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STOCHASTIC DIFFERENTIAL EQUATIONS ON TWO-DIMENSIONAL THEORY SPACE AND MORSE THEORY*

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The renormalization group equations are shown to be saddle points of the action of a superparticle moving in the presence of a Kähler potential. This allows us to view the "theory space" in the language of Morse theory where the Kähler potential in the Morse function. In the case of two-dimensional field theory, Zamolodchikov's c function can be used as a Morse function. The Morse polynomial tr (t^F) can be computed entirely from the universal Virasoro data at the various fixed points and this provides a topological characterization of the space of potential functions. This idea is applied to the two-dimensional theory space which contains the $c < 1$ minimal models.

1. Introduction

In the past few years of activity in string theory, conformal invariance has clearly emerged as a central organizing principle. Conformally invariant field theories (CIFT) in two dimensions with zero central charge correspond to classical ground states of the string theory.[1]

A two-dimensional CIFT is characterized by (a) its central charge, (b) the primary fields V_i with scaling dimensions $(h_i, \bar{h}_i)$ and (c) C_{ijk}, the structure constants of the operator algebra of V_i. We shall call this collection the Virasoro data.

CIFT are, in particular, scale invariant and hence correspond to fixed points of the renormalization group (RG) in the space of all two-dimensional cutoff hamiltonians. We parametrize the space of coordinates g_i (i.e., the couplings) on a manifold M. (The label i may in principle be continuous.) The RG equations are

$$\frac{dg^i}{dt} = \beta^i(g), \tag{1}$$

where $\beta^i(g)$ is the beta function, and $t = \log a$, a being the position space cutoff. Scale invariance is achieved at the fixed points g^{*i}

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$$\beta^i(g^*) = 0. \tag{2}$$

In the neighbourhood of a fixed point, the Hamiltonian may be parametrized by[2]

$$H = H^* + \sum_i \bar{g}^i V_i \qquad (\bar{g}^i \equiv g^i - g^{i*}). \tag{3}$$

The manifold M may be equipped with a Riemannian metric G_{ij} defined as[3,4,5]

$$G_{ij} \equiv [x^{x_i + x_j} \langle V_i(z, \bar{z}) V_j(0) \rangle]_{x = x_0} . \tag{4}$$

Here $(z, \bar{z})$ are complex coordinates on the $2d$ plane, $x^2 \equiv z\bar{z}$ and x_0 is a renormalization point; n_i is the canonical dimension of the operator V_i. Near a fixed point, one can choose a parametrization in which

$$G_{ij} = \delta_{ij} + O(\bar{g}^2) \tag{5}$$

so one can show that

$$\beta^i = \sum_i (h_i + \bar{h}_i - 2)\bar{g}^i + \sum_{ijk} C_{ijk}\bar{g}^j \bar{g}^k + \dots \tag{6}$$

The total symmetry of C_{ijk}, together with (5) enables one to write, to this order,

$$\beta^i = -G^{ij} \frac{\partial S}{\partial g^j}, \tag{7}$$

where

$$S = S^* - \frac{1}{2} \sum (h_i + \bar{h}_i - 2)\bar{g}^i \bar{g}^j - \frac{1}{3} \sum C_{ijk} \bar{g}^i \bar{g}^j \bar{g}^k + \dots \tag{8}$$

Zamolodchikov[3] has shown that in the space of two-dimensional field theories, there exists a function $S(g)$ such that, near fixed points, $S(g)$ is given by (8) and $S^* = c$ is the central charge. It is *not* known whether (7) is valid everywhere, or equivalently, whether a RG scheme can be chosen so that (7) is valid everywhere. Subsequently, we shall argue that the requirement (7) is natural if one wishes to "quantize" (1).

2. Stochastic Quantization

The equation (1) gives us paths in M that interpolate between the various solutions of $\beta^i = 0$. The solutions of (1) are reminiscent of instantons in a fictitious time $t = \log a$. Does there exist a lagrangian formalism which incorporates (1) as saddle point equations? We show below that this is indeed the case: the lagrangian turns out to be the supersymmetric quantum mechanics in M with $t = \log a$. We arrived at this conclusion by constructing a quantum theory of the classical dynamical system $\beta^i = 0$ by the procedure of stochastic quantization in which t itself is the extra time. In this way, we can address the question of whether the system is quantizable.

Following Wilson, let us interpret (1) as the equation of motion of a point particle with coordinates $g^i(t)$. As $t \to \infty$, $g^i(t) \to g^*$, i.e., the particle asymptotically reaches a definite position. If one is interested in quantization, the final position is not definite, but is characterized by a probability distribution $P_\infty(g^*)$. One thus considers a probability function $P(g^i, t; g_0^i, t_0)$ such that $P(g^i, t; g_0^i, t_0)\, d\mu(g)$ is the probability of finding the particle in a neighbourhood of g characterized by the measure $d\mu(g)$ at time t, given the position g_0 at time t_0. The averages

$$\begin{aligned} \langle dg^i \rangle &\equiv \int d\mu(g)(g^i - \bar{g}_0^i)\, P(g, t; g_0, t_0) \\ \langle dg^i dg^j \rangle &\equiv \int d\mu(g)(g^i - \bar{g}_0^i)\,(g^j - g_0^j) P(g, t; g_0, t_0), \end{aligned} \tag{9}$$

with $t = t_0 + dt$, satisfy the Langevin equations[6]

$$\begin{aligned} \langle dg^i \rangle &= b^i(g)dt + O((dt)^2) \\ \langle dg^i dg^j \rangle &= \hbar G^{ij}(g)dt + O((dt)^2), \end{aligned} \tag{10}$$

where $\hbar$ is the Planck constant and $b^i(g)$ is the drift term. Note that because of the stochastic nature of the quantities, $b^i(g)$ is *not* a covariant object. The measure $d\mu(g) = dg\sqrt{G}$, the natural measure on M. Henceforth, we will restrict discussion to situations having time translation invariance

$$P(g, t; g_0, t_0) = P(g, g_0 | \tau) \qquad \tau \equiv t - t_0 .$$

Equations (9) and (10) immediately lead to the equation of motion for the average of a given function on M. With

$$\langle f(g)\rangle_\tau \equiv \int d\mu(g) f(g) P(g, g_0|\tau),$$

we get for a Markov process (i.e., for $P(g_1, t_1; g_2, t_2) = \int d\mu(\bar{g}) P(g_1, t_1; \bar{g}, \bar{t}) P(\bar{g}, \bar{t}; g_2, t_2)$)

$$\frac{\partial}{\partial \tau} \langle f(g)\rangle_\tau = \langle \mathscr{L}^+ f(g)\rangle, \tag{11}$$

where

$$\mathscr{L}^+ = b^i \frac{\partial}{\partial g^i} + G^{ij} \frac{\partial}{\partial g^i} \frac{\partial}{\partial g^j}.$$

Integration by parts of (11) leads to the Kolmogorov-Fokker-Planck (KFP) equation[7] for P

$$\frac{\partial}{\partial \tau} P(g, g_0|\tau) = H_{FP} P(g, g_0|\tau) = \frac{1}{2}\hbar \Delta P - \nabla_i(\tilde{\beta}^i P), \tag{12}$$

where $\Delta = \nabla_i \nabla^i$ is the laplacian on functions, ∇_i is the covariant derivative, and

$$\tilde{\beta}^i \equiv b^i + \frac{1}{2}\hbar \Gamma^i, \qquad \Gamma^i = \Gamma^i_{jk} G^{jk}.$$

$\tilde{\beta}^i$ is a proper vector field in M, so that (12) is a covariant equation. In the classical limit, i.e., $\hbar \to 0$ becomes equivalent to a classical equation of motion of the form (1) — provided we identify $\tilde{\beta}^i = \beta^i$.

3. Detailed Balance

It is reasonable to require that the stochastic process described by the Langevin equations satisfies the principle of detailed balance

$$\begin{aligned} P(g, t; g', t') P_\infty(g') &= P_\infty(g) P(g', t; g, t'), \\ P_\infty(g) &\equiv \lim_{\tau \to \infty} P(g, g_0|\tau). \end{aligned} \tag{13}$$

Introducing the notation

$$P(g, t; g', t') = \langle g| e^{-(t-t')H_{FP}} |g'\rangle,$$

this means

$$\langle g|e^{-(t-t')H_{FP}}|g'\rangle P^{-1}{}_{\infty}(g) = P^{-1}{}_{\infty}(g')\langle g'|e^{-(t-t')H_{FP}}|g\rangle, \tag{14}$$

which says that H_{FP} is required to be hermitian with respect to the measure $d\mu(g)P_{\infty}^{-1}$. This happens if, and only if,

$$\beta_i = \frac{\hbar}{2}\frac{\partial}{\partial g^i}\ln P_{\infty} \equiv -\frac{\partial}{\partial g^i}h \qquad h = -\frac{\hbar}{2}\ln p_{\infty}\,. \tag{15}$$

We shall call the function $h(g)$ the "height function" for reasons which will become clear later. The manifestly hermitian $\hat{H}_{FP}$ is given by a similarity transformation on H_{FP}

$$\begin{aligned}\hat{H}_{FP} &= P_{\infty}^{-1/2}\frac{1}{2}\nabla^i(\hbar\nabla_i - 2\beta^i)P_{\infty}^{-1/2}\\ &= (\hbar\nabla^i + \partial^i h)(\hbar\nabla_i - \partial_i h)\\ &= \hbar^2\nabla_i\nabla^i - (\partial_i h)(\partial^i h) - \hbar\nabla^i\partial_i h.\end{aligned} \tag{16}$$

Using (11) and (15), we arrive at the equation of motion for h,

$$\partial_\tau\langle h\rangle = \left\langle \hbar\nabla_i\nabla^i h - (\nabla^i h + \frac{1}{2}h\Gamma^i)\nabla_i h\right\rangle. \tag{17}$$

In the classical limit as $h \to 0$, (17) becomes

$$\partial_\tau\bar{h} = -\nabla^i\bar{h}\nabla_i\bar{h} = -\beta^i\beta^j G_{ij}\,. \tag{18}$$

$\bar{h} = h(\bar{g})$ and $g = \bar{g}$ satisfy the classical equations of motion (*viz.* the renormalization group equations). (18) is similar to the equation satisfied by Zamolodchikov's function. In particular, if G_{ij} is a metric of positive signature, $\partial_\tau\bar{h} < 0$, i.e., the classical value of h decreases with τ. It is important to note that quantum fluctuations (i.e., the diffusion term in (17)) violate this condition.

4. Supersymmetric Quantum Mechanics

If the manifold is flat, $G_{ij} = \delta_{ij}$, the path integral representation of $P(g; g_0, \tau)$ can be written as supersymmetric quantum mechanics of a superparticle moving in a super magnetic field (or Kähler potential) $h(g)$, *a lá* Parisi and Sourlas.[8] The presence of a non-trivial metric G_{ij} makes this procedure difficult. Instead we generalize the final result: i.e., consider a superparticle moving in a *curved* space in the presence of a Kähler Potential $h(g)$.

We thus introduce the fermionic partners of the bosonic coordinates, $\psi^i(t)$, $\psi^{*i}(t)$ which are single component grassmann variables. The corresponding operators satisfy

$$(\psi^i)^2 = (\psi^{*i})^2 = 0; \qquad \{\psi^i, \psi^{*j}\} = G^{ij}.$$

Introducing the supercharges

$$Q = \sum_i \psi^i(\hbar\nabla_i + \beta_i), \qquad Q^* = -\sum_i \psi^{*i}(\hbar\nabla_i - \beta_i), \qquad \beta_i = -\partial_i h,$$

it is easy to check that $Q^2 = Q^{*2} = 0$. It is worth noticing that this is true if, and only if, β_i is curl free, i.e., $\beta_i = -\partial_i h$. Now we complete the supersymmetry algebra by introducing the hamiltonian

$$\hat{H} = \frac{1}{2}(QQ^* + Q^*Q) = \hbar^2\Delta + G^{ij}\partial_i h \partial_j h + \hbar\nabla_i\nabla_j h[\psi^{*i}, \psi^j]\,. \tag{19}$$

When $\hat{H}$ acts on a wave function containing p fermions, Δ is the Laplacian on p-forms. This follows from the well known correspondence between fermionic fock space and the de Rham complex. It may be easily verified that acting on scalar functions $\hat{H}$ reduces to $\hat{H}_{FP}$. For flat space, H is the hamiltonian for supersymmetric quantum mechanics obtained by the Parisi-Sourlas procedure. In general, (19) is the correct generalization to curved space.

The path integral corresponding to (19) is well known[9]

$$Z \sim \int \mathscr{D}g^i \mathscr{D}\psi^i \exp\left[i\int_{\tau_1}^{\tau_2} \mathscr{L}(g, \psi)dt\right],$$

where

$$\mathscr{L} = \frac{1}{2}G_{ij}\left(\frac{dg^i}{dt}\frac{dg^j}{dt} - \partial^i h \partial^j h\right) + i\bar{\psi}^i \frac{D}{Dt}\psi^j G_{ij}$$

$$+ \frac{1}{4}R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\bar{\psi}^l - \nabla_i\nabla_j h\bar{\psi}^i\bar{\psi}^j \tag{20}$$

with

$$\frac{D}{Dt}\psi^i = \frac{d\psi^i}{dt} + \Gamma^i_{jk}\frac{dg^j}{dt}\psi^k .$$

The saddle points, as $\hbar \to 0$, of the supersymmetric lagrangian (20) are precisely the RG equations (1). The relevant part of the action is the bosonic part, which becomes, after a Wick rotation,

$$\frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau G_{ij}\left(\frac{dg^i}{dt}\frac{dg^j}{dt} + \partial^i h \partial^j h\right) = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau \left(\frac{dg^i}{dt} \pm \partial^i h\right)\left(\frac{dg_i}{dt} \pm \partial_i h\right)$$

$$\mp (h(\tau_2) - h(\tau_1)) . \tag{21}$$

Hence, for a fixed height difference, the euclidean action is minimised by

$$\dot{g}^i = \pm \partial^i h, \tag{22}$$

the $+(-)$ sign denotes anti-instantons(instantons).

5. Asymptotic Probability Distribution

From the path integral (20), we can extract the probability distribution which satisfies the KFP equation by projecting out the purely bosonic part

$$P(g, g_0|\tau) = \int \prod_i d\psi_i(0) d\psi_i(\tau)\psi_i(0)\psi_i(\tau) Z(g, g_0; \tau; \psi(0), \psi(\tau)). \tag{23}$$

Clearly the asymptotic probability distribution must be a normalizable zero energy state of $\hat{H}_{FP}$

$$\hat{H}_{FP}P(g, g_0|\infty) = 0. \tag{24}$$

A *necessary* condition for this is

$$\hat{H}Z(\tau \to \infty) = 0,$$

which is equivalent to

$$QZ(\tau \to \infty) = 0. \tag{25}$$

The existence of a normalizable probability thus requires unbroken supersymmetry. This means that the Witten index $\mathrm{tr}(-1)^F$ must be non-zero.[10] In addition, the zero mode of Q must have a nonzero bosonic part.

An asymptotic probability distribution, if it exists, is of the form

$$P_\infty(g) \sim e^{-k(g)},$$

which corresponds to the quantization of the classical equations $\beta^i = 0$. The saddle point equations (22) certainly describes non-perturbative fluctuations which interpolate (in fictitious time, τ between the critical points $\beta^i = 0$). Note that the latter are extrema of the action, not necessarily of the potential, of the dynamical theory described by $h(g)$. Thus the saddle points are not *a priori* related to usual instantons; rather, they are instantons in the fictitious time τ.

6. Height Function and Zamolodchikov's Function

Our discussion of the stochastic quantization of the RG equations is applicable to critical phenomena in any number of dimensions, provided $\beta_i = \partial_i h$. As mentioned above, Zamolodchikov has shown that this is true for two-dimensional theories near a fixed point. Since the coefficients of the β-function are universal only near criticality — it may indeed be possible to have $\beta_i = \partial_i h$ globally by a suitable choice of the RG scheme. Then the Zamolodchikov function $S(g)$ may be identified with the height function $h(g)$.[11] It is not known whether a version of Zamolodchikov's result is valid in higher dimensions. Two-dimensional critical phenomena have, of course, special significance since amongst the fixed points there are classical solutions of string theories. With the above identification, $P_\infty(g)$ describes a quantum string theory.

Having identified the height function with Zamolodchikov function, a critical point in the Morse theory sense, i.e., $\partial_i h = 0$ corresponds to a fixed point of the RG. (Henceforth, "critical point" shall be used in the Morse theory sense.) Around a critical point, h has an expansion of the form (8) with the identifications

$$h(g^*) = c, \qquad \partial_i\partial_j h|_{g^*} = \delta_{ij}(h_i + \bar{h}_i - 2), \qquad \partial_i\partial_j\partial_k h|_{g^*} = C_{ijk}. \tag{26}$$

Positive, zero and negative eigenvalues of the Hessian matrix $\partial_i\partial_j h$ correspond to scaling fields which are irrelevant, marginal or relevant, respectively. In the following, we shall assume, for simplicity, that there are no marginal directions.

The number of negative eigenvalues, i.e., directions of instability at a critical point, is called the Morse index of that critical point. The RG formalism tells us that perturbing the system at a fixed point by a relevant operator takes it away to another fixed point with a lower value of the height h^*, by virtue of Zamolodchikov's theorem. (The latter fixed point may not correspond to any critical phenomena, e.g., the high temperature fixed point.) The description of this phenomena in the language of supersymmetric quantum mechanics (19) and (20) is as follows.[12]

In (17), $(dh)^2$ is the potential energy which vanishes at g^* and the Hessian becomes the mass matrix of fermions. In the harmonic approximation, i.e., when only the quadratic piece of $S(g)$ is retained, we can solve the problem around each g^*. If the Morse index of g^* is p, the ground state wave function around g^* is a p-form

$$\Psi(g^* + \bar{g}, \psi_i) = \Psi_0(\bar{g}_i)\psi_1^* \dots \psi_p^*|0\rangle, \tag{27}$$

where $\Psi_0(\bar{g}_i)$ is the usual gaussian of the bosonic coordinates and $|0\rangle$ is the vacuum of the fermionic fock space. This state has zero energy and is annihilated by Q. Now consider another critical point with Morse index $(p-1)$. Once again the zero energy ground state in the harmonic approximation is a $(p-1)$-form. There must be an energy barrier between these two critical points since the energy is positive and we have assumed that there are no flat directions.

The instantons which tunnel between the two ground states cause a non-zero overlap between ψ_p and ψ_{p-1}. If one approximates the path integral (20) by this instanton, the Dirac operator D/Dt has a zero mode which is simply the supersymmetry transform of the instanton solution: $\psi_0^i = \bar{\varepsilon}\dot{g}_{cl}^i$. This means that the functional measure (20) must be modified by insertion of this zero mode and one can now calculate

$$\langle\psi_{p-1}|\psi_p\rangle \sim \exp[-|h(\tau = -\infty) - h(\tau = \infty)|]\,. \tag{28}$$

If these were the only two critical points on the manifold, this overlap would give rise to a non-zero shift of the ground state energy and thus cause supersymmetry breaking. As noted above, this implies that there is no asymptotic probability distribution. In general, however, there are other critical points and other instantons so that there may be a zero energy ground state in spite of the overlaps leading to unbroken supersymmetry. Now one has to see whether this ground state has a bosonic component — if not, an asymptotic distribution does not exist.

7. An Application to Two-dimensional Critical Phenomena

Zamolodchikov[13] and Cardy[14] have argued that the minimal models of FQS have a Landau-Ginsburg description. Since our purpose here is only illustrative, we shall assume this to be true. The order parameter is a single scalar field ϕ and the hamiltonian is parametrized by a polynomial interaction

$$H = \partial_z\phi\partial_{\bar{z}}\phi + P_{2p-2}(\phi) \tag{29}$$

where

$$P_{2p-2}(\phi) = r_1\phi + r_2\phi^2 + \ldots + r_{2p-2}\phi^{2p-2}.$$

p is related to the central charge by the formula $c = 1 - 6/(p(p+1)), p \geq 3$. The set of couplings $(r_1, r_2, \ldots r_\infty)$ parametrize the theory space of the general hamiltonian (29), i.e., under renormalizations, the polynomial interaction is closed. Call this theory space M_ϕ. The operators ϕ^k, $k = 1,2, \ldots, (2p-4)$ are relevant perturbations. Now we consider two possibilities. First, when the interaction in (29) is restricted to be even, $P_{2p-2}(\phi) = P_{2p-2}(-\phi)$. Let us call this theory space M_ϕ. We are assuming a cutoff scheme that will preserve this symmetry. At the pth critical point in M_ϕ, there are $(p-1)$ relevant perturbations. Hence the Morse index of the pth critical point is $n_p = (p-2)$. For example, for $p = 3$ (Ising model), $n_3 = 1$; $p = 4$ (tricritical Ising model) $n_4 = 2$; $p = 5$ (three state Potts model) $n_5 = 3$, etc. Hence as the central charge increases from 1/2 to 1, the Morse index increases in steps of 1. Using this, we may calculate the Morse Polynomial of the height function h (which we have taken to be the Zamolodchikov function)

$$m(t, h) = \mathrm{tr}(t^F) = 1 + \sum_{p=3}^{\infty} t^{n_p}. \tag{30}$$

The extra 1 in (30) corresponds to the contribution of the high temperature fixed point. Using the values of n_p,

$$m(t, h) = 1 + t + t^2 + \ldots = \frac{1}{1-t} \tag{31}$$

and the Morse inequality states that $P(t) < m(t, h)$, where $P(t) = \Sigma_p t^p B_p$ is the Poincare polynomial of M_ϕ, B_p are the betti numbers. We also have for the Witten index

$$\mathrm{tr}(-1)^F = m(-1, h) = \frac{1}{2}. \tag{32}$$

In the case of Z_2 even interactions, Morse theory is not powerful enough to tell us all the betti numbers: it simply provides an upper bound for these. However, if we consider the space M_ϕ which includes both the odd and even polynomials, $n_p = 2(p-2)$ and we have a stronger result. In thise case, the Morse polynomial is

$$m(t, h) = 1 + t^2 + t^4 + \ldots = \frac{1}{1-t^2}.$$

Using the Lacunary principle of Morse theory,[15] we can conclude that

$$m(t, h) = P(t)$$

and hence one knows the betti numbers $B_p = 0$ for p odd and $B_p = 1$ for p even.

We would like to point out that the fixed points in the full space M_ϕ are not really isolated points, but lines, since for the fixed points for any given p the operator ϕ^{2p-3} is a redundant operator. This has important consequences in the RG flow patterns in theory space. For example, the tricritical Ising model has four relevant operators in M_ϕ, ϕ, ϕ^2, ϕ^3 and ϕ^4, while ϕ^5 is a redundant operator. The Ising model has two relevant operators ϕ and ϕ^2, while ϕ^3 is redundant. It may be easily seen that if one makes a perturbation which is a general combination of ϕ^3 and ϕ^4, RG flows end up at different points on the line parallel to the r_3 axis containing the point $r_4 = 0$. However, all points of the latter line correspond to the Ising model since for this critical point ϕ^3 is a redundant perturbation. This suggests that a more suitable theory space for the whole set of odd and even perturbations is the space spanned by the r_i modulo the symmetry transformations which correspond to the redundant perturbations. In this particular case, this latter theory space would be a quotient space.

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