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CRITICAL BEHAVIOR IN TWO-DIMENSIONAL QUANTUM GRAVITY AND EQUATIONS OF MOTION OF THE STRING

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We show how consistent quantization determines the renormalization of couplings in a quantum field theory coupled to gravity in two dimensions. The special status of couplings corresponding to conformally invariant matter is discussed. In string theory, where the dynamical degree of freedom of the two-dimensional metric plays the role of time in target space, these renormalization group equations are themselves the classical equations of motion. Time independent solutions, like classical vacuua, correspond to the situation in which matter is conformally invariant. Time dependent solutions, like tunnelling configurations between vacuua, correspond to special trajectories in theory space. We discuss an example of such a trajectory in the space containing the $c < 1$ minimal models. We also discuss the connection between this work and the recent attempts to construct non-perturbative string theories based on matrix models.

1. Introduction

This paper continues to investigate the formulation of string theory that quantizes the two-dimensional metric.[1,2] In Ref. 1 this problem was considered in the conformal gauge. The two main points made in that paper were that (i) working in the conformal gauge one can argue that the sigma model couplings receive gravitational dressings and the renormalized couplings develop a dependence on the Liouville mode, and (ii) under certain circumstances, the Liouville mode can be interpreted as a time variable. In Ref. 2, besides developing a deeper understanding of the ideas in Ref. 1, the emergence of the Virasoro amplitudes of the old closed bosonic string in the framework of the light-cone gauge[3] (which provides an unambiguous local quantization of the two-dimensional metric) was established beyond doubt. This work also demonstrated the existence of a scalar field (related in a non-local way to the metric component h_{++}), that describes the gravitational dressing. Both these works have emphasized that the organizing principle of string theory is reparametrization invariance instead of the prescription of conformal invariance.

This paper is mainly devoted to an understanding of the results and the formulae obtained in Refs. 1 and 2, and the general question of renormalization of couplings and the universality of critical behavior in quantum gravity.

2. The Tachyon Coupling in the Bosonic String

Let us begin with the tachyon coupling in the d-dimensional bosonic string theory. The reparametrization invariant action is

$$S=\frac{1}{8\pi}\int d^2x\sqrt{g}\partial_a\phi^i\partial_b\phi^j g^{ab}\delta_{ij}+\int d^dP\ T(P)\int d^2x\sqrt{g}e^{iP^i\phi^i(x)}\,, \tag{1}$$

where ϕ^i are the matter fields, g_{ab} is the two-dimensional metric and we have chosen to write a general interaction in terms of its Fourier modes. $T(P)$ denotes the coupling corresponding to the operator $e^{iP^i\phi^i}$.

In the light-cone gauge of the two-dimensional metric, it is postulated that the gravitationally dressed tachyon vertex operator is a primary field of the constrained SL(2, R) current algebra, labelled by the SL(2, R) spin λ, $V_\lambda(P, \phi(x))$. λ is determined by the KPZ equation[3]

$$\Delta_0+\frac{\lambda(\lambda-1)}{(K+2)}-(\lambda-1)=1\,, \tag{2}$$

where Δ_0 is the dimension of the operator in flat two-dimensional space, i.e., $\Delta_0=1/2\,P^2$ in this case.

The central charge K of the SL(2, R) current algebra is determined by the anomaly vanishing condition

$$\frac{3K}{(K+2)}-6K+d-28=0\,. \tag{3}$$

In terms of $\varepsilon=i\sqrt{2/f}\ (1-\lambda)$, where $f\ =\ K+2$, (2) becomes

$$-\varepsilon(\varepsilon+Q)+P^2=2, \tag{4}$$

where $Q\equiv -i\sqrt{2/f}\,(1-f)=\ \sqrt{(25-d)/3}$. This equation has two solutions, $\varepsilon_\pm$, for a given P. Thus

$$\int d^2x\sqrt{g}e^{iP^i\phi^i}\ T(P)\to\int d^2xV_\lambda(P,\phi(x))\,T(P);\quad \lambda=\lambda_\pm(P^2,d). \tag{5}$$

It was shown in Ref. 2 that within the correlation functions one can make a canonical transformation to free fields X^i involving another free field X^0, which is a mode of the metric, such that

$$V_\lambda(P,\phi(x))\approx e^{\sqrt{\frac{2}{f}}(\lambda-1)X^0}\ e^{iP_iX^i}\,. \tag{6}$$

Equations (4) and (5) then imply that one may write the gravitationally dressed interaction in the form

$$\int dP_i d\varepsilon\int d^2x\, e^{i\varepsilon X^0+iP_iX^i}\ T(P,\varepsilon)\equiv\int d^2x\,T(X^i,X^0)\,, \tag{7}$$

provided $T(X^i, X^0)$ defined above satisfies

$$\left(-\frac{\partial}{\partial X^i}\frac{\partial}{\partial X_i}+iQ\frac{\partial}{\partial X^0}+\frac{\partial}{\partial X^{02}}-2\right)T(X^i,X^0)=0. \tag{8}$$

We wish to emphasize that this is a result of ensuring consistent reparametrization invariance in the theory. Equation (2) is obtained by imposing the conditions T_{++} and $J^- \approx 0$ on physical states, while K is obtained in (3) by ensuring that T_{++} satisfies the Virasoro algebra with a zero central charge. In the light-cone gauge, T_{++} and J^- generate the remaining reparametrizations which maintain the gauge condition. Equation (8), or equivalently (2) or (4), tells us how reparametrization invariance determines the gravitational dressing of the tachyon operator.

It is worthwhile to understand how this result is obtained in the conformal gauge, $g_{ab}=e^{\sigma(x)}\hat{g}_{ab}$. Here $\sigma(x)$ is the Liouville mode, and $\hat{g}_{ab}$ is the fiducial metric. A choice of $\hat{g}_{ab}$ constitutes gauge fixing for local two-dimensional reparametrizations. All quantities must depend on $\hat{g}_{ab}$ only in a reparametrization invariant way. More importantly, however, because $\sigma(x)$ is a dynamical field, two choices of $\hat{g}_{ab}$ which differ from each other only by a conformal factor are indistinguishable. Hence all quantities must be independent of the conformal mode of $\hat{g}_{ab}$.[4] Let us apply this requirement to

$$\langle \int d^2x\sqrt{g}\,T(\phi(x))\rangle=\int d^dP\int d^2x\sqrt{\hat{g}}\,T(P)\langle e^{\sigma(x)}e^{iP^i\phi^i(x)}\rangle. \tag{9}$$

To evaluate (9) we first do the ϕ integration, using a regularization which is general coordinate invariant *a la* the *full* metric g_{ab}. On a spherical topology one can locally choose $\hat{g}_{ab}=e^{\sigma_0(x)}\delta_{ab}$ and then the result is

$$\langle e^{iP^i\phi^i}\rangle_\phi=\exp\left(-\frac{1}{2}P^2\left(\sigma+\sigma_0\right)\right). \tag{10}$$

Equation (9) now becomes

$$\langle \int d^2x\sqrt{g}\,T\rangle=\int d^dP\int d^2x\,T(P)\,e^{-\frac{1}{2}(P^2-2)\sigma_0}\langle e^{-\frac{1}{2}(P^2-2)\sigma(x)}\rangle, \tag{11}$$

where the expectation value is now in the theory of $\sigma(x)$. Averaging over $\sigma(x)$ is a complicated matter since it is not a Gaussian field. With some hindsight, we can, however, assume that it is possible to transform to a new field $\eta(x)$ which has the action[4]

$$S=\frac{1}{8\pi}\int d^2x\sqrt{\hat{g}}\left[\partial\eta\partial\eta+Q\hat{R}\eta\right]. \tag{12}$$

The field $\eta(x)$ can be identified with iX^0, where X^0 is the free field introduced in (6).

The quantity within the expectation value in (11) becomes some functional of η, $f(\eta(x))$. In principle, one may imagine determining this functional – provided the exact transformation to $\eta(x)$ is known. However, $f(\eta)$ must be such that the quantity (9) does not depend on σ_0 as we have argued earlier. This requirement is sufficient to determine $f(\eta)$ in this case. We perform a standard background field decomposition into a classical background $\eta_0(x)$ and the quantum fluctuation $\tilde{\eta}(x)$. Integrating over $\tilde{\eta}(x)$ with a cutoff which respects the reparametrization of the fiducial metric $\hat{g}_{ab}$ one has

$$\langle f(\eta(x))\rangle = \exp\left(\frac{1}{2}\sigma_0(\partial^2_{\eta_0} + Q\partial_{\eta_0})\right)f(\eta_0). \tag{13}$$

Thus (11) is independent of σ_0 provided

$$f(\eta(x)) = e^{\epsilon\eta(x)}, \tag{14}$$

where ε satisfies (4). Once again we can think of $T(\phi(x))$ as receiving a gravitational dressing to become $T(\phi, \eta)$, the η dependence being dictated by two-dimensional general covariance via the differential equation

$$(\partial^2_\eta + Q\partial_\eta + \partial^2_\phi + 2)\,T(\phi, \eta) = 0. \tag{15}$$

So far we have discussed how vertex operators get dressed in the *free* theory. This is however equivalent to a discussion of how the *action S* of the full interacting theory gets dressed at the linearized level in T. The strategy to find the dressed action to higher orders is now clear. In a conformal gauge treatment, we replace $T(\phi(x))$ by $T(\phi(x), \eta(x))$ in the action and determine the latter by requiring that the effective action be independent of σ_0. The above calculation is however identical to the calculation of "beta functions" of the theory with dynamical fields ϕ and η living in a fixed background metric $\hat{g}_{ab}$. We shall refer to these as B-functions to distinguish them from the usual beta functions of the original matter field theory in some *fixed* metric (i.e., gravity is not quantized). $T(\phi, \eta)$ is determined by requiring that the B-function is zero. As for the calculation of the tachyon beta function in the usual fixed metric field theory, this calculation may be done in a double expansion in $T(\phi, \eta)$ and $(\partial^2_\eta + Q\partial_\eta + \partial^2_\phi + 2)T$. The result is

$$B(T) = (\partial^2_\eta + Q\partial_\eta + \partial^2_\phi + 2)\,T(\phi, \eta) + T^2(\phi, \eta) = 0. \tag{16}$$

As is well known this equation has to be supplemented by the requirement that B-functions for all the other couplings, like those for the operator $\partial\eta\partial\eta$ or $\partial\phi\partial\phi$, be zero.

In terms of the *couplings* of the original theory $T(P)$ (16) means

$$(\partial^2_\eta + Q\partial_\eta - P^2 + 2)\,T(P, \eta) + \int dP'\,T(P', \eta)\,T(P - P', \eta) = 0. \tag{17}$$

The couplings have acquired a dependence on η which is related to the scale of the original metric g_{ab}. In this sense we can consider $T(P, \eta)$ as the "renormalized" coupling. We shall make this notion more precise in a later section.

For the theory of ϕ and $\hat{g}_{ab}$, defined on the fixed metric $\hat{g}_{ab}$, the couplings are

$T(P, \varepsilon)$ which are the Fourier components of $T(P, \eta)$ in the η space. It might appear that for each coupling of the original theory $T(P)$ there is a one-parameter infinity of couplings $T(P, \varepsilon)$. What (17) says is that this is not true. In fact it determines $T(P, \varepsilon)$ given P and ε. To this order there are two such solutions, since the differential equation is second order. In higher orders of the double expansion, higher derivatives would appear in general, and the order of the differential equation would then determine the number of possible dressings.

3. Strings in Gravitational Backgrounds

Consider now the bosonic string coupled to a gravitational and a dilaton background. The reparametrization invariant action is

$$S = \frac{1}{8\pi}\int d^2x\sqrt{g}\ \partial_a\phi^i\partial_b\phi^j G_{ij}(\phi)g^{ab} + \int d^2x\sqrt{g}\ R^{(2)}D(\phi) \tag{18}$$

in the standard notation. In the conformal gauge this becomes

$$S = \frac{1}{8\pi}\int d^2x\sqrt{\hat{g}}\ \partial_a\phi^i\partial_b\phi^j G_{ij}(\phi)\hat{g}^{ab} + \int d^2x\sqrt{\hat{g}}\ \hat{R}D(\phi) - \int d^2x\sqrt{\hat{g}}\ \hat{g}^{ab}\,(\partial_i D)\partial_a\sigma\partial_b\phi^i\,. \tag{19}$$

In the quantum theory we go over to the Gaussian field $\eta(x)$ and one has to discuss the general renormalization of the theory in a fixed metric $\hat{g}_{ab}$. A little thought shows that one must begin with general couplings $G_{ij}(\phi, \eta)$, $D(\phi, \eta)$ as in the tachyon case. However, in this case one also has to consider the general couplings $G_{00}(\phi, \eta)$ and $G_{0i}(\phi, \eta)$ which are coefficients of $\partial\phi\partial\phi$ and $\partial\eta\partial\phi$ respectively. Classically $G_{0i} = \partial_i D$ and $G_{00} = 1$. However, all these couplings undergo *independent* renormalizations in the quantum theory and the classical correspondence is lost.

Defining $X^\mu \equiv (\phi^i, \eta)$, the vanishing of the corresponding B-functions leads to the familiar equations[5]

$$\begin{aligned} B(G_{\mu\nu}) &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu D = 0\,, \\ B(D) &= (25-d) + 3\alpha'(-R + 4\nabla_\mu\nabla^\mu D - 4\nabla_\mu D\nabla^\mu D) = 0\,, \end{aligned} \tag{20}$$

to lowest order in the loop expansion parameter α'. (For example for the trivial background $G_{\mu\nu} = \delta_{\mu\nu}$ and $D = 1/2\sqrt{\alpha'}\,Q\eta$, the second equation of (20) reduces to (3).)

As discussed in Refs. 1 and 2, $\eta(x)$ may be interpreted as a time-like coordinate, at least for $d \geq 25$. It is then natural to start with d space-like coordinates ϕ^i, and in the final theory $X^0 = -i\eta$ becomes the single time direction.

Equations (20) are generally covariant in $(d+1)$ dimensions if $D(\phi, \eta)$ is a scalar. This is, however, not necessarily so. For example, for the free string, $D = 1/\sqrt{2\alpha'}Q\eta$ which is not a scalar function and the $(d+1)$ general covariance is lost. It is, however, fruitful to think of this as a breaking due to expansion around a particular symmetry-breaking solution.

4. Conformal Invariance and Static Solutions in String Theory

Although we considered the bosonic string in the above discussion, the result holds for any quantum field theory (hereafter referred to as "matter") coupled to gravity. Consider a term in the action $G_i \int d^2x \sqrt{g}\, O_i(\phi(x))$ where $O_i(\phi(x))$ is a local operator constructed from the matter fields. After coupling to gravity, this term becomes $\int d^2x\; G_i[\eta(x), \phi(x)]$, the η-dependence of which is determined by the corresponding B-function equation $B_i(G_j) = 0$.

The dilaton coupling $D(\eta(x), \phi(x))$ in (20) plays a special role in the theory since it couples to the conformal anomaly. As we have seen above, this coupling is present even in the *free* string theory and is the crucial term which allows consistent strings to exist in other than critical dimensions. This term is present in any field theory coupled to gravity.

A special situation arises when the couplings G_i, except the dilaton, do not get dressed, i.e., do not acquire any η-dependence. In this case the B-functions become the ordinary beta functions in a fixed metric theory. Since the B-functions are always zero, this can happen only if the undressed couplings G_i correspond to values for which the fixed metric theory is conformally invariant. This may be easily seen from the tachyon Eq. (17). When $T(\phi, \eta)$ is independent of η one simply has the equation $\beta_T = 0$ of the model defined on a fixed two-dimensional metric.

The case of the target space gravitational background is a bit subtle. This is because the process of renormalization generates *independent* couplings G_{0i} and G_{00}. As follows from the above discussion it is necessary to have a *large* piece in the dilaton D to accommodate a $d \neq 25$. We thus write, in general,

$$D(\phi, \eta) = \frac{1}{2\sqrt{\alpha'}} Q\eta + \tilde{D}(\phi, \eta). \tag{21}$$

In the following discussion we shall consider the case $\tilde{D} = 0$ without any loss of generality. Equations (20) then become[1]

$$R_{\mu\nu} + \frac{1}{\sqrt{\alpha'}} Q\Gamma^0_{\mu\nu} = 0\,, \tag{22}$$

$$(25 - d - 3Q^2G^{00}) + 6\sqrt{\alpha'}QG^{\mu\nu}\Gamma^0_{\mu\nu} - 3\alpha' R = 0\,. \tag{23}$$

Note that the various terms in (22) and (23) are in different orders in α'. Thus in a consistent loop expansion, they must be separately set to zero. Then (22) would mean

$$R_{\mu\nu} = 0\,, \tag{24}$$

$$\Gamma^0_{\mu\nu} = 0\,, \tag{25}$$

and the only other independent condition obtained from (23) is

$$25 - d - 3Q^2G^{00} = 0\,. \tag{26}$$

Let us now look for η-independent solutions of these equations. This is like looking for zero mode solutions in a Kaluza-Klein theory. We introduce the notation $G_{0i} = A_i$ and $G_{00} = \Phi$ with the warning that A_i is not quite a gauge field in the d-dimensional viewpoint since there is no full $(d + 1)$-dimensional general covariance unless $Q = 0$. Nevertheless, (24) are simply the *vacuum* Einstein equations in $d + 1$ dimensions. For the η-independent metric $G_{\mu\nu}$ they become, as in any Kaluza-Klein theory,

$$\overline{R}_{ij} - \frac{1}{2} G_{ij} \overline{R} = T_{ij} , \tag{27}$$

$$\overline{\nabla}^j F_{ij} = 0 , \tag{28}$$

$$\overline{\nabla}^2 \Phi = 0 , \tag{29}$$

where $F_{ij} = \overline{\nabla}_i A_j - \overline{\nabla}_j A_i$ is the field strength for A_i and T_{ij} is the sum of the energy-momentum tensors of the fields A_i and Φ. Equations (25) become

$$\Gamma^0_{ij} = \frac{1}{2} G^{00} (\overline{\nabla}_i A_j + \overline{\nabla}_j A_i) = 0 , \tag{30}$$

$$\Gamma^0_{0i} = \frac{1}{2} (G^{00} \partial_i \Phi + G^{0k} F_{ki}) = 0 , \tag{31}$$

$$\Gamma^0_{00} = -\frac{1}{2} G^{0k} \partial_k \Phi = 0 , \tag{32}$$

where the bars mean that the corresponding covariant quantities are constructed out of the spatial metric G_{ij} only.

Let us first consider the case $Q \neq 0$, i.e., $d \neq 25$. Since the spatial metric has been assumed to be of positive signature, the consistent solution of (29) on closed manifolds or on open manifolds with smooth asymptotic properties is Φ = constant. Furthermore, (26) independently implies that G^{00} = constant. These two facts, together with Eqs. (28) and (30) imply that $F_{ij} = 0$. Going back to Eq. (27), the energy-momentum tensor T_{ij} vanishes for constant Φ and A_i so that one gets the vacuum Einstein equations in d dimensions, i.e., $\overline{R}_{ij} = 0$. This, however, is precisely the condition that the theory we started with is conformally invariant in a fixed two-dimensional metric.

For $Q = 0$, Eqs. (30) – (32) need not be satisfied, as is evident from the original Eqs. (22) and (23). Furthermore (26) is also trivially satisfied since $d = 25$. We are left only with the condition $R_{\mu,\nu} = 0$, i.e., vacuum Einstein equations in 26 dimensions. Now there is a full 26-dimensional general coordinate invariance. Suppose there is a coordinate system in which the metric is η-independent. Since there is no matter to which the 26-dimensional metric couples, in such a coordinate system, $\eta \rightarrow -\eta$ must be a good symmetry. This means that the components of the metric G_{0i} must vanish, or $F_{ij} = 0$ in Eqs. (27) – (29). Once again, $G_{00} = \Phi$ must be constant by virtue of (29) and the Euclidean signature of space. This then implies, once again $\overline{R}_{ij} = 0$.

In string theory, at least for $d \geq 25$, $X^0 = -i\eta$ is interpreted as a time variable, with the ϕ^i denoting the space variables. The B-function vanishing conditions are nothing but the full classical equations of motion of the string in space-time. The preceding discussion then implies that *time-independent* solutions of the classical equations of motion correspond to conformally invariant matter coupled to gravity in two dimensions. Classical *vacuua* are a special class of such time-independent solutions. In a light-cone gauge description, conformally invariant matter coupled to gravity has an enhanced symmetry of the SL(2,R) current algebra.[3] *Classical ground states are thus distinguished from time-dependent classical solutions in string theories in which the two-dimensional metric is properly quantized: they correspond to points of higher symmetry.* The higher symmetry is, however, explicit in only the light-cone gauge. In formulations of string theory in which the two-dimensional metric is not quantized there is no such distinction between time-dependent and static solutions from the two-dimensional field theory point of view: they both correspond to conformal field theories in fixed metric.

What about time-dependent solutions? These may be of the plane-wave type, giving particle-like excitations or may have some other complicated time dependence. If the time variable is analytically continued so that one has an Euclidean $d + 1$ space-time, one also has *instanton* solutions which represent tunnelling between classical ground states in the full quantum theory. In the theory space, which is the space of (undressed) couplings G_i, such instantons have a special meaning: they are *trajectories* $G_i(\eta)$ which asymptotically interpolate between conformally invariant *points*. In principle, such trajectories contain some nonperturbative information and would contribute factors like exp $(-1/g^2)$, where g denotes the string coupling. In the final section we shall provide an explicit example of such a trajectory in the space containing the $c < 1$ minimal models.

5. Gravitational Dressing and Scale Transformations

In the conformal gauge, arbitrary changes of $\eta(x)$ correspond to changes of scale. In particular, constant shifts of η are global scale changes. The dressed couplings like $T(P, \eta)$ in Eq. (17) describe, in some sense, how couplings change with global scale changes. We shall now try to make this notion more precise.

We would like to emphasize that we are *not* talking of scale changes due to changes of any cutoff in the effective theory of dynamical fields like $\phi(x)$ and $\eta(x)$. Such a cutoff is clearly necessary to perform any computation. The B-functions precisely determine the dependence of couplings on this cutoff. However, as we have seen, the fact that we are dealing with a theory of gravity means that all such B-functions must necessarily be zero, giving us differential equations for the couplings $G_i(\eta)$. Thus, for such admissible $G_i(\eta)$, the scale coming from the necessity of a cutoff has disappeared from the theory, as is expected in a theory of gravity.

Nevertheless, one may think of flows of couplings G_i in terms of finite size scaling. Let us define the partition function in a finite *invariant* area A which is classically given by

$$A = \int d^2x \sqrt{g}\,.$$

In the conformal gauge, after transformation to the Gaussian field $\eta(x)$, this becomes

$$A = \int d^2x \sqrt{\hat{g}}\, f[\eta(x)]\,, \tag{33}$$

where the functional form of $f[\eta]$ is determined by requiring that $\langle f[\eta(x)]\rangle$ does not depend on the conformal mode of the fiducial metric $\hat{g}_{ab}$. If we shift the integration variable by a constant ρ,

$$\eta'(x) = \eta(x) + \rho, \tag{34}$$

we describe the same partition function with a different invariant area A'.

In general f is a complicated function. To illustrate the concept, it is, however, sufficient to consider the case of a conformally invariant theory coupled to gravity. The free energy is

$$F_0(A) = \log \int \mathcal{D}\eta\, \mathcal{D}\phi\, e^{-S_0(\eta,\phi)}\, \delta[A - \int d^2x \sqrt{\hat{g}}\, f(\eta(x))]\,. \tag{35}$$

The function $f(\eta)$ may, in this case, be easily determined to be

$$f[\eta(x)] = e^{\alpha_0 \eta(x)}\,; \quad \alpha_0(\alpha_0 + Q) = -2\,. \tag{36}$$

Performing the change of variables in (34) one has[4]

$$F_0(A) = F_0(Ae^{-\alpha_0 \rho}) + \text{constant}\,, \tag{37}$$

the constant being due to the presence of a background charge Q. In fact this constant determines the string susceptibility. Equation (37) shows that *for a conformally invariant theory coupled to gravity the free energy is invariant (upto a constant) under rescalings of the area.* One can alternatively think in terms of a canonical ensemble in which the area is integrated with a weight $e^{-\mu A}$. In this case the cosmological constant μ provides the scale.

Consider now a small perturbation term:

$$S_{\text{int}} = G_i \int d^2x \sqrt{g}\, O_i\,. \tag{38}$$

In the linearized approximation, the dressed object $G_i[\eta(x)]$ satisfies the $B = 0$ equation:

$$[\partial_\eta^2 + Q\partial_\eta + 2(1 - \Delta_0^i)]G_i(\eta) = 0\,, \tag{39}$$

where Δ_0^i is the dimension of the operator O_i in the flat theory. To this order the two allowed dressings are

$$G_i^{\pm}(\eta(x)) \approx \exp[\beta_i^{\pm}\eta(x)], \tag{40}$$

where β_i^{+} are the roots of the equation

$$\beta_i^2 + Q\beta_i + 2(1 - \Delta_0^i) = 0 \tag{41}$$

and the free energy satisfies the relation

$$F(A, G_i) = F(\varepsilon A, G_i \varepsilon^{(\Delta_i - 1)}), \tag{42}$$

where ε is an arbitrary parameter. Here the gravitational scaling dimension Δ^i is given by

$$\Delta_i = 1 - \frac{\beta_i}{\alpha_0}. \tag{43}$$

Δ_i is related to the SL(2, R) spin of the operator O_i in the light-cone gauge description by the formula[6]

$$\Delta_i = \frac{\lambda}{(1 - \lambda_\upsilon)},$$

where λ_υ is a solution of the KPZ Eq. (2) with $\Delta_0 = 0$.

Equation (42) shows that the theory defined on an invariant area A is equivalent to one defined on a rescaled A but with redefined couplings G_i. This clearly defines a flow in the space of G_i's. The equations for the flows are given by the vanishing B-function conditions.

It is natural to extend this notion beyond the linearized approximation. Under rescalings of A the couplings no longer rescale simply, but change in a complicated manner. Nevertheless, their flows $G_i(\eta)$ still satisfy the zero B-function equations.

The stability properties of the conformally invariant point are determined by the scaling dimensions Δ_i. The coupling G_i is stable, marginal or unstable under the increases of the length scale if Δ_i is greater than, equal to, or less than 1. Clearly there would be a critical surface such that under these scale changes a point on the surface would be attracted to the conformally invariant theory. All points on the critical surface have the same large A behavior; i.e., gravitational exponents which relate to asymptotic A behavior are universal and determined by the Δ_i's of the unstable couplings.

The above discussion shows that it is instructive, near a conformally invariant point, to go over to the new variable defined by $\tau \equiv \eta\alpha_0$. Now shifts of τ directly correspond to scale changes of the invariant area and the corresponding $G_i(\tau)$ give the flow of couplings. *In this sense the B-function = 0 equations are precisely the renormalization group equations of the gravity-coupled theory.*

These renormalization group equations differ from those on fixed two-dimensional metric in an important respect: they involve higher order derivates with respect to the scale. Near a conformally invariant point they are second order differential equations. Thus *a priori* there are two sets of flows leading to two sets of exponents. Away from a conformally invariant point, they are, in general, higher order, at least in perturbation theory. Thus, in general, there may be many possible flows, all of which must *asymptotically* merge with either of the two solutions near the conformally invariant points. Thus, near the critical surface, but far away from the conformally invariant point, the exponents are still given, as they should be, by the $\beta_\pm$ of the attractor. This is because the large A behavior is controlled by the conformal theory. In some specific situations, like the $c < 1$ minimal models, there are additional physical requirements which pick out one of the solutions. In string theory, however, it is absolutely essential that both solutions be retained, since they correspond to positive and negative energy particle excitations.

It is fruitful to rewrite equations like (39) in terms of first order equations. For example consider the case where no higher order derivatives in τ appear in the non-linear terms. Then one may write the equation as

$$\frac{\partial P_i}{\partial \tau} = F_i(P_j, G_j), \quad \frac{\partial G_i}{\partial \tau} = P_i, \tag{44}$$

and look at the trajectories in the phase space (P_i, G_i). Clearly the singular points of the differential equations, i.e., points where both $\partial_\tau P_i$ and $\partial_\tau G_i$ vanish, are precisely the conformally invariant points. The stability properties of the phase space trajectories are governed by the appropriate Δ_i's. It is interesting to enquire whether limit cycles or bifurcations can appear, and to investigate their implications for the two-dimensional theory. A further interesting question relates to the meaning of quantities like the Poincaré index in this context.

5.1. *An example from minimal models*

We provide a concrete realization of the above ideas in the space containing the minimal models with central charges

$$c = 1 - \frac{6}{m(m+1)}$$

in regions where m is very large.

In flat two-dimensional space such a model has an operator O ($\equiv O_{(1,3)}$ in standard notation) which has a $\Delta_0 = 1 - 4/m$; i.e., it is relevant but almost marginal. Furthermore, this operator generates a closed algebra with itself and the identity operator I:

$$O \times O = I + O. \tag{45}$$

Let us add a small perturbation to the conformal theory

$$\kappa \int d^2x\, O. \tag{46}$$

The ordinary beta function of κ is given by[7,8]

$$\beta(\kappa) = h\kappa - b\kappa^2 + O(\kappa^3), \tag{47}$$

where we have defined $h \equiv 2(1-\Delta_0)$ and b is the operator product coefficient C_{000}. Equation (47) shows that there is another conformal theory at

$$\kappa = \frac{h}{b},$$

on which the flow emerging from $\kappa = 0$ terminates. Since h is very small and b is finite, this conclusion is consistent with the perturbative calculation. If $\kappa = 0$ corresponds to the minimal model labelled by m, the one at $\kappa = h/b$ is labelled by $(m-1)$.

When we couple this model to gravity, the interaction term becomes

$$\int d^2x \sqrt{\hat{g}}\ \kappa(\eta(x))O \tag{48}$$

and the function $\kappa(\eta)$ has to be determined by requiring that the corresponding B-function vanish. The B_κ-function may be calculated in a double expansion of κ and $(\partial_\eta^2 + Q\partial_\eta + h)$, where Q is appropriate to the $\kappa = 0$ theory. The calculation is similar to but much simpler than that for the tachyon. This is because the operator algebra ensures that only the operators O and I are induced by renormalization. The identity operator contribution actually means that the coupling of the $\hat{R}$ term gets renormalized, which means that Q also becomes a function of η, but the difference from Q at $\kappa = 0$ is at least $O(\kappa^2)$. The contribution of the O operator in the operator algebra gives rise to the B_κ-function

$$B_\kappa(\kappa) = (\partial_\eta^2 + Q\partial_\eta + h)\kappa(\eta) - b\kappa^2(\eta) = 0\,. \tag{49}$$

We shall ignore any higher derivatives on κ which are necessarily of higher order in the double expansion.

Equation (49) describes the motion of a particle in η in a potential

$$V(\kappa) = \frac{1}{2}h\kappa^2 - \frac{1}{3}b\kappa^3 \tag{50}$$

and a very large damping $Q\partial_\eta\kappa$. This is because Q is much larger than h. The η-independent solutions are the two conformal theories at $\kappa = 0$ and $\kappa = h/b$. If one lets the particle drop from the maximum at $\kappa = h/b$ with a very small velocity, the particle rolls down the hill. Because the damping is very large and exceeds the critical value, it will settle down at the minimum at $\kappa = 0$ for $\eta = \infty$ without any oscillation. One thus has a trajectory joining the two conformal theories at *every point of which the model is consistently coupled to gravity.*

It is instructive to look at solutions of (49) in the two asymptotic regimes. In accordance with our earlier discussion, it is useful to change over to the variable $\tau = \eta\alpha_0$ where α_0, the solutions of (36), are

$$\alpha_{0,\pm} = \frac{1}{2}(-Q \pm \sqrt{Q^2 - 8})\,. \tag{51}$$

Since these models lie in the weak coupling regime of quantum gravity, there must be correspondence with the semiclassical limit. This picks out $\alpha_{0,+}$ as the right solution. Note that $\alpha_{0,+}$ is negative, so that direction of τ is opposite to that of η. The sign of the damping term is also reversed. The particle now climbs up the potential hill from $\kappa = 0$ to $\kappa = h/b$; this is possible because of the large anti-damping. Near $\kappa = 0$, the nonlinear term in (49) may be ignored and one has the solution

$$\kappa(\tau) \sim e^{\omega\tau}; \quad \omega_\pm = \frac{1}{2\alpha_{0+}}[-Q \pm \sqrt{Q^2 - 4h}].$$

Once again the semiclassical correspondence picks out ω_+ which for small h becomes

$$\omega_+ \approx -\frac{h}{Q\alpha_{0+}}.$$

Of course ω is nothing but $(1 - \Delta)$, where Δ is the gravitational scaling dimension of the operator O in the $\kappa = 0$ conformal theory coupled to gravity. Since $\alpha_{0+} < 1$, $\omega_+ > 0$, i.e., $\Delta < 1$, as is characteristic of an unstable direction. In fact for the $c < 1$ models, all $\Delta_0 < 1$ lead to $\Delta < 1$; i.e., the dressed operators of relevant operators of the flat theory become unstable in the gravity-coupled theory. Clearly as $\tau \to -\infty$, $\kappa(\tau) \to 0$.

Near $\kappa = h/b$ one can similarly linearize Eq. (49). Strictly speaking, the Q to be used should be the Q of the $\kappa = h/b$ theory. However, the difference in the central charges of the two conformal field theories is of $O(\kappa^3)$.[8] Since Q^2 is determined in terms of the central charge by the zero anomaly condition, one easily sees that the difference in the two Q's is also $O(\kappa^3)$. This difference may, therefore, be ignored in (49). The linearized equation around $\kappa = h/b$ is the same as that around $\kappa = 0$ with the sign of h flipped. This corresponds to the fact that for the $\kappa = h/b$ theory O is an irrelevant operator. One thus has the asymptotic solution

$$\kappa = \frac{h}{b} - Ae^{-\omega_+\tau},$$

which approaches h/b for $\tau \to \infty$. This confirms the qualitative picture of the flow interpolating between the two conformal theories coupled to gravity.

In this example the meaning of the trajectory is clear. The exponents measured at any point of the trajectory between the two conformal points have the values corresponding to the point at $\kappa = h/b$, while to obtain the exponents of the $\kappa = 0$ theory one has to sit *exactly* at $\kappa = 0$.

Finally, we would like to point out that we had a single trajectory in this example even though the differential equation was second order. This happened because we could use the correspondence principle which picked out one of the solutions. In the generic case, this situation does not hold.

6. Concluding Remarks

Much of the work in string theories in the last five years has dealt with string perturbation theory in which the two-dimensional metric is *not* quantized. Thus these theories necessarily live in the critical number of dimensions, 26 for the bosonic string and 10 for the superstring and the heterotic string. Most of the beautiful properties of string perturbation theory which have, on the one hand, given us for the first time a consistent quantum theory of gravity and, on the other hand, made grand unified strings phenomenologically promising, have been obtained in this formalism. In a parallel and initially seemingly unrelated development several authors have investigated the original Polyakov formulation of string theory (in which the two-dimensional metric is quantized) by discretizing the world sheet in terms of random triangulations.[9] In this discretized formulation, the sum over all possible triangulations of a surface of given topology simulates the integration over metrics, and the hope is that if one is able to construct a continuum limit of these models one would have a rigorous *definition* of the sum over metrics, leading to a correct calculational framework. Although so far such a continuum limit has not been satisfactorily constructed in physical number of dimensions, the agreement between the results for critical exponents for all rational $c \leq 1$ models in the discretized approach with the predictions of KPZ theory strongly vindicates this hope.

Nevertheless, till recently it was not obvious what these theories of non-critical strings (the continuum KPZ approach or the discretized approach) have got to do with the standard theory of unified strings. Non-critical strings, of course, have the possibility of describing other phenomena which have a string description but do not contain massless excitations, like the strings of quantum chromodynamics.

The recent progress in the quantization of continuum two-dimensional gravity has changed the situation drastically. Indeed, it has now become clear[1,2] that the only correct way to construct a string theory is to quantize the two-dimensional metric properly. The idea is to start with d space-like directions ϕ^i and couple these to gravity. The dynamical degree of freedom of gravity then plays the role of time; its kinetic energy term has the sign appropriate to a time-like coordinate for $d \geq 25$. For $d = 25$ one regains the usual critical string in $(25 + 1)$ dimensions and its attendant symmetries. In Ref. 2 this statement has been proved rigorously in the light-cone gauge framework and the correct string amplitudes have been shown to be reproduced from the first principles for the first time. In the present work we have shown how a much richer structure emerges when one considers this theory in general background fields. Furthermore, this continuum approach provides a way to calculate critical exponents for general models coupled to gravity and to understand universality of critical behavior. In a sense, these works have bridged the gap between the efforts in critical and non-critical string theories.

The integration over all two-dimensional metrics is, of course, the basic reason why one sums over all topologies in string theory. While our considerations have been based on *tree level* string theory, we believe that a better understanding of the integration over two-dimensional metrics on higher genus surfaces will provide a better understanding of string perturbation theory.

However, some of the basic issues in string theory like supersymmetry breaking, appearance of mass scales at low energies, etc. seem to be non-perturbative in nature. In this respect, the continuum approach we have studied does not appear to be useful at this moment. This is because such an approach is essentially perturbative in nature: one is always restricted to two-dimensional surfaces. It is possible that this is also the reason why very little is understood about the physics of the theory outside of the critical dimension.[10] A non-perturbative formalism presumably liberates one from description in terms of surfaces, just as non-perturbative field theory does not deal with Feynman diagrams. In the past several years, a lot of effort has gone into construction of second quantized string field theories – but with very limited success. In fact, so far there is no satisfactory candidate for closed string field theory. For open strings such a theory does exist but so far no non-perturbative result has been obtained in this framework. It is quite possible that incorporation of dynamical two-dimensional gravity will help in overcoming the outstanding hurdles in this area.

The discrete approach to non-critical strings looks a lot more promising in this regard. This is because summing over triangulations of surfaces of various genus may be regarded as the large-N perturbation expansion of models of $N \times N$ matrices. The matrix models, of course, exist independently of their large-N expansion. It is thus quite natural to regard the matrix models *per se* as *definitions* of non-perturbative string theory.[11] What our work has shown is that it is indeed reasonable to use this approach to try to construct non-perturbative theories of grand unified strings.

Note Added

While this paper was being written we received a preprint by T. Banks and J. Lykken[12] which has much overlap with Ref. 1.

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