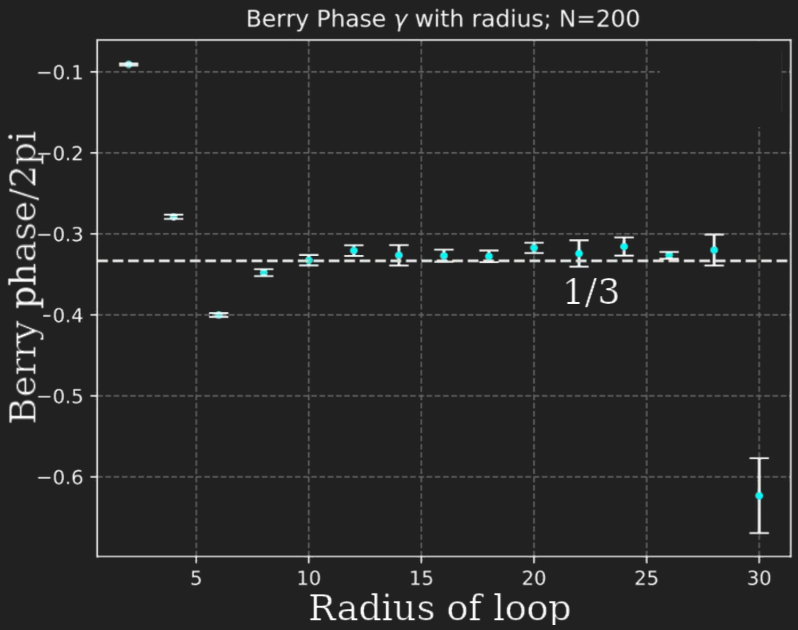


Low energy states of LLL Coulomb
have the same quantum numbers as
the low energy states of the model Hamiltonian

Berry phase, charge of localized excitations



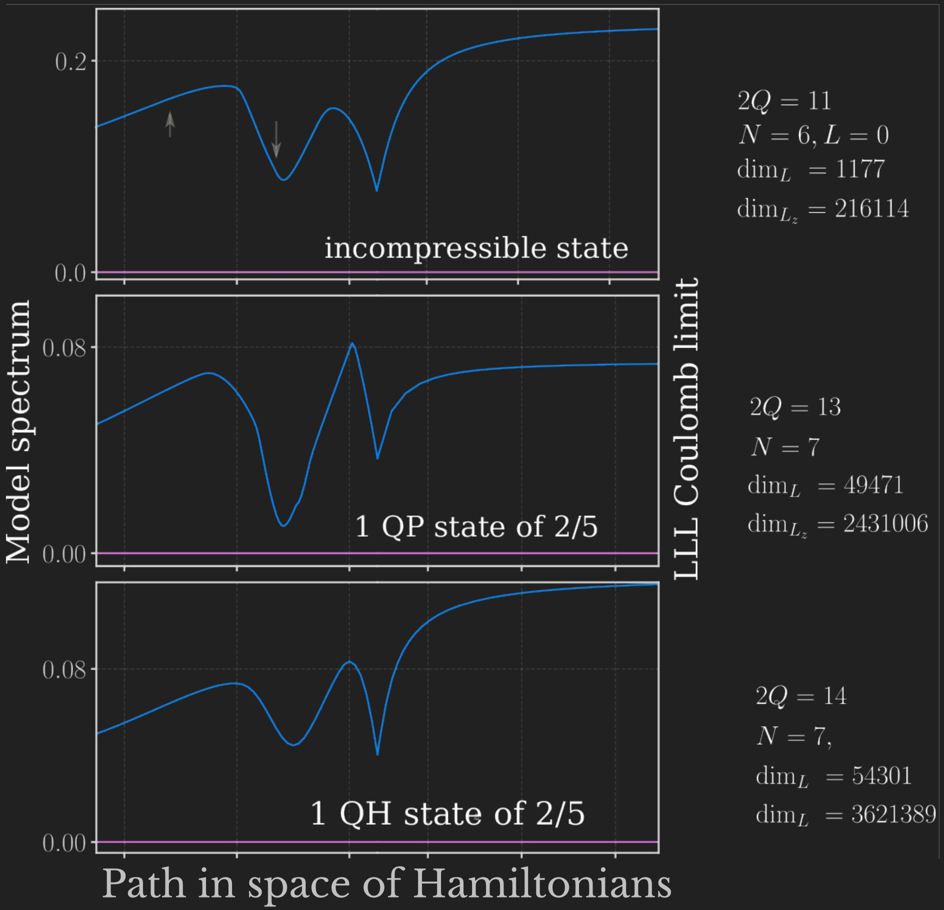
Charge inside a disc around 1 or 2 localized QPs of the model



Berry phase from winding one localized QP around a QH of 1/3

A Anand, unpublished

Adiabatic continuity: From model to *LLL Coulomb*

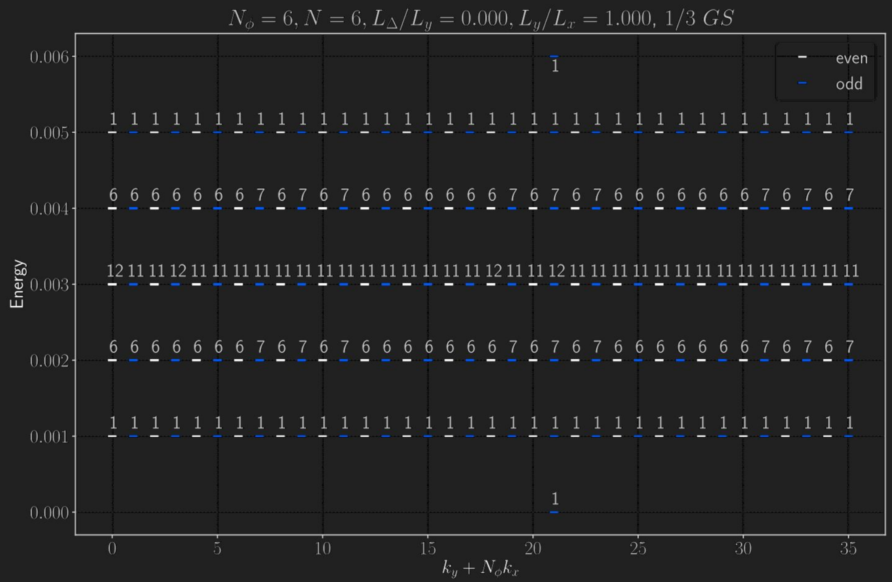


For all finite systems that we could study,
ground state, single quasihole state and single quasiparticle state of the model are adiabatically connected with corresponding LLL Coulomb states.

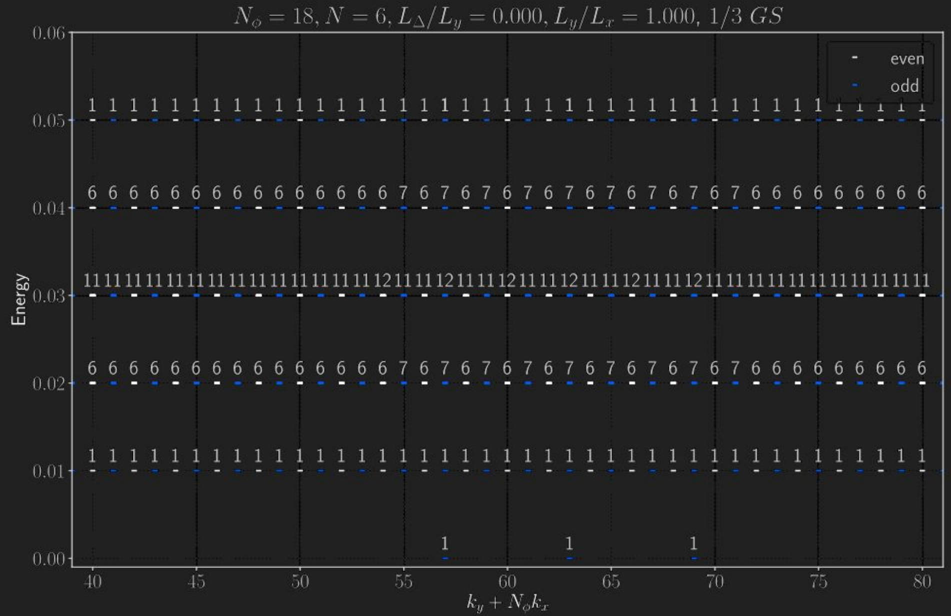
Generalization to the Torus geometry

Though the model was originally written in terms of angular momenta, the interaction is local.
So we expect that the model should be generalizable to geometries without any rotational symmetry.

Indeed this can be generalized to the torus geometry. *(Haldane, 1990; Haldane 1985)*



IQH spectrum



FQH spectrum

A Anand, unpublished

Conclusions

- An infinitely strongly interacting model with exactly solvable spectra - not just GS and QH but also QP, exciton states and all excited states can be constructed.
  - Gapped GS at Jain sequence *$\nu=n/(2pn+1)$*
  - Low energy quantum numbers, charge, statistics, topological degeneracies, shifts of incompressible states all same as LLL Coulomb
  - Microscopic model with an exact mapping between FQHE of electrons at B and IQHE of CFs at B*

- An unusual limit: Large interaction compared to the cyclotron gap
  - Instead of the usual single particle constraint of restricting to LLL
  - This has a many particle constraint wherein the low energy Hilbert space is made of highly correlated many body states
  - KE splits this space to produce LLL Coulomb-like low energy spectra.

- Numerics clearly show that the low energy quantum numbers expected from the exact solutions hold in all geometries - torus, sphere, disk, cylinder.
  - Exact solutions exist only on systems with an open boundary - cylinder and disk.
  - Attempts at writing the solutions on closed manifolds fail → Wavefunctions “spill out of the Hilbert space” on the sphere *(Greiter 2011)*
  - On the torus necessary boundary conditions fail to be satisfied (Anand, Pu et al unpublished)

Anand, Jain, Sreejith, PRL 126, 136601 (2021), Anand, Pu, et al (unpublished)

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