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THE ROLE OF QUANTIZED 2-DIM. GRAVITY IN STRING THEORY

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We present a formulation of string theory in which the 2-dim. metric is exactly quantized in the framework of SL(2, *R*) current algebra. In this way we replace the conformal invariance prescription by the principle of reparametrization invariance. The theory is formulated in arbitrary number of dimensions since the usual restriction of fixed matter central charge is not present. As a concrete illustration of our approach, we show that in 25 Euclidean dimensions the usual amplitudes of the 26-dimensional bosonic string theory arise. The extra time-like dimension emerges as a mode of the 2-dim. metric and the gravitational dressing of vertex operators gives rise to their time dependence.

1. Introduction

The σ-model approach to string theory[1] considers the motion of a string coupled to the backgrounds which can be thought of as condensates of the modes of the string. An essential role is played by the 2-dim. metric g_{ab} because the string couplings are governed by 2-dim. reparametrization invariance. However, the Weyl mode of g_{ab} is treated classically and one imposes the constraint of anomaly cancellation to decouple it from the theory. This requirement of Weyl invariance is elevated to the status of a principle and used to derive constraints on the backgrounds, and develop a modular invariant perturbation theory around them. It does not, however, provide a framework to formulate string theory beyond the semiclassical method – the necessity of which cannot be underestimated. An understanding of the foundations of string theory may enable us to make further progress in this direction.

Here we investigate the formulation of string theory that quantizes the 2-dim.

metric exactly. This maintains exact reparametrization invariance and hence the vanishing of all components of the stress tensor. In this way we replace the conformal invariance prescription by the principle of reparametrization invariance in 2-dim. As a first step we discuss the problem of d-scalar fields interacting with 2-dim. gravity, extending the light-cone gauge method of Polyakov and Knihznik, Polyakov and Zamolodchikov.[2] We demonstrate that at $d = 25$, one can indeed obtain all the amplitudes of the usual 26 dim. closed string theory, with Lorentzian signature of space-time. The 26th 'time' coordinate emerges from the 2-dim. metric. We also explain the global mobius factor and comment on the generic non-vanishing of the partition function in this approach. In this theory, the vertex operators depend both on the matter and 2-dim. metric degrees of freedom. It is therefore natural to conclude that the string couplings (for non-trivial backgrounds) also have such a dependence. In this sense the present paper puts previous work in this direction on a firm footing.[3]

We mention here that for $d > 25$ we find that the time coordinate is linearly coupled to a background charge. The "spectrum" that emerges and the physics of this theory seem to be identical to the recent work of Antoniadis *et al.*[4] In the present paper, however, we shall restrict ourselves to the $d = 25$ case.

2. *d*-Scalar Fields Interacting with 2-Dim. Gravity

Denote the scalar fields by $\phi^i(x)$, $i = 1,\ldots, d$. The reparametrization invariant action is

$$S = \frac{1}{8\pi}\int d^2x\sqrt{-g}\,\partial_a\phi^i\partial_b\phi^i g^{ab}\delta_{ij}\,. \tag{1}$$

$g_{ab}(x)$ is the 2-dim. metric and $g = \det g_{ab}$. Our present discussion is in the light-cone gauge, where a reparametrization invariant ultraviolet cutoff is possible, when one integrates over both ϕ^i and g_{ab}. Introducing light-cone coordinates $x^\pm = x^0 \pm x^1$ and the light-cone gauge defined by $(ds)^2 = dx^+dx^- + h_{++}(x^+, x^-)(dx^+)^2$, which is possible when the 2-dim. space-time has Minkowski signature, the action (1) becomes

$$S = \frac{1}{4\pi}\int dx_+dx_-(\partial_+\phi^i\partial_-\phi^i - h_{++}(\partial_-\phi^i)^2)\,. \tag{2}$$

The presence of massless matter fields in the classical Lagrangian leads to the "chiral" invariance

$$\delta\phi^i = \varepsilon_+\partial_-\phi^i,\ \delta h_{++} = (\partial_+ - h_{++}\partial_- + \partial_-h_{++})\varepsilon_+\,, \tag{3}$$

where $\varepsilon_+ = \varepsilon_+(x^+, x^-)$. However, the quantum effective action is invariant under a smaller symmetry because of the trace anomaly.

2.1. *Emergence of SL(2, R) local symmetry*

The quantum effective action is defined by integrating over ϕ^i using a reparametrization invariant regulator,

$$i\Gamma(h_{++}) = \ln(\int_\phi \exp(iS)) \tag{4}$$

and

$$\delta\Gamma(h_{++}) = \int \delta h_{++} T_{--} , \tag{5}$$

where T_{--} is a component of the 2-dim. stress tensor, satisfying the conservation law

$$\nabla_+ T_{--} + \partial_- \theta = 0 . \tag{6}$$

Since the trace of the stress tensor $\theta \propto R$, being the 2-dim. curvature scalar,

$$\nabla_+ T_{--} \propto \partial_- R = \partial_-^3 h_{++} . \tag{7}$$

Using (5) and (7), we can find the "chiral" invariance that survives the quantization of the scalar fields,

$$\delta\Gamma(h_{++}) \propto \int \partial_-^3 \varepsilon_+ h_{++} .$$

This vanishes if $\partial_-^3 \varepsilon_+ = 0$. This means that Γ is invariant only under the "chiral" transformations parametrized by

$$\varepsilon_+(x^+, x^-) = \varepsilon^+(x^+) - 2x^- \varepsilon^0(x^+) + (x^-)^2 \varepsilon^-(x^+) . \tag{8}$$

Now using the equation of motion $\delta\Gamma/\delta h_{--} = T_{--} = 0$, we get $\partial_-^3 h_{++} = 0$ implying the decomposition

$$h_{++} = J^+(x^+) - 2x^- J^0(x^+) + (x^-)^2 J^-(x^+) . \tag{9}$$

Substituting (8) and (9) into (3) we find a residual SL(2, R) local symmetry

$$\begin{aligned} \delta\phi^i(x^+, x^-) &= 2\eta_{\alpha\beta} \varepsilon^\alpha(x^+) l^\beta_{\lambda=0} \phi^i(x^+, x^-) \\ \delta J^\alpha(x^+) &= \partial_+ \varepsilon^\alpha(x^+) + 2 f^\alpha_{\beta\gamma} J^\beta(x^+) \varepsilon^\gamma(x^+) . \end{aligned} \tag{10}$$

Here $\eta_{\alpha\beta}$ is the SL(2, R) metric ($\eta_{00} = -1$, $\eta_{-+} = \eta_{+-} = 1/2$) and the $l^\alpha_\lambda (\alpha = 0, \pm)$ defined by

$$\begin{aligned} l^+_\lambda &= (x^-)^2 \partial_- + 2\lambda x^- , \\ l^0_\lambda &= x^- \partial_- + \lambda , \\ l^-_\lambda &= \partial_- , \end{aligned} \tag{11}$$

satisfy a SL(2, R) Lie algebra

$$\begin{aligned} [l^\alpha_\lambda, l^\beta_\lambda] &= -f^{\alpha\beta}_\gamma l^\gamma_\lambda , \\ f^{+-}_0 &= 2, f^{0+}_+ = -1, f^{0-}_- = +1 . \end{aligned} \tag{12}$$

The parameter λ is the SL(2, R) spin. In (10) we have inserted the "classical values" of the SL(2, R) spin which are in general modified by quantum fluctuations of h_{++}.

2.2. *The matter sector*

First let us do some classical analysis. In addition to the "chiral" invariance (3), the action (1) is invariant under the following transformations:

$$\delta\phi^i = \omega^i(x^+), \quad \delta h_{++} = 0\,, \tag{13a}$$

$$\delta\phi^i = \eta_-(x^+)\partial_+\phi^i, \quad \delta h_{++} = \partial_+(\eta_-(x^+)h_{++})\,. \tag{13b}$$

The first of these, the holomorphic shifts of the scalar fields, are generated by the holomorphic current

$$K_+^i = \partial_+\phi^i - h_{++}\partial_-\phi^i, \quad \partial_- K_+^i = 0\,. \tag{14}$$

Note that (at least classically)K_+^i is invariant under SL(2, R) transformations. Moreover, under the variations $\delta\phi^i = \omega^i, \delta h_{++} = 0$ we have $\delta_\omega K_+^i = \partial_+\omega^i$, which is equivalent to the Heisenberg current algebra

$$K_+^i(x^+)K_+^j(y^+) = \frac{-\delta^{ij}}{(x^+ - y^+)^2}\,. \tag{15}$$

Further it is possible to derive the Ward identities from which one can extract the operator product expansion of K_+^i with the string modes, e.g., for the tachyon one has

$$K_+^i(x^+)\cdot e^{iP_i\phi^i(y^+,y^-)} = \frac{-iP^i}{(x^+ - y^+)}e^{iP_i\phi^i(y^+,y^-)}\,. \tag{16}$$

Under the second symmetry transformation, the holomorphic Weyl rescalings of h_{++}, (13b),[5] the K_+^i changes as $\delta_\eta K_+^i = \partial_+(\eta K_+^i)$. Using (15) one can show that the corresponding generator is the matter stress tensor

$$T_{++}^m = -\frac{1}{2}:\ K_+^i K_+^i:\,. \tag{17}$$

More generally, one can also derive ward identities for these symmetry transformations. These show that the corresponding generator is the non-anomalous part of the total (matter + ghost + gravity) stress tensor (see the next section). Finally, we note that the trace of the matter stress tensor vanishes, i.e.,

$$\theta^m = 0\,. \tag{18}$$

2.3. *Quantization of the metric h_{++}*

The presence of the local SL(2, R) symmetry enables an unambiguous quantization of the 2-dim. gravitational field and the quantum theory is defined by:

(i) the SL(2, R) current algebra

$$J^{\alpha}(x^{+})J^{\beta}(y^{+}) = \frac{-1(K/2)}{(x^{+}-y^{+})^{2}}\eta^{\alpha\beta} + f^{\alpha\beta}{}_{\gamma}\frac{J^{\gamma}(y^{+})}{(x^{+}-y^{+})} + \cdots \tag{19}$$

(ii) the operators of the theory are characterized, besides other quantum numbers, by an SL(2, R) spin λ,

$$J^{\alpha}(x^{+})\cdot V_{\lambda}(y^{+},y^{-}) = l^{\alpha}_{\lambda}(y^{-})\frac{V_{\lambda}(y^{+},y^{-})}{(x^{+}-y^{+})} + \cdots \tag{20}$$

(iii) the gravitational stress tensor (which is a reflection of the contribution due to the presence of regulators) is

$$T^{\mathrm{gr}}_{++} = -\frac{1}{K+2}\eta_{\alpha\beta}\colon\ J^{\alpha}J^{\beta}\colon\ + \partial_{+}J^{0} \tag{21}$$

$$\theta^{\mathrm{gr}} = \partial_{-}^{2}h_{++} = 2J^{-}\,. \tag{22}$$

The total stress tensor including matter and ghost contributions are

$$T_{++} = T^{m}_{++} + T^{\mathrm{gh}}_{++} + T^{\mathrm{gr}}_{++},\quad \theta = \theta^{\mathrm{gr}} \tag{23}$$

T_{++} and θ generate the residual reparametrizations in the light-cone gauge and hence are weakly set to zero

$$T_{++} \approx 0,\quad \theta \approx 0\,. \tag{24}$$

(iv) A set of observables in a theory of gravity are correlation functions with their arguments integrated over. In the light-cone gauge we have

$$S(1,2,\cdots,n) = \int_{-\infty}^{+\infty}\prod_{i=1}^{n}dx_{i}^{+}dx_{i}^{-}\langle V_{\lambda_1}(x_1)\cdots V_{\lambda_n}(x_n)\rangle\,. \tag{25}$$

In the definition of $\langle V_{\lambda_1}(x_1)\cdots V_{\lambda_n}(x_n)\rangle$ we do not divide by the partition function. A more precise meaning to this expression will be given later. It is worth mentioning that the constraints (24) will be interpreted as operative within the integrated correlation functions. As we shall see later under special circumstances (25) can be interpreted as S-matrix.

2.4. *Energy-momentum relation of string states*

The total stress tensor of the matter, ghost and gravity system

$$T_{++} = -\frac{1}{2}\colon K^{i}_{+}K^{i}_{+}\colon + T^{\mathrm{gh}}_{++} - \frac{1}{K+2}\eta_{\alpha\beta}\colon J^{\alpha}J^{\beta}\colon + \partial_{+}J^{0} \approx 0 \tag{26}$$

satisfies a Virasoro algebra. The total central charge of the Virasoro algebra must vanish for consistency. This gives the following formula for the central charge K of the SL(2, R) current algebra

$$K + 2 = \frac{d - 13 \pm \sqrt{(1-d)(25-d)}}{12} . \tag{27}$$

The choice of sign depends on whether we are defining the semiclassical limit by $d \to -\infty$ or $d \to +\infty$. At $d = 25$, however, there is a unique value $K + 2 = 1$. To derive the energy-momentum relation, say for the string state created by the operator $V_{P,\lambda} = e^{iP_i\phi^i}$, (the tachyon) we will assume that the operator receives gravitational dressing characterized by the SL(2, R) spin λ. Since all our calculations are to be done within integrated correlators of the type (25), it is convenient to introduce the notion of a physical operator

$$V_\lambda^{\text{ph}}(x^+) = \int_{-\infty}^{+\infty} dx^- V_\lambda(x^+, x^-) \tag{28}$$

and note the properties

$$\begin{aligned} J^-(x^+)\cdot V_\lambda^{\text{ph}}(y^+) &= 0 , \\ J^0(x^+)\cdot V_\lambda^{\text{ph}}(y^+) &= (\lambda - 1)\frac{V^{\text{ph}}(y^+)}{x^+ - y^+} , \\ J^+(x^+)\cdot V_\lambda^{\text{ph}}(y^+) &= \frac{2(\lambda-1)}{(x^+ - y^+)} \int dy^- y^- V(y^+, y^-) . \end{aligned} \tag{29}$$

In deducing the above we have freely integrated by parts and dropped the surface terms. It is crucial to note that this can be done only when it is possible to introduce an SL(2, R) invariant infrared regulator. In a non-rational conformal field theory coupled to gravity, like the string model under consideration, this is indeed possibly by analytically continuing the momenta and the SL(2, R) spin to complex values. The introduction of infrared cutoffs in the form of mass terms or finite area violate SL(2, R) invariance.

It is easy to calculate

$$T_{++}(x^+)V_\lambda^{\text{ph}}(y^+) = \frac{\gamma V_\lambda^{\text{ph}}(y^+)}{(x^+ - y^+)^2} + \frac{\partial_+ V_\lambda^{\text{ph}}(y^+)}{(x^+ - y^+)} + \cdots$$

$$\gamma = \frac{P_i P_i}{2} + \frac{(\lambda-1)^2 + (\lambda - 1)}{K+2} - (\lambda - 1) . \tag{30}$$

Now since we require the constraint $T_{++} \approx 0$, within correlation function, i.e.,

$$\int dy^+ T_{++}(x^+)V_\lambda^{\text{ph}}(y^+) = 0 \tag{31}$$

we have to set $\gamma = 1$, which is the KPZ dimension formula.[2,6,7] At $d = 25$, this leads to

$$\frac{P_i^2}{2} + (\lambda - 1)^2 = 1 . \tag{32}$$

Introducing the particle energy by $\gamma = 1 + i\varepsilon/\sqrt{2}$ (complex SL(2, R) spin), we get the energy momentum relation

$$\varepsilon^2 - P_i^2 = -2 . \tag{33}$$

The same method can be used to derive the energy momentum relation for all the other modes of the string.

2.5. *Operator equations of motion*

As we have indicated, the theory under consideration has 3 infinite dimensional algebras associated with it: the matter current algebra (15), the SL(2, R) current algebra (19) and the Virasoro algebra generated by (26). We will require that all the operators of the theory, are primary fields with respect to the semidirect product of these algebras. For illustration we consider the tachyon

$$\begin{aligned} K_+^i(x^+)V(y^+,y^-) &= -iP_i\frac{V(y^+,y^-)}{(x^+-y^+)} , \\ J^\alpha(x^+)V(y^+,y^-) &= \frac{l_\lambda^\alpha V(y^+,y^-)}{(x^+-y^+)} , \\ T_{++}(x^+)V(y^+,y^-) &= \frac{\tilde{\Delta}_\lambda V(y^+,y^-)}{(x^+-y^+)^2} + \frac{\partial_+ V(y^+,y^-)}{(x^+-y^+)} . \end{aligned} \tag{34}$$

The Sugawara relation

$$\oint T_{++}\frac{dx^+}{2\pi i} = L_{-1} = -K_{-1}^i K_0^i - \frac{1}{K+2}2\eta_{\alpha\beta}J_{-1}^\alpha J_0^\beta \tag{35}$$

implies that a primary field is degenerate and leads to the operator equation of motion

$$\partial_+ V_\lambda(x^+,x^-) - iP^i{:}K_+^i(x^+)V_\lambda(x^+,x^-){:} + \frac{2}{f}\eta_{\alpha\beta}l_\lambda^\alpha{:}\, J^\beta(x^+)V_\lambda(x^+,x^-){:} = 0 , \tag{36}$$

where the normal ordering is defined using the operator product expansion for primary fields, e.g.

$$\begin{aligned} \eta_{\alpha\beta}l_\lambda^\alpha J_{-1}^\beta V(y) &= \lim_{x\to y}\left[\eta_{\alpha\beta}l_\lambda^\alpha J^\beta(x)V(y) - \frac{\lambda^2-\lambda}{f}\frac{V(y)}{(x^+-y^+)}\right] \\ &\equiv \eta_{\alpha\beta}l^\alpha{:}\, J^\beta V{:} \end{aligned}$$

and we have introduced the notation $K + 2 = f$.

We now proceed to analyze the operator equation (36). Since we know that the K_+^i and J^α operator product is non-singular, we may consider the decomposition

$$V_\lambda(x^+, x^-) = V_+(x^+)V_-(x^+, x^-)$$

such that the following operator product relations are satisfied

$$K^i_+(x^+)V_-(y^+,y^-) = 0,\quad K^i_+(x^+)\cdot V_+(y^+) = \frac{-iP^i}{(x^+-y^+)}V_+(y^+)$$
$$J^\alpha(x^+)\cdot V_+(y^+) = 0,\quad J^\alpha(x^+)\cdot V_-(y^+,y^-) = \frac{l^\alpha_\lambda}{(x^+-y^+)}V_-(y^+,y^-)\,. \tag{37}$$

Then (36) implies the pair of equations

$$\partial_+V_- + \frac{2}{f}\eta_{\alpha\beta}l^\alpha_\lambda\!: J^\beta V_-\!: = 0\,, \tag{38}$$

$$\partial_+V_+ - iP^i\!: K^i_+V_+\!: = 0\,. \tag{39}$$

2.6. *Gravitational dressing*

Let us analyze (38), the equation which determines the gravitational dressing of the operator $V(x^+,x^-)$. Since the gravitational dressing is proportional to $(\lambda-1)$, the eigenvalue of J^0 (Eq. (29)), we isolate this piece in (38):

$$\partial_+V_- + \frac{2}{f}l_{\lambda=1}\!: J\,V_-\!: + \frac{(\lambda-1)}{f}\!: \partial_-h_{++}V_-\!: = 0\,. \tag{40}$$

In the above equation we have used the convenient notion $\eta_{\alpha\beta}l^\alpha_\lambda J^\beta = l_\lambda\cdot J$. We now look for a solution of (40) of the form

$$V_-(x^+,x^-) = :\tilde{V}_-(x^+,x^-)e^{-\frac{(\lambda-1)}{f}\phi^0(x^+,x^-)}:\,, \tag{41}$$

where

$$\partial_+\tilde{V}_- + \frac{2}{f}l_{\lambda=1}\!: J\tilde{V}_-\!: = 0\,, \tag{42}$$

and the dressing field ϕ^0 satisfies the inhomogeneous equation

$$\partial_+\phi^0 + \frac{2}{f}l_{\lambda=0}\!: J\phi^0\!: = \partial_-h_{++}\,. \tag{43}$$

Now define the field $X^0(x^+,x^-)$ by

$$\sqrt{2f}X^0(x^+,x^-) = :U_{\lambda=0}(x^+,x^-)\phi^0(x^+,x^-):\,, \tag{44}$$

where U_λ is the path-ordered exponential

$$U_\lambda(x^+,x^-) = P\exp\left[\frac{2}{f}\int_{-\infty}^{x^+}dy^+J(y^+)\cdot l_\lambda(x^-)\right]. \tag{45}$$

The definition of path ordering is such that the point x^+ is at the extreme right and the point $-\infty$ at the extreme left. In the inverse of U_λ it is, of course, just the opposite. Then (43) implies

$$\sqrt{2f}\,\partial_+ X^0 = :U_{\lambda=0}\partial_- h_{++}: \,, \tag{46}$$

while from (44) we get

$$\sqrt{2f}\,\partial_- X^0 = :U_{\lambda=1}\partial_-\phi^0: \,. \tag{47}$$

Both (46) and (47) lead to

$$\sqrt{2f}\,\partial_-\partial_+ X^0 = U_{\lambda=1} 2J^- \approx 0\,. \tag{48}$$

This equation, together with the observation that $\partial_- X^0 \neq 0$ (Eq. (47)), implies that within physical observables defined in (25) one may use the decomposition.

$$X^0(x^+, x^-) = X^0_+(x^+) + X^0_-(x^-)\,, \tag{49}$$

where $X^0_\pm(x^\pm)$ satisfy the free-field operator product relations,

$$X^0_\pm(x^\pm) X^0_\pm(y^\pm) \sim \ln(x^\pm - y^\pm)\,. \tag{50a}$$

Furthermore, (42) is solved by

$$\tilde{V}_-(x^+, x^-) = U^{-1}_{\lambda=1} V_-(x^-)\,, \tag{50b}$$

where the operator $V_-(x^-)$ is blind to 2-dim. gravity and will be fixed later. Putting together the above results, we get

$$V_-(x^+, x^-) = :e^{-\frac{(\lambda-1)}{f}\phi^0(x^+,x^-)} U^{-1}_{\lambda=1}(x^+, x^-) V_-(x^-): \,.$$

We may equivalently write the above as

$$V_-(x^+, x^-) =: U_{\lambda=1} e^{-\frac{(\lambda-1)}{f}\phi^0} U^{-1}_{\lambda=1} V_-(x^-):$$

since only the identity part contributes in the first U factor inside integrated correlation functions. Thus,

$$\begin{aligned} V_-(x^+, x^-) &= e^{-\frac{(\lambda-1)}{f}(U_{\lambda=0}\phi^0)} V_-(x^-): \\ &= :e^{-\sqrt{\frac{2}{f}}(\lambda-1)X^0} : V_-(x^-)\,. \end{aligned}$$

At $d = 25, f = 1$ and using $\lambda = 1 + i\varepsilon/\sqrt{2}$, the above equation becomes

$$V_-(x^+, x^-) = :e^{-i\varepsilon X^0(x^+,x^-)}: V_-(x^-)\,. \tag{51}$$

Equation (51) isolates the gravitational dressing of $V_-(x^+, x^-)$. It is this form of the dressed operator that makes it natural to interpret $X^0(x^+, x^-)$ as the time coordinate.

We will now solve (39) and fix $V_-(x^-)$ in (51). Let us introduce the coordinates $X^i(x^+, x^-)$ by the definition

$$X^i(x^+, x^-) = :U_{\lambda=0}(x^+, x^-)\phi^i(x^+, x^-): . \tag{52}$$

Then,

$$\partial_+ X^i = :U_{\lambda=0} K^i_+: \tag{53a}$$

and

$$\partial_- X^i = :U_{\lambda=1} K^i_-: , \tag{53b}$$

where we have defined $\partial_- \phi^i \equiv K^i_-$. Using the holomorphic property of K^i_+, i.e., $\partial_- K^i_+ = 0$ (which also implies the operator equation for $K^i_-, \partial_+ K^i_- + \frac{2}{f} l_{\lambda=1}\!: JK^i_-: = 0$, see (38)), we deduce from either of (53) that

$$\partial_- \partial_+ X^i = 0 . \tag{54}$$

So we may write

$$X^i(x^+, x^-) = X^i_+(x^+) + X^i_-(x^-) , \tag{55}$$

where $X^i_\pm(x^\pm)$ satisfy the free field operator products

$$X^i_\pm(x^\pm)\cdot X^j_\pm(y^\pm) \sim -\delta^{ij}\ln(x^\pm - y^\pm) . \tag{56}$$

Equation (39) is then solved by

$$V_+(x^+) = :e^{iP^i X^i_+(x^+)}: . \tag{57}$$

The operators $V_-(x^-)$ are constructed out of the $X^i_-(x^-)$ introduced above. In particular, for the tachyon under consideration, $V_-(x^-) = :e^{iP^i X^i_-(x^-)}:$. The choice of this operator ensures that the gravitational dressing does not renormalize the spin. In summary, at $d = 25$, the effective tachyon operator is

$$\begin{aligned} V(x^+, x^-) &= :e^{iP^i X^i_+(x^+)}: :e^{-i\varepsilon(X^0_+(x^+)+X^0_-(x^-))}: :e^{iP^i X^i_-(x^-)}: \\ &= :e^{i(P^i X^i - \varepsilon X^0)}: \end{aligned} \tag{58}$$

which is precisely the tachyon operator in the 25+1 dim. closed bosonic string with energy-momentum relation $\varepsilon^2 - P^{i2} = -2$. One can similarly show that the other dressed vertex operators reduce to their 25+1 dim. counterparts.

2.7. *The S-matrix*

Now that we know the formula (58) for the effective tachyon vertex, we can easily calculate the scattering amplitude

$$S(1, 2, \ldots, n) = \frac{1}{\text{Vol}} \int \prod_{r=1}^{n} dx_r^+ dx_r^- \prod_{i<j} (x_i^+ - x_j^+)^{-p_i \cdot p_j} \prod_{k<l} (x_k^- - x_l^-)^{-p_k \cdot p_l} , \tag{59}$$

where $p_i^\mu = (\varepsilon_i, \mathbf{P}_i)$ and $p_i p_j = \varepsilon_i \varepsilon_j - \mathbf{P}_i \cdot \mathbf{P}_j$. In the above expression "Vol" is the volume of global SL(2, R) ⊗ SL(2, R) group. The reason for the presence of this factor is the following. The correlation function $\langle V_{\lambda_1} \cdots V_{\lambda_n} \rangle$ which enters the definition of $S(1, 2, ..., n)$ in (25) is defined by the corresponding functional integral *divided* by the volume of the local symmetry group parametrized by the SL(2, R) parameter $\varepsilon_+(x^+, x^-)$ (3) and the holomorphic weyl scalings $\eta_-(x^+)$ (13b). These local symmetries are generated respectively by the SL(2, R) currents and T^{sug}_{++}, which satisfy closed algebras with central terms. Since the vertex operators create highest weight states of SL(2, R), the positive modes of the currents and so also the positive modes of $T^{\text{sug.}}_{++}$ vanish on physical states. In the functional integral this "gauge-fixing" accounts for the local part of the gauge volume. The zero modes of the currents and $T^{\text{sug.}}_{++}$ satisfy a global SL(2, R) ⊗ SL(2, R) algebra without central term. This is the remaining global symmetry of the n-point amplitudes and the presence of "Vol" factor in (59) just accounts for this global part of the gauge volume. If $n \geq 3$ we can fix the gauge further and the "Vol" factor can be taken care of as usual by fixing three points. For $n \leq 2$ (for example, for the partition function) one has to do additional gauge fixing on h_{++}. It is important to point out here that in this correct treatment the partition function is generically non-zero. In the "old" string theory the vanishing of the partition function can be seen from an overdivision of gauge volumes; the "Weyl volume" actually contains the "volume" of the conformal killing modes.

Using the methods outlined above, it is easy to calculate the S-matrix of all the modes of the string at $d = 25$. The graviton vertex for example is given by

$$V(x^+, x^-) = \sum_{\mu\nu} h_{\mu\nu}(\mathbf{P}, \varepsilon) \colon K^\mu_+(x^+) K^\nu_-(x^+, x^-) e^{iP^i\phi_i(x^+, x^-)} \colon . \qquad (60)$$

The time components $K^0_\pm$ are defined by $K^0_+ = \frac{1}{\sqrt{2}} \partial_- h_{++}$, $K^0_- = \frac{1}{\sqrt{2}} \partial_- \phi^0$, where ϕ^0 is the dressing field introduced in (43). The gravitational dressing introduces these new operators, which are beyond those present in the original conformal field theory. Using the constraint (31) it can be seen that

$$p^\mu h_{\mu\nu} = 0, \quad \varepsilon^2 - \mathbf{P}^2 = p^\mu p_\mu = 0 , \qquad (61)$$

which are equivalent to the linearized version of the Ricci flat condition, $R_{\mu\nu} = 0$ for the traceless part of $h_{\mu\nu}$. The vertex operator (60) can be written within correlation functions in terms of the free fields X^i and X^0:

$$V(x^+, x^-) = \sum_{\mu\nu} h_{\mu\nu}(\mathbf{P}, \varepsilon) \colon \partial_+ X^\mu \partial_- X^\nu e^{ip^\mu X_\mu} \colon . \qquad (62)$$

In the case when $d \neq 25$ ($K + 1 \neq 0$), the time-coordinate is coupled to a background charge and the constraint (31) leads to linearized version of the equations derived in Ref. 3, namely $R_{\mu\nu} = Q\Gamma^\phi_{\mu\nu}$. The significance of the full nonlinear equations in relation to theory space[8] is discussed in detail in Ref. 9.

2.8. *Energy conservation*

Finally, in this section we will show that in (25) energy is conserved, i.e., the sum

$\sum_{i=1}^{n}\varepsilon_i$ vanishes. This is because at $d = 25$ ($K + 1 = 0$), the vacuum carries vanishing J^c spin. In general, for $d \neq 25$ ($k + 1 \neq 0$), $\sum_i \varepsilon_i$ is proportional to ($K + 1$). This means that the scalar field X^0 has a background charge proportional to ($K + 1$). To see this let us rewrite the T_{++} stress tensor in terms of the new variables, X^i and X^0. The gravitational part of the stress tensor, T_{++}^{gr}, is given by[2,7]

$$T_{++}^{gr} = \frac{1}{2f}\!:\!\left[\frac{1}{2}(\partial_- h_{++})^2 - h_{++}\partial_-^2 h_{++}\right]\!: -\frac{1}{2}\partial_+\partial_- h_{++} + \frac{1}{2}x^-\partial_+\partial_-^2 h_{++}\,.$$

Using (45) and (46) this can be rewritten as

$$\begin{aligned} T_{++}^{gr} &= :U_{\lambda=0}^{-1}\left[\frac{1}{4f}(U_{\lambda=0}\partial_- h_{++})^2 - \frac{1}{2}\partial_+(U_{\lambda=0}\partial_- h_{++}) + U_{\lambda=0}x^-\partial_+ J^-\right]: \\ &= :U_{\lambda=0}^{-1}\left[\frac{1}{2}(\partial_+ X^0)^2 - \sqrt{\frac{f}{2}}\partial_+^2 X^0 + \gamma\partial_+ J^-\right]:\,, \end{aligned}$$

where

$$\gamma \equiv U_{\lambda=0}x^-\,.$$

Similarly, it can be shown that

$$\begin{aligned} T_{++}^{m} &= -\frac{1}{2}\!:K_+^i K_+^i: \\ &= -\frac{1}{2}\!:U_{\lambda=0}^{-1}(\partial_+ X^i)^2:\,. \end{aligned}$$

The total T_{++} stress tensor can, therefore, be rewritten as

$$T_{++} = :U_{\lambda=0}^{-1}\left[-\frac{1}{2}(\partial_+ X^i)^2 + \frac{1}{2}(\partial_+ X^0)^2 - \sqrt{\frac{f}{2}}\partial_+^2 X^0 + \gamma\partial_+ J^- + \eta_{++}\partial_-\varepsilon_- + \zeta\partial_+\varepsilon_+\right].$$

Hence, the constraint $T_{++} \approx 0$ gets translated into

$$\tilde{T}_{++} \equiv :\left[-\frac{1}{2}(\partial_+ X^i)^2 + \frac{1}{2}(\partial_+ X^0)^2 - \frac{Q}{2}\partial_+^2 X^0 + \gamma\partial_+ J^- + \eta_{++}\partial_-\varepsilon_- + \zeta\partial_+\varepsilon_+\right]: \approx 0\,.$$

In the above we have changed the coefficient of the linear piece in X^0 from $-\sqrt{f/2}$ to $-Q/2$ since the normal ordering is now with respect to the new fields. Q is fixed by the total central charge vanishing condition. To derive this condition we note that (γ, J^-) form a central charge 2 system. This is because γ is a $\lambda = 0$ operator, and so the central term in the operator product of $:\gamma\partial_+ J^-:$ with itself is given by

$$(:\gamma\partial_+ J^-:)_x (:\gamma\partial_+ J^-:)_y \sim \frac{(\partial_-\gamma)_x \cdot (\partial_-\gamma)_y}{(x^+ - y^+)^4}$$

$$= \frac{(U_{\lambda=1}1)_x \cdot (U_{\lambda=1}1)_y}{(x^+ - y^+)^4}\,.$$

Now using the SL(2, R) current algebra it is easy to show that $(U_{\lambda=1}1)_x \cdot (U_{\lambda=1}1)_y \sim 1$ as $x \to y$. It follows, therefore, that (γ, J^-) form a central charge 2 system. The total central charge vanishing condition is then

$$d + 1 + 3Q^2 + 2 - 28 = 0$$

which gives $Q = -i\sqrt{2/f}\,(K+1)$. Thus, in general, $\sum_i \varepsilon_i = Q \propto (K+1)$. Only at $d = 25$ this vanishes, giving us energy conservation.

3. Concluding Remarks

In this letter we have presented a formulation of string theory that is more general than the usual approach, by exactly quantizing the 2-dim. world sheet metric. This replaces the conformal invariance prescription by the principle of reparametrization invariance and also liberates us from the restriction of fixed matter central charge within the usual σ-model approach, and also provides for new operators beyond those available in the bare conformal field theory. This formulation therefore has the inherent possibility of surveying all the theory space.[9]

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