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CLASSICAL SOLUTIONS OF 2-DIMENSIONAL STRING THEORY

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We present an exact one-parameter family of solutions to the classical graviton-dilaton system in two dimensions. The solution can be identified as a black hole. We present the solution both in a Schwarzschild-like gauge and in the target space conformal gauge. We discuss possible relations with matrix models.

1. Introduction

Recently there has been considerable interest in the problem of quantized 2-dimensional gravity coupled to a scalar field, which in the absence of gravity would be a $c = 1$ system. There are presently two approaches to this problem — the matrix model[1, 3–9] and the continuum Liouville theory.[10–12] There are encouraging results which indicate that the weak coupling tree level S-matrices computed in the matrix model can be obtained by Liouville theory calculation.[a] One of the interesting results of the continuum Liouville theory is that this system has a 2-dimensional target space, in that the general couplings are functions of two variables one of them being the conformal mode of the 2-dimensional metric.[2] The reasoning that led to this proposal was similar to that which leads to the σ-model approach to critical

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[a]By a Liouville theory here we mean a string in 2-dimensional target space with the following backgrounds: $G_{\mu,\nu} = \eta_{\mu\nu}$ (metric), $\Phi = Q\eta/2$ (dilaton) and $T = a\exp(Q\eta/2)$ (tachyon), where η is the Liouville mode. However, the 2-D gravity theory coupled to matter can admit of more general backgrounds having more complicated dependences on η. The reasoning that led to this proposal[2] is similar to the one that led to the σ-model approach in critical string theory.

string theory. One considers d scalar fields x^i (d-dimensional Euclidean space) coupled to 2-dimensional gravity. In the case where the "tachyon background" is initially set to zero, one can study the spectrum of allowed operators using standard methods. In particular in the conformal gauge there is the $(d+1)$-dimensional "graviton"-like operator

$$\int d^2\xi \sqrt{\hat{g}}\hat{g}^{ab}h_{\mu\nu}(x^i, \eta)\partial_a x^\mu \partial_b x^\nu , \tag{1}$$

where $\mu, \nu = 1, 2, \ldots, d+1$, $x^\mu = (x_i, \eta)$, $i = 1, 2, \ldots, d$; $(-i\partial_\mu + Q_\mu)h_{\mu\nu}(x^i, \eta) = 0 = -i\partial^\mu(-i\partial_\mu + Q_\mu)h_{\lambda\rho}(x^i, \eta)$, $Q^\mu = (Q^i, Q) = (0, Q)$, $Q = \sqrt{(25-d)/3}$ is the background charge, and the dilaton

$$\int d^2\xi \sqrt{\hat{g}}\hat{R}^{(2)} d(x^i, \eta) . \tag{2}$$

If we perturb the original system by these operators, the new couplings become

$$G_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}(x^i, \eta) \text{ and } \Phi = Q\eta + d(x_i, \eta) . \tag{3}$$

Now one can imagine solving the system with these new couplings and find the new spectrum of operators which shift the above and so on. With hindsight from the σ-model approach to critical string theory we then begin with the most general σ-model with couplings that depend on (x^i, η) and require the corresponding β-functions to vanish. In this way the 2-dimensional gravity coupled to d scalar fields can be regarded as a critical string theory in $D = d+1$ target space dimensions.

The σ-model action is

$$A = \frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}G_{\mu\nu}(x)\partial_a x^\mu \partial_b x^\nu - \alpha' \hat{R}^{(2)}\Phi(x^\mu) + T(x^\mu)\right) \tag{4}$$

and the various equations of motion are obtained by setting the various β-functions to zero[13–15]:

$$R_{\mu\nu} - 2\nabla_\mu\nabla_\nu\Phi + \nabla_\mu T \nabla_\nu T = 0 , \tag{5}$$

$$R + 4(\nabla\Phi)^2 - 4\nabla^2\Phi + (\nabla T)^2 + V(T) + c = 0 , \tag{6}$$

$$-2\nabla^2 T + 4\nabla\Phi\nabla T + V'(T) = 0 , \tag{7}$$

$c = (D-26)/(3\alpha') = -8/\alpha'$ since in our case $D = 2$. $V(T) = -(2/\alpha')T^2 + O(T^3)$.

These equations can be derived from a target space action

$$S = \int d^2x \exp(-2\Phi)\sqrt{G}\,[R - 4(\nabla\Phi)^2 + (\nabla T)^2 + V(T)] . \tag{8}$$

Our main task is to solve this coupled set of equations for $G_{\mu\nu}$, Φ, and T. This, however, is a very difficult problem, so to begin with let us attempt to solve the system of equations by setting $T = 0$.

The relevant equations become

$$R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \Phi = 0 \,, \tag{9}$$

$$R + 4(\nabla\Phi)^2 - 4\nabla^2\Phi + c = 0 \,. \tag{10}$$

We proceed in two ways to solve this set of equations.

2. First Method: Schwarzschild-Like Gauge

Let us choose a gauge in which the dilaton is proportional to one of the coordinates. It is possible to choose such a gauge in a local neighborhood that excludes critical points of the dilaton. Thus

$$\Phi = \frac{Q\eta}{2} \,. \tag{11}$$

Then choosing the time coordinate orthogonal to Φ, the metric takes the form

$$G_{\mu\nu} = \text{diag}\,[G_{tt},\, G_{\eta\eta}] \,. \tag{12}$$

Now combining Eq. (9) with the fact that $R_{\mu\nu} - (1/2)RG_{\mu\nu} = 0$ identically in two dimensions we get

$$\nabla_\mu \nabla_\nu \Phi - \frac{1}{2} G_{\mu\nu} \nabla^2 \Phi = 0 \,. \tag{13}$$

Using (11) and (12) in the above gives $\partial_t G_{\eta\eta} = 0$ as well as $\partial_\eta \ln(G_{tt} G_{\eta\eta}) = 0$, which implies that the metric is of the form

$$G_{\mu\nu} = \text{diag}\,[f(t)g(\eta),\, g(\eta)^{-1}] \tag{14}$$

$f(t)$ can be absorbed in the definition of the time coordinate to give

$$G_{\mu\nu} = \text{diag}\,[\epsilon g(\eta),\, g(\eta)^{-1}] \,, \tag{15}$$

where $\epsilon = \text{sgn}(f(t))$. We choose $\epsilon = -1$ for Minkowski space. Substitution in (9) gives

$$g'' = Qg' \tag{16}$$

with solution $g(\eta) = f - a\exp(Q\eta)$. The constant f can be set to unity by a suitable scaling of the target space coordinates. Now substitution in the dilaton β-function (10) gives $Q^2 = 8/\alpha'$.

We summarize the one-parameter solution we have obtained[b]

$$ds^2 = -g(\eta)dt^2 + \frac{1}{g(\eta)}d\eta^2, \quad g(\eta) = 1 - a\exp(Q\eta) \,, \tag{17}$$

[b]The black hole character of our solution was pointed out to us by Witten who has independently found the solution in his study of gauged WZW models. The solution has also been found by Rocek and collaborators.

$$\Phi(\eta,t)=\frac{Q\eta}{2},\quad Q^2=\frac{8}{\alpha'}\text{ and }T(\eta,t)=0\ . \tag{18}$$

The corresponding scalar curvature is

$$R(\eta)=g''=-aQ^2\exp(Q\eta)\ . \tag{19}$$

A few facts about the solution are worth noting:

(i) $Q^2=8/\alpha'$ admits of two roots $Q=\pm2\sqrt{2}/\sqrt{\alpha'}$ and hence there are two solutions which are related to each other by a parity transformation.

(ii) For positive a, $g(\eta)$ has a zero at real $\eta=-\ln a/Q$. This indicates the presence of a horizon at $\eta^*=-\ln a/Q$ just like in the Schwarzschild solution in 4-dimensional general relativity. The parameter a is therefore related to the 'mass' of the black hole.

(iii) The solution has a curvature singularity $|R|=\infty$ at $\eta=+\infty$ for $Q=+\sqrt{8/\alpha'}$ and at $\eta=-\infty$ for $Q=-\sqrt{8/\alpha'}$.

(iv) A limiting case of our solution is in the limit when a is small, so that

$$ds^2=-(1-a\exp(Q\eta))dt^2-(1+a\exp(Q\eta))d\eta^2=(dt^2-d\eta^2)-a\exp(Q\eta)\times(dt^2+d\eta^2)\ . \tag{20}$$

This metric perturbation corresponds to the vertex operator $\int d^2\xi\sqrt{\tilde{g}}(\partial t\bar{\partial}t+\partial\eta\bar{\partial}\eta)\exp(Q\eta)$ for the discrete state coming from the graviton-dilaton sector in $c=1$ Liouville theory.[10] (Recall that this vertex operator cannot be gauged away unlike the ones with generic values of momenta.)

3. Second Method: Target Space Conformal Gauge

We now discuss solving the system of equations in the conformal gauge, defined by

$$ds^2=e^\sigma(dx^2-dy^2)=e^\sigma du dv\ , \tag{21}$$

where u and v are the light-cone coordinates $u=x+y$, $v=x-y$. We list below the formulae for the metric, Christoffel connections and curvature (our convention for Riemann tensor as defined in Ref. 14):

$$G_{uu}=G_{vv}=0,\quad G_{uv}=\frac{1}{2}\exp(\sigma)\ . \tag{22}$$

All components of the connection $\Gamma^\lambda_{\mu\nu}$ vanish except for

$$\Gamma^u_{uu}=\partial_u\sigma,\quad \Gamma^v_{vv}=\partial_v\sigma\ . \tag{23}$$

For the Ricci tensor we have

$$R_{uu}=R_{vv}=0 \tag{24}$$

and

$$R_{uv}=\partial_u\partial_v\sigma\ . \tag{25}$$

The scalar curvature is

$$R = 4\exp(-\sigma)\partial_u\partial_v\sigma \ . \tag{26}$$

In this gauge Eq. (9) becomes

$$R_{uv} = 2\partial_u\partial_v\Phi \ , \tag{27}$$

$$R_{uu} = 2\nabla_u\partial_u\Phi \ , \tag{28}$$

$$R_{vv} = 2\nabla_v\partial_v\Phi \ . \tag{29}$$

Equation (27) implies

$$\partial_u\partial_v\sigma = 2\partial_u\partial_v\Phi \tag{30}$$

with solution

$$\sigma - 2\Phi = F(u) + G(v) \ ,$$

where F and G are arbitrary functions. We can choose $F(u) = G(v) = 0$ by using the residual gauge symmetry of conformal reparametrizations in the conformal gauge. Hence we have the important relation

$$\sigma = 2\Phi \ . \tag{31}$$

With this the dilaton β-function can be written purely in terms of Φ. The result is

$$\partial_u\partial_v(e^{-2\Phi}) = \frac{2}{\alpha'} \ . \tag{32}$$

Combining this with (28), viz.

$$0 = \nabla_u\partial_u\Phi \Rightarrow \partial_u^2(e^{-2\Phi}) = 0 \ , \tag{33}$$

and a similar equation for v coming from (29), we get

$$e^{-2\Phi} = \frac{2}{\alpha'}uv + \alpha u + \beta v + \gamma \tag{34}$$

which after a shift of u, v by constants can be put in the form

$$e^{-2\Phi} = \frac{2}{\alpha'}uv + A \ . \tag{35}$$

Here A is an arbitrary constant signifying a one-parameter family of solutions again. One can locally match with the solutions in the previous gauge to identify $A = a$. Using $\sigma = 2\Phi$ we obtain

$$ds^2 = \frac{dudv}{2uv/\alpha' + a} \ . \tag{36}$$

This form of the solution is similar to that of a black hole in Kruskal–Szekeres coordinates.[17] The horizon is given by the lines $uv = 0$.

The curvature is given by

$$R = \frac{-8a/\alpha'}{2uv/\alpha' + a}$$

which shows that the curvature singularity occurs at $uv = -a\alpha'/2$.

It is important to note that by Eq. (35) the string coupling $g_{\rm st}^2 = \exp(2\Phi)$ (cf. Eq. (9)) also grows infinitely large at the location of the curvature singularity.

It is possible to obtain our two solutions from one another by a system of coordinate transformations $(\eta, t) \leftrightarrow (u, v)$ summarized by

$$u = \exp[-Q/2(\eta' + t)], \quad v = -\epsilon \exp[-Q/2(\eta' - t)]\,,$$
$$(\alpha'/2)\exp(Q\eta') = \frac{\exp(Q\eta)}{|1 - a\exp(Q\eta)|}\,. \tag{37}$$

Here we have taken a as positive, ϵ is $+1$ inside the horizon and -1 outside the horizon. Like in the 4-dimensional case the u, v space is a two-fold cover of the η, t-space.

Note that

$$\partial_{\eta'}\Phi = (1 - a\exp(Q\eta))\partial_\eta \Phi \tag{38}$$

vanishes at the horizon $1 - a\exp(Q\eta) = 0$. This means that the most general solution described in the conformal gauge cannot be put in a gauge where the gradient of the dilaton is held constant.

3.1. *The tachyon equation in the background of a black hole*

Let us now study the tachyon equation in the background of the black hole. The tachyon equation (neglecting nonlinear terms in the tachyon potential) in a gravity-dilaton background is

$$\nabla^2 T - 2\nabla T \nabla \Phi + (2/\alpha')T + O(T^2) = 0\,. \tag{39}$$

Defining $T = \tilde{T}\exp(\Phi)$ we get

$$\nabla^2 \tilde{T} + (\nabla^2\Phi - (\nabla\Phi)^2)\tilde{T} + (2/\alpha')\tilde{T} + O(\tilde{T}^2) = 0\,. \tag{40}$$

Using the dilaton β-function (19), (40) becomes

$$\nabla^2\tilde{T} + \frac{R}{4}\tilde{T} + O(\tilde{T}^2) = 0\,. \tag{41}$$

Introducing 'polar' coordinates $u = r\exp(\theta)$, $v = -r\exp(-\theta)$ and taking $\tilde{T}$ to be of the form $\tilde{T} = f(r)\exp(\omega\theta)$ we get

$$-\frac{\omega^2}{r^2} f + \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f\right) + \frac{2a/\alpha'}{(a - 2r^2/\alpha')^2} f + O(f^2) = 0\,. \tag{42}$$

Let us study the behavior of the solution near the 'future' black hole singularity (in the u, v plane) so we can restrict ourselves to $r > 0$ (we are using the convention that $u = x + y$, $v = x - y$ with y identified with time, so that $u > 0$, $v < 0$ near the future singularity). The case $r < 0$ can be dealt with identically. Thus we write $a = 2b/\alpha'$ and introduce the variable $r = b\exp(\xi)$. Equation (42) becomes

$$\frac{d^2 f}{d\xi^2} \approx -\frac{f}{4\sinh^2\xi} + \omega^2 f \tag{43}$$

which near $\xi = 0$ has a solution

$$f \approx \sqrt{\xi}(\alpha + \beta \ln \xi) \tag{44}$$

hence

$$\tilde{T} \approx \sqrt{a + \frac{2uv}{\alpha'}}\left(\ln\left(a + \frac{2uv}{\alpha'}\right) + \text{ constant}\right)e^{\omega\theta} \,. \tag{45}$$

This implies that the tachyon T and its derivatives become very large at the singularity and cannot be neglected. Hence it is important to consider the back reaction of the tachyon into the graviton-dilaton system.

We should note that our analysis so far has been upto leading order in α'. We do not know how the solutions we have presented here might be modified if higher orders in α' are taken into account.

3.2. *Relationship to matrix model*

It has been suggested by Witten[16] that the black hole solution would be destablized by Hawking radiation whose final state would be described by the standard $c = 1$ Liouville theory or equivalently the $c = 1$ matrix model. Presumably an analysis of Eq. (41) in the spirit of its 4-dimensional analog would lead to an estimate of this Hawking radiation. Note that the $a \to 0$ limit of our solution (Eqs. (17) and (18)) corresponds to flat space and a linear dilaton background which is identified with $c = 1$ matter plus Liouville (see remarks in the introduction). There is evidence that this theory corresponds to the weakly coupled (large cosmological constant) matrix model. (More precisely the tree-level scattering amplitudes calculated in Ref. 11 agree with perturbative calculations in the matrix model.[7-9]) Combining these two observations imply that the black hole solution rolls down to the $c = 1$ matrix model. It is important to note that the latter theory is a completely stable and finite theory of non-relativistic fermions, with a unique ground state given by the filled Fermi sea. The tachyon is an elementary excitation above the Fermi sea.

As we remarked (Eq. (20)), the limiting black hole solution ($a \to 0$) precisely corresponds to the discrete state[10] in the graviton-dilaton sector of Liouville theory. It would be interesting if the matrix model can see this discrete state and hence the limiting black hole. Presumably the non-perturbative (small cosmological constant) regime of the matrix model corresponds to theories more complicated than with flat space and a linear dilaton. It would be extremely interesting to extract the appropriate degrees of freedom suitable for describing this regime and to understand their dynamics.

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